Kernelization Using Structural Parameters on Sparse Graph Classes

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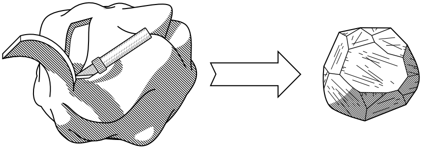
Brief history

Kernelization



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- Goal: obtain polynomial or linear kernels (whenever possible).

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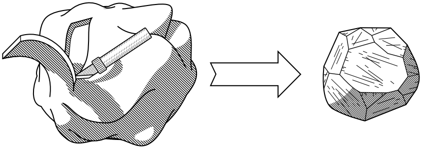

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Basic technique

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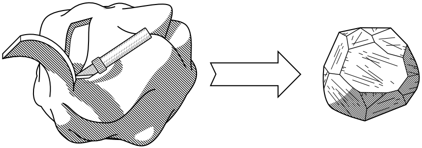

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Algorithmic meta-theorems: algorithms for problem classes

Previous work

- Framework for planar graphs.
  Guo and Niedermeier: *Linear problem kernels for NP-hard problems on planar graphs*

Meta result for graphs . . .

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- ... of bounded genus.
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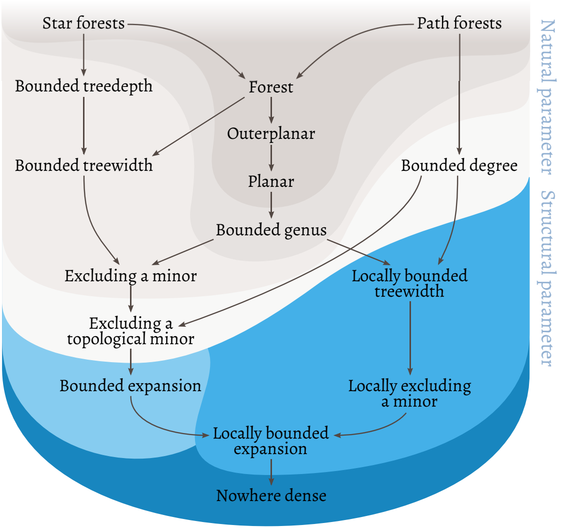
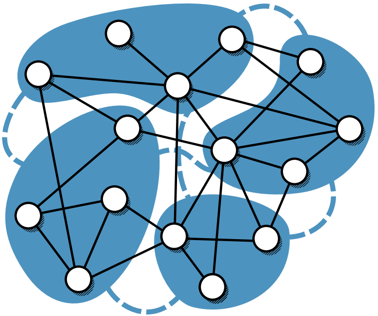
- . . . of bounded expansion, locally bounded expansion and nowhere-dense graphs using structural parameterization.

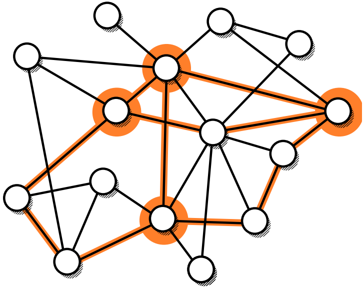
Sparse graphs

The big picture

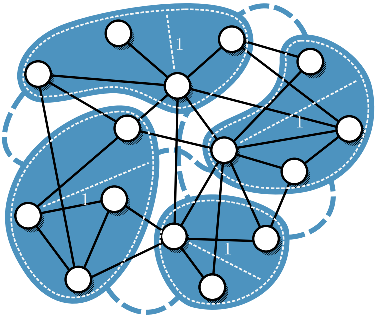


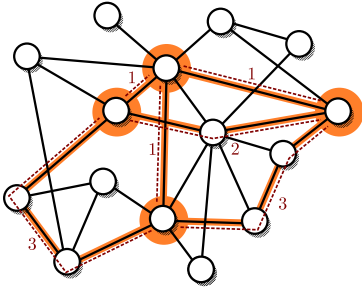
Minors and topological minors





Shallow minors and shallow topological minors





Bounded expansion

$G \nabla r$ denotes the set of the r-shallow minors of G.

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The greatest reduced average density of a graph G is defined as

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A graph class $\mathcal{G}$ has bounded expansion if for some function f and all $r \in \mathbf{N}$

$$\nabla_r(\mathcal{G}) \leq f(r).$$

Excluded minors vs Bounded Expansion

Excluded minors	Bounded Expansion
d-degenerate (depends on excluded minor).	$f(0)$-degenerate (depends on expansion).

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Techniques from H-topo-minor-free graphs don't work! (They use large (non-shallow) topological minors.)

Natural parameters

The problem

Treewidth-t Deletion

Input:	A graph G, an integer k.
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An $f(k)$ kernel on bounded expansion graphs implies $f(k)$ kernel on general graphs.

A kernel on general graphs from sparse graphs

1. Treewidth closed under subdivision of edges.

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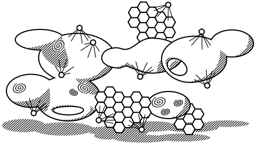
For a meta-kernel result, the parameter must not be closed under edge subdivision!

Structural parameters

The natural view

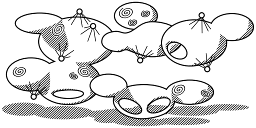
Bounded Expansion ?

$\cup$



H-Topological-Minor-Free *Treewidth-bounding*

$\cup$



H-Minor-Free *Bidimensional* *+separation property*

$\cup$

Bounded Genus *Quasi-compact*

The structural view

Bounded Expansion ?

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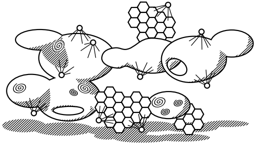

H-Topological-Minor-Free *Treewidth-t Modulator*

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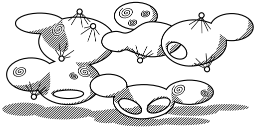

H-Minor-Free *Treewidth-t Modulator* *(implied by Lemma 3.2)*

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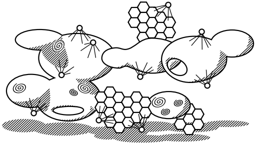
Bounded Genus *Treewidth-t Modulator* *(implied by Lemma 9)*

The structural view

Bounded Expansion *Treedepth-d Modulator*

$\cup$



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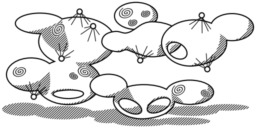

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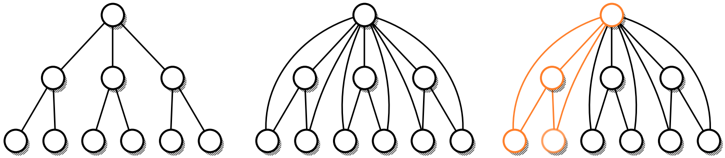
Tree*depth*?



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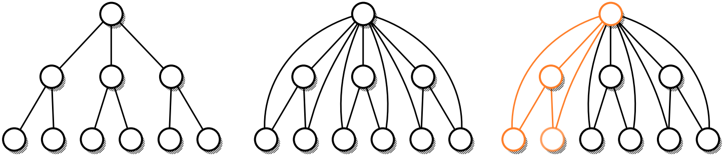

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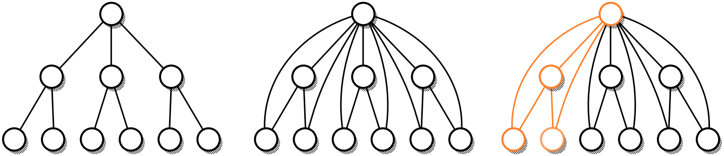

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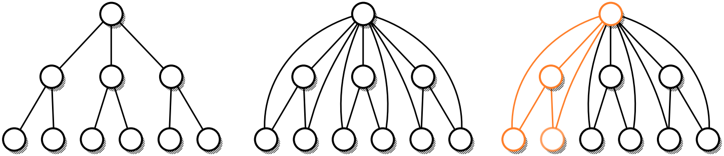

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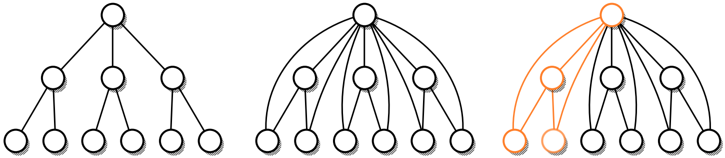

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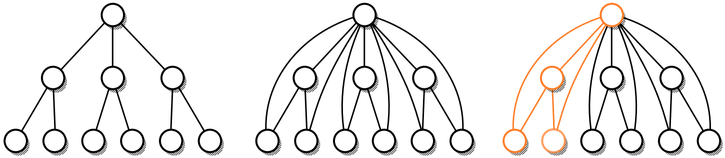

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If X is a treedepth-d-modulator, $G - X$ does not contain long paths.

Theorem

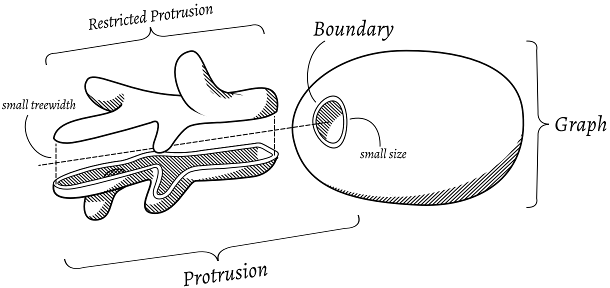
Any graph-theoretic problem that has finite integer index on graphs of constant treedepth admits linear kernels on graphs of bounded expansion if parameterized by a modulator to constant treedepth.

Kernelization possible in linear time.

And ...

- ... quadratic kernels on graphs of locally bounded expansion;
- ... polynomial kernels on nowhere dense graphs.

Protrusion anatomy

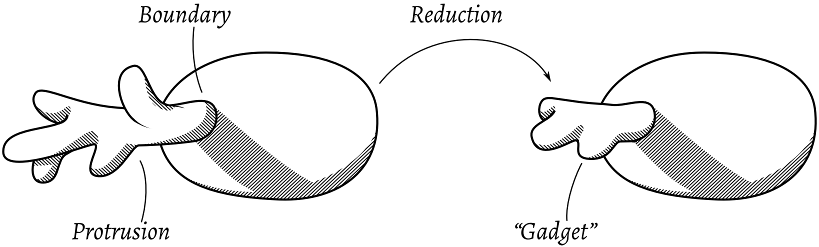


Definition

$X \subseteq V(G)$ is a t-*protrusion* if

1. $|\partial(X)| = |N(X) \setminus X| \leq t$ (small boundary)
2. $\mathbf{tw}(G[X]) \leq t$ (small treewidth)

The magic reduction rule



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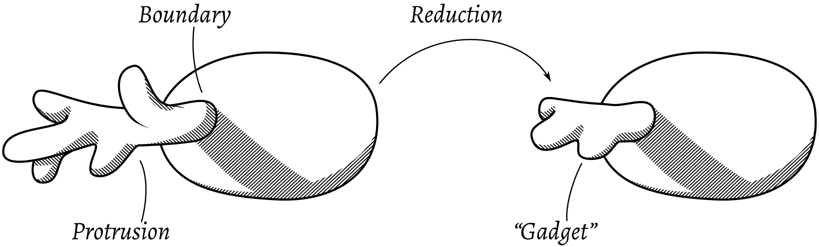

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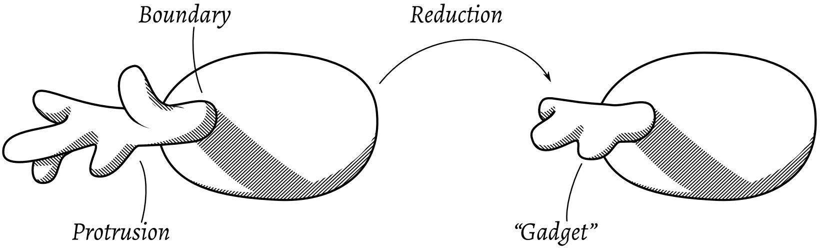

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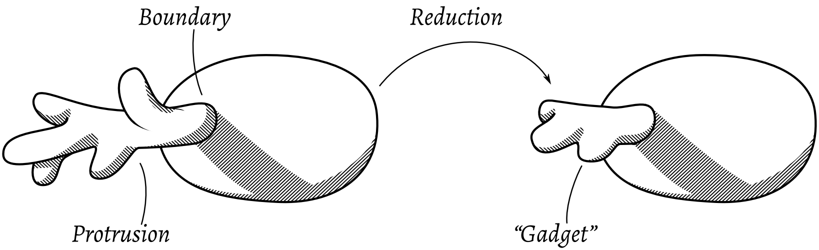

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- Lots of technicalities omitted . . .

The magic reduction rule . . .

Our results assume finite integer index on **graphs of bounded treedepth**.

- How does one ensure that the graph obtained by replacing protrusions is in the same class?

The magic reduction rule . . .

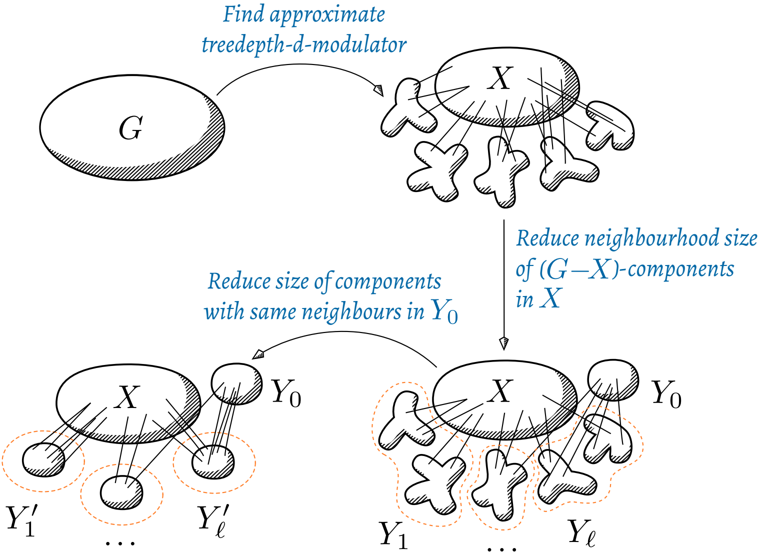
We can show that the replacements are always **induced subgraphs of the original protrusions**.

The magic reduction rule . . .

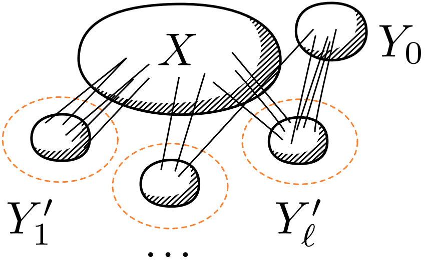
We can show that the replacements are always **induced subgraphs of the original protrusions**.

- Graphs of treedepth d are well-quasi-ordered wrt the induced subgraph relation [Nešetřil and Ossona de Mendez, Sparsity].
- Every equivalence class of the FII-relation is partitioned into a finite number of posets.
- The minimal elements of the posets of each equivalence class are its representatives.

Proof Idea: By a picture



Using sparseness



- Each Y_i' for $1 \le i \le \ell$ is a protrusion and has constant size (after protrusion reduction).
- $|Y_0| = O(|X|)$ (follows from degeneracy of 2^d-shallow minors).
- $\ell = O(|Y_0|) = O(|X|)$ (ditto).
- Hidden constants depend on expansion $\nabla_{2^d}(\mathcal{G}) \le f(2^d)$.

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- Structural parameter helps to include decision problems like 3-Colorability and Hamiltonian Path.

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- Kernels for problems like Treewidth and Longest Path.
- Structural parameter helps to include decision problems like 3-Colorability and Hamiltionian Path.
- Quadratic kernels on graphs of locally bounded expansion.

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- Structural parameter enables us to relax the FII condition.
- Kernels for problems like Treewidth and Longest Path.
- Structural parameter helps to include decision problems like 3-Colorability and Hamiltonian Path.
- Quadratic kernels on graphs of locally bounded expansion.
- Polynomial kernels on nowhere dense graphs.

Consequences

The problems. . .

Dominating Set, Connected Dominating Set, r-Dominating Set, Efficient Dominating Set, Connected Vertex Cover, (Connected) Vertex Cover, Hamiltonian Path/Cycle, 3-Colorability, Independent Set, Feedback Vertex Set, Edge Dominating Set, Induced Matching, Chordal Vertex Deletion, Interval Vertex Deletion, Odd Cycle Transversal, Induced d-Degree Subgraph, Min Leaf Spanning Tree, Max Full Degree Spanning Tree, Longest Path/Cycle, Exact s,t-Path, Exact Cycle, Treewidth, Pathwidth

. . . parameterized by a treedepth-modulator have . . .

- . . . linear kernels on graphs of bounded expansion
- . . . quadratic kernels on graphs of locally bounded expansion
- . . . polynomial kernels on nowhere-dense graphs

Conclusion

Interpretation of meta-theorems

For meta-theorems up until H-topo-minor-free graphs, a **small treewidth modulator** is crucial:

- quasi-compactness on bounded genus graphs, and
- bidimensionality + separability on H-minor-free graphs

are tangible properties which guarantee this on these classes.

Larger graph classes need stronger (structural) parameters.

Treedepth-modulator is a useful parameter (generalization of vertex cover).

Open questions

- Which problems admit kernels on these classes with a natural parameter?
- Problem categories: closed under subdivision vs. not closed. Weaker parameterization for latter?
- Linear kernels for graphs with locally bounded treewidth?
- Lower bounds!

Thanks!