Lower Bounds on the Complexity of MSO_1 Model-Checking

Somnath Sikdar

Joint work with

Robert Ganian Petr Hliněný Alexander Langer
Jan Obdržálek Peter Rossmanith

Theoretical Computer Science,
RWTH Aachen University, Germany.

Faculty of Informatics,
Masaryk University, Brno, Czech Republic.

Outline

Courcelle's Theorem . . .

. . . states that problems expressible in a certain logic can be efficiently solvable on graphs with a "nice" decomposition structure.

More specifically,

Theorem (Courcelle, 1990)

If φ is a graph problem expressible in MSO$_2$, then "solving" this problem on a graph G takes time $f(tw(G), |\varphi|) \cdot |G|$.

Here

- $\text{tw}(G)$ denotes the treewidth of G;
- $|\varphi|$ denotes the number of quantifier alternations;
- f is a computable function.

MSO_2 and MSO_1

Monadic Second Order (MSO) logic

- allows one to quantify over sets of objects.
- MSO_2: quantifications over vertex and edge sets.
- MSO_1: quantifications over vertex sets only.

MSO_2 has strictly more expressive power than MSO_1.

- Hamiltonian Cycle can be expressed in MSO_2 but not in MSO_1.

Courcelle's Theorem . . .

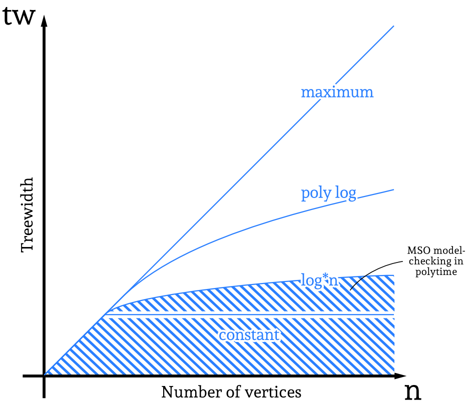
immediately gives us linear-time algorithms for several NP-hard problems on graphs of "small" treewidth:

- Hamiltonian Cycle;
- Vertex Cover;
- 3-Colorability.

Of immense practical use in parameterized complexity:

- can check whether φ is fixed-parameter tractable wrt the treewidth as parameter.

Courcelle's Theorem: Lower Bounds



Kreutzer and Tazari [2010] show that Courcelle's Theorem fails if the treewidth grows poly-logarithmically with the graph size.

An Overview of Kreutzer and Tazari's Result

Theorem (Kreutzer and Tazari, 2010)

Let $\mathcal{C}$ be a graph class that is

- *closed under subgraphs, and*
- *has polylogarithmically unbounded treewidth.*

Then solving an MSO_2-expressible problem φ on a graph G from $\mathcal{C}$ is not in XP, unless SAT can be solved in subexponential time.

- XP: solvability in time $|G|^{f(|\varphi|)}$.
- Exponential Time Hypothesis: SAT cannot be solved in subexponential time.

Aspects of Kreutzer & Tazari's Theorem

I. Threshold for treewidth is more-or-less strict.

- $\exists$ subgraph-closed classes with $\text{tw}\,(G) = \log|G|$ that can be model-checked in XP-time [Makowski and Mariño, 2003].

II. The proof requires certain witness structures to be constructed efficiently.

- Constructibility issues make the proofs very technical.

Outline

The Setting

We consider:

- Vertex-labeled graphs, where the labels are from a fixed, finite set.
- Problems expressible in MSO_1 on such vertex-labeled graphs.
- Non-uniform ETH: SAT, 3-Colorability are not in $2^{o(n)}$ time with subexponential advice.

Main Theorem

Theorem

Let L be some fixed set of labels and let $\mathcal{C}$ be a graph class that is

- *closed under subgraphs;*
- *and has polylogarithmically unbounded treewidth.*

Then deciding whether a vertex-labeled graph G from $\mathcal{C}$ with labels from L models an MSO_1-expressible problem φ is not in XP, unless 3-Colorability is in time $2^{o(n)}$ with subexponential advice.

Major Differences Between the Two Results

I. We use a diferent logic.

- **Our result:** MSO_1 model-checking on vertex-labeled graphs.
- **K & T's:** MSO_2 model-checking on unlabeled graphs.

The two logic classes not comparable: consider Hamiltonian Cycle and Red-Blue Dominating Set.

II. We assume that witnesses are given as advice:

- No constructibility issues but a stronger complexity assumption: Nonuniform ETH.
- Proofs are shorter and easier.

ETH versus Nonuniform ETH (NETH)

Exponential Time Hypothesis [Impagliazzo, Paturi, and Zane, 2001]:

- n-variable 3-SAT cannot be solved in $2^{o(n)}$ time.
- Can be formulated using other problems such as Vertex Cover or 3-Colorability.

NETH: n-variable 3-SAT not solvable in $2^{o(n)}$ time using:

- an algorithm that receives oracle advice which depends only on the input length n and has $2^{o(n)}$ bits.
- Can be formulated in terms of Vertex Cover or 3-Colorability.

Outline

Main Theorem

Theorem

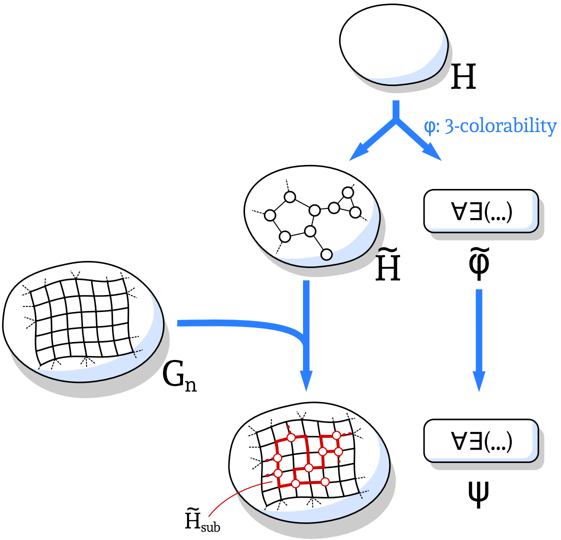
Let L be some fixed set of labels and let $\mathcal{C}$ be a graph class that is

- *closed under subgraphs;*
- *and has polylogarithmically unbounded treewidth.*

Then deciding whether a vertex-labeled graph G from $\mathcal{C}$ with labels from L models an MSO_1-expressible problem φ is not in XP, unless 3-Colorability is in time $2^{o(n)}$ with subexponential advice.

Proof. A multistage reduction from 3-Colorability.

Proof Outline



Step 1: Reducing to a Subcubic Planar Graph

Given

- φ: MSO_1 formula expressing 3-Colorability.
- H: n-vertex graph, instance of 3-Colorability.

Reduce $(H, \varphi) \to (\widetilde{H}, \tilde{\varphi})$ in polynomial-time such that:

- $\widetilde{H}$ is planar and has vertices of degree only 1 or 3;
- $H \models \varphi$ iff $\widetilde{H}_{\mathsf{sub}} \models \tilde{\varphi}$ for every subdivision $\widetilde{H}_{\mathsf{sub}}$ of $\widetilde{H}$;
- $\tilde{\varphi}$ depends only on φ and $|\tilde{\varphi}| = O(|\varphi|)$.

$\widetilde{H}$ may not be in the class $\mathcal{C}$.

Goal: In order to contradict the XP model-checkability of $\mathcal{C}$, want a graph in $\mathcal{C}$ that "contains" $\widetilde{H}$.

Step 2: Finding a Graph in $\mathcal{C}$ containing $\widetilde{H}$

$|H| = n$ and $|\widetilde{H}| \leq n^b$, for some constant b.

Polylogarithmic unboundedness of tw $(\mathcal{C})$

- $\exists G \in \mathcal{C}$ s.t. $\log^c |G| \leq \text{tw}(G)$ and $|G| = 2^{n^\epsilon}$.

Grid-like subgraphs [Reed and Wood, 2008]

- $\log^c |G| \leq \text{tw}(G)$ and $|G| = 2^{n^\epsilon}$ implies $n^{O(1)} \leq \text{tw}(G)$.
- $n^{O(1)} \leq \text{tw}(G)$ implies G contains a **grid-like subgraph** Γ_n **of order** n: Γ_n "contains" a subdivision $\widetilde{H}_{\text{sub}}$ of $\widetilde{H}$.

Closure of $\mathcal{C}$ under subgraphs

- $\Gamma_n \in \mathcal{C}$.

Summary so far

- Can "embed" $\widetilde{H}$ in a graph from $\mathcal{C}$ of size $2^{o(n)}$.

Step 3: Using Subexponential Advice

Supexponential advice

- Γ_n has size $2^{o(n)}$ and depends only on n: supplied as advice.

Using vertex labels to identify $\widetilde{H}_{\text{sub}}$ in Γ_n

- Γ_n "contains" $\widetilde{H}_{\text{sub}}$: can construct a vertex labeling λ and a formula $\psi \in \text{MSO}_1[L]$ s.t.

$$\widetilde{H}_{\text{sub}} \models \varphi \text{ iff } (\Gamma_n, \lambda) \models \psi.$$

Model-checking $\mathcal{C}$ in XP implies

- deciding $(\Gamma_n, \lambda) \models \psi$ in $|\Gamma_n|^{f(|\psi|)}$ time;
- thereby deciding $H \models \varphi$ in $|\Gamma_n|^{f(|\psi|)} = 2^{o(n) \cdot f(|\psi|)} = 2^{o(n)}$ time, contradicting nonuniform ETH.

Outline

Summary

- Contributed to Kreutzer and Tazari's result.
- **Open.** Can the result of Kreutzer and Tazari be extended to (unlabeled) MSO_1?

Thank You!