Linear Kernels for Graphs Excluding a Topological Minor

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Theoretical Computer Science

RWTH AACHEN

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Contents

Linear kernels in sparse graphs

Overview

- Dominating Set in planar graphs
  Alber, Fellows, Niedermeier. *Polynomial-time data reduction for Dominating Set*, 2004.
- Framework for planar graphs
  Guo and Niedermeier: *Linear problem kernels for NP-hard problems on planar graphs*, 2007.
- Meta-result for graphs of bounded genus
  Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh and Thilikos: *(Meta) Kernelization*, 2009.
- Meta-result for graphs excluding a fixed graph as a minor
  Fomin, Lokshtanov, Saurabh and Thilikos: *Bidimensionality and kernels*, 2010.
- **Our contribution**: a meta-result for graphs excluding a fixed graph as a *topological minor*.

Main theorem

Theorem

Let Π be a parameterized graph problem that has

1. *finite integer index, and*
2. *is treewidth-bounding,*

both on the class of H-topological-minor-free graphs. Then Π admits a linear kernel.

The undefined terms

1. A **parameterized graph problem** Π is a set of pairs (G, k), where G is a graph and k a non-negative integer s.t.

$$G_1 \cong G_2 \text{ implies } (G_1, k) \in \Pi \text{ iff } (G_2, k) \in \Pi.$$

2. A parameterized graph problem Π is **treewidth-bounding** if $\exists$ constants c, t such that $(G, k) \in \Pi$ implies that

$$\exists X \subseteq V(G) \text{ s.t. } |X| \leqslant c \cdot k \text{ and } \mathbf{tw}(G - X) \leqslant t.$$

3. The property of **finite integer index** allows us to replace large "protrusions" by smaller gadgets.

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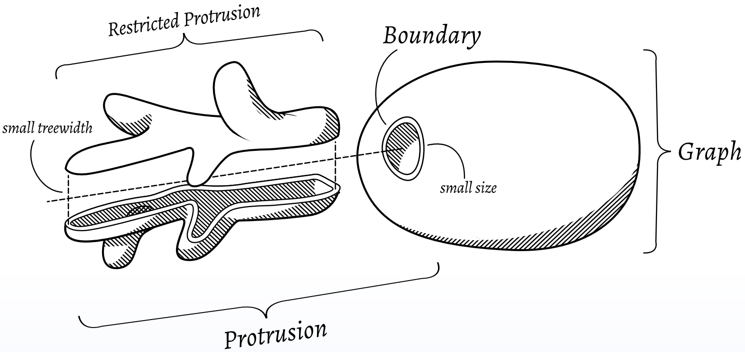
   $\exists X \subseteq V(G)$ s.t. $|X| \leqslant c \cdot k$ and $\mathbf{tw}(G - X) \leqslant t$.

3. The property of **finite integer index** allows us to replace large "protrusions" by smaller gadgets.

We assume that the gadgets are given. Our algorithm is nonuniform.

Reduction via protrusions

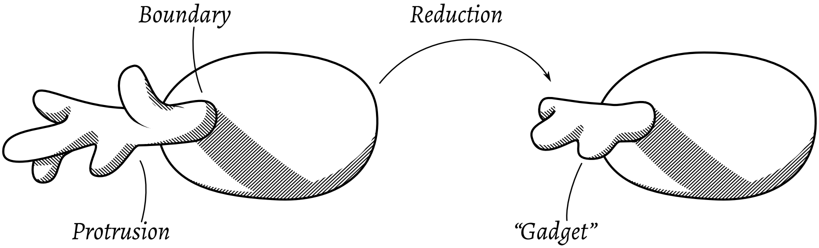
Protrusion anatomy



Definition

$X \subseteq V(G)$ is a t-*protrusion* if

1. $|\partial(X)| = |N(X) \setminus X| \leqslant t$ (small boundary)
2. $\mathbf{tw}(G[X]) \leqslant t$ (small treewidth)

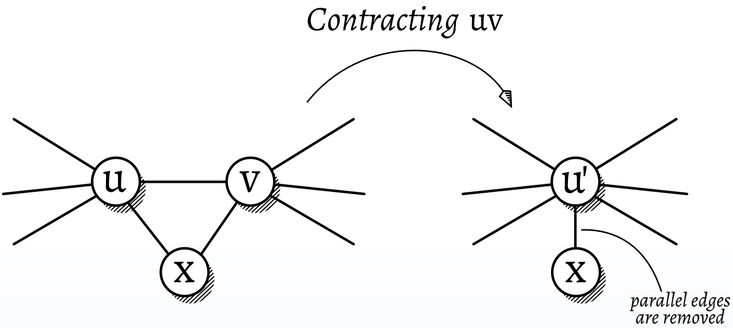


We want to replace a large protrusion by a smaller gadget.

1. Requires that the problem has finite integer index.
2. The gadgets can always be chosen such that the parameter does *not* increase.
3. This is the only reduction.

(Topological) Minors

Edge contraction



Graph relations

(now with contractions!)

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Relation	Operations
induced subgraph	delete vertices
subgraph	
topological minor	
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Graph relations

(now with contractions!)

Relation	Operations
induced subgraph	delete vertices
subgraph	delete vertices and edges
topological minor	take a subgraph, contract edges *incident to a degree-2 vertex*
minor	delete vertices and edges, contract edges

Properties of $\mathcal{H}$-topological-minor-free graphs

Let G be a graph excluding H as a topological minor.

Properties of H-topological-minor-free graphs

Let G be a graph excluding H as a topological minor.

- Not interested in structure of H, but its size $r = |H|$.
- In particular: K_r not a topological minor of G.

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Important properties:

1. $m \leqslant \frac{1}{2}\beta r^2 n$ (for some $\beta < 10$);
2. no. of cliques $\leqslant 2^{\tau r \log r} n$ (for some $\tau < 4.51$);
3. Closed under taking topological minors.

Our result and how it works

On the treewidth-bounding property

Definition (Treewidth bounding)

A parameterized graph problem Π is called *treewidth bounding* if $\exists$ constants c, t such that for every $(G, k) \in \Pi$, $\exists\, S \subseteq V(G)$ s.t.

1. $|S| \leqslant ck$;
2. $\mathbf{tw}(G - S) \leqslant t$.

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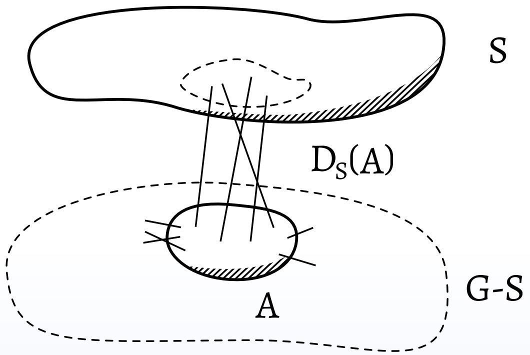
1. $|S| \leqslant ck$;
2. $\mathbf{tw}(G - S) \leqslant t$.

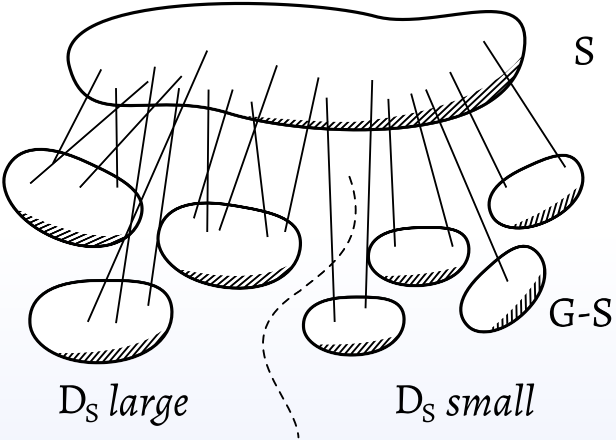
- S usually is the solution set.
  - Vertex Cover, Feedback Vertex Set in general graphs.
  - Chordal Vertex Deletion in graphs with bounded clique-size.

A little bit of notation

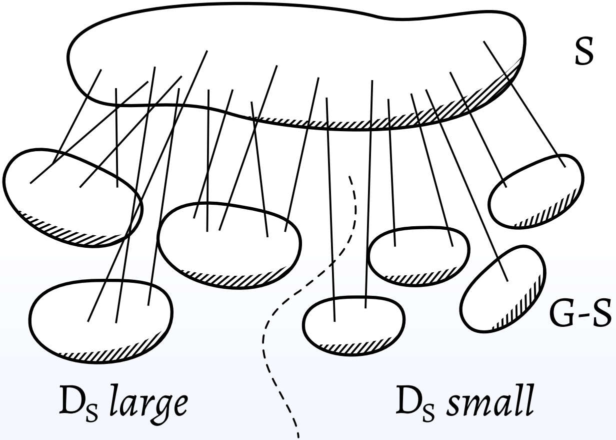


For disjoint vertex sets $S, A \in V(G)$, $D_S(A) = |N_G(A) \cap S|$.

A decomposition

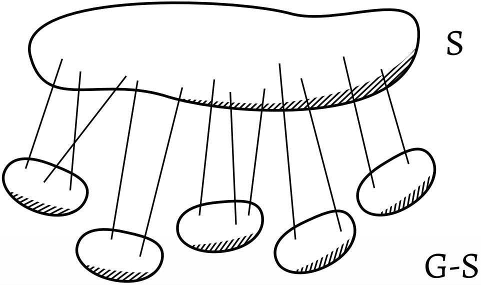


A decomposition



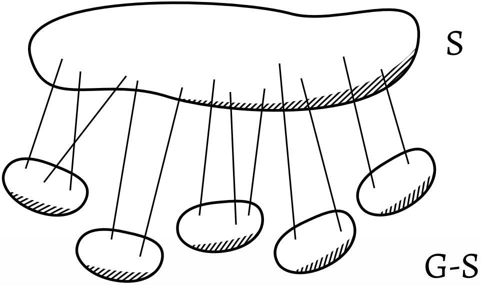
Reduced instance: large protrusions are gone

Small-degree components



- $D_S(C) < r$, therefore each component C has a boundary of size r.

Small-degree components



- $D_S(C) < r$, therefore each component C has a boundary of size r.
- C has constant treewidth (problem is treewidth bounding).

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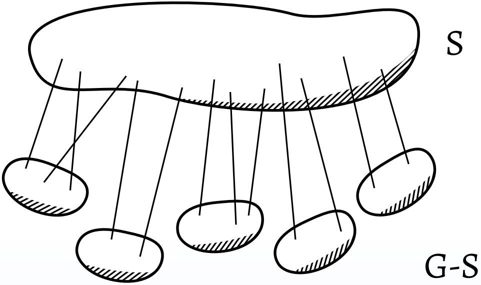

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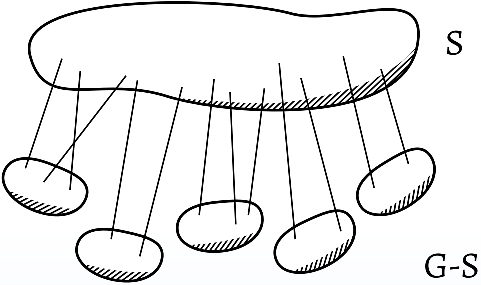

- $D_S(C) < r$, therefore each component C has a boundary of size r.
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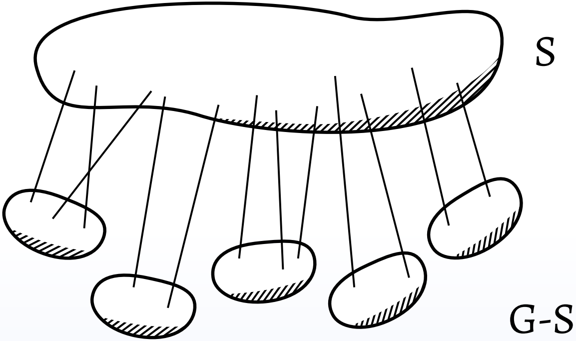
$\Rightarrow$ Each small-degree component has constant size (reduced instance).

- What about the *number* of small-degree components?

Small-degree components



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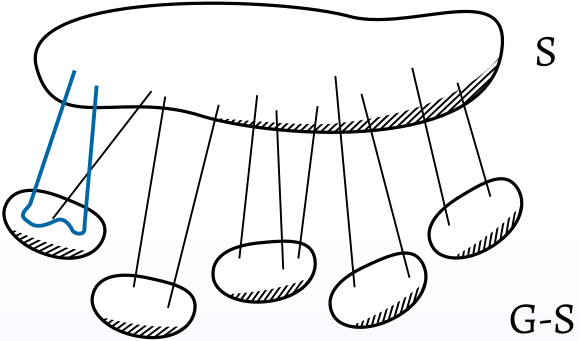

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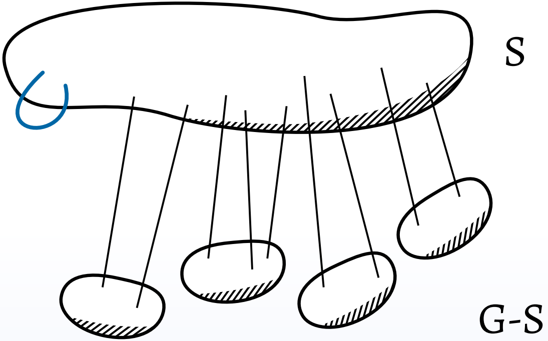

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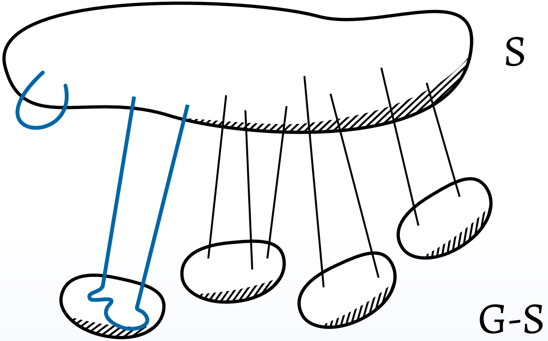

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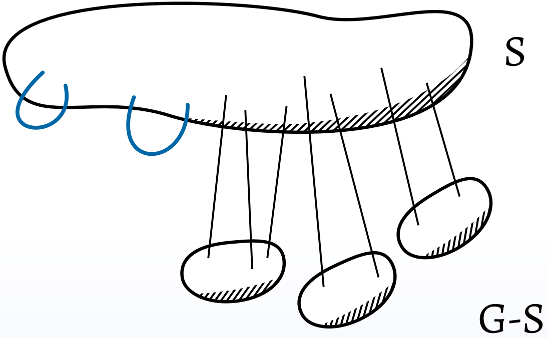

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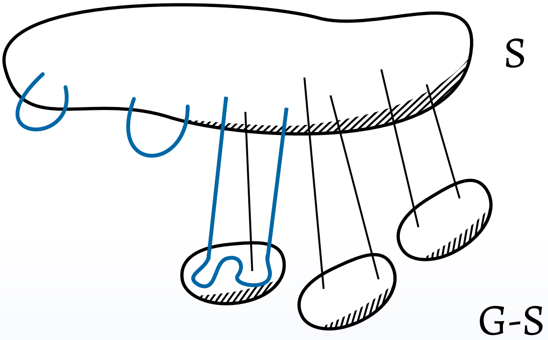

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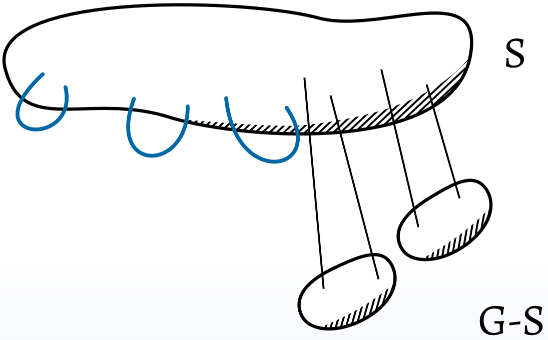

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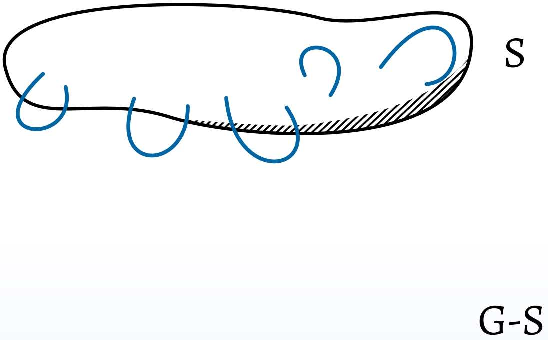

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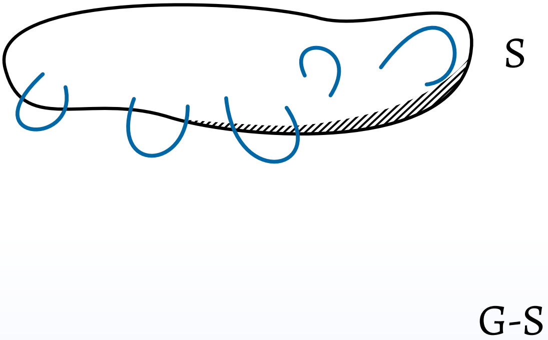
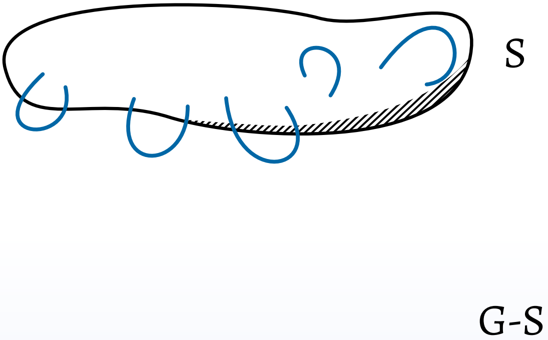


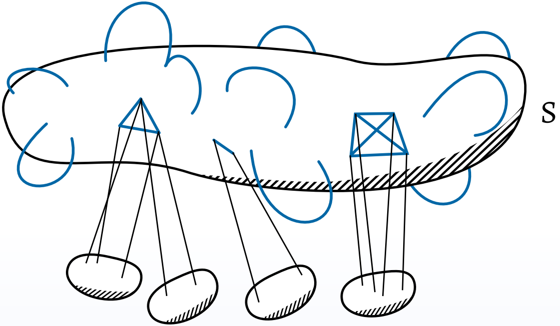
- How often can we do this?

Small-degree components



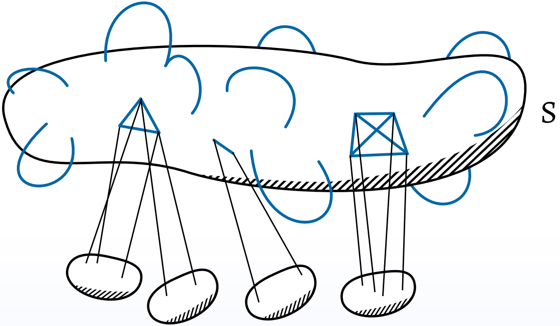
- How often can we do this?
- Is it exhaustive?

Small-degree components



- Components now connected to cliques (or not finished)

Small-degree components



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- $G[S]$ is H-topological minor free, therefore...

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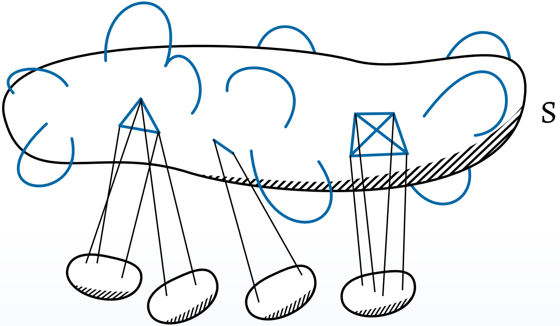

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  ... $O(|S|) = O(k)$ cliques

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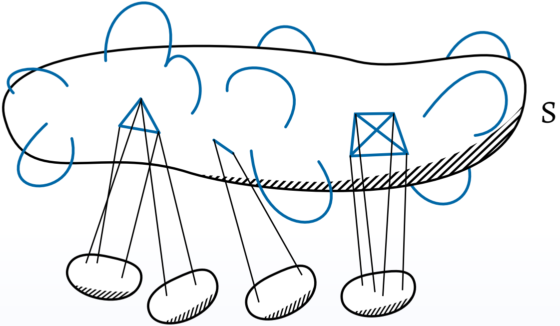

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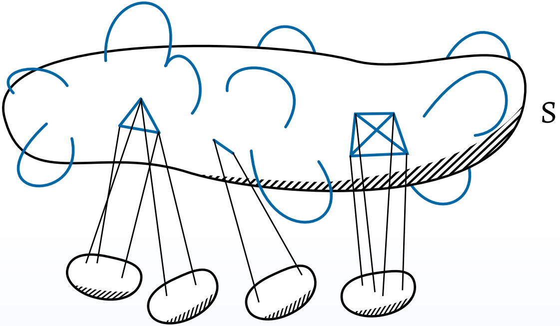

- Components now connected to cliques (or not finished)
- $G[S]$ is H-topological minor free, therefore...
  - ... $O(|S|) = O(k)$ cliques
  - ... $O(|S|) = O(k)$ edges
- Constant number of vertices in components connected to a common clique (or large protrusion in G)

Total size of small degree components

$O(k)$ vertices in small-degree components.

Large-degree components

Two ingredients:

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1. At most $O(k)$ connected *subgraphs* with $D_S \geqslant r$.

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2. Decompose the large degree components into subgraphs
   - of constant size; and
   - with a large degree in X.

Since there are $O(k)$ such subgraphs, their total size is $O(k)$.

Large-degree components

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2. Decompose the large degree components into subgraphs
   - of constant size; and
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Since there are $O(k)$ such subgraphs, their total size is $O(k)$.

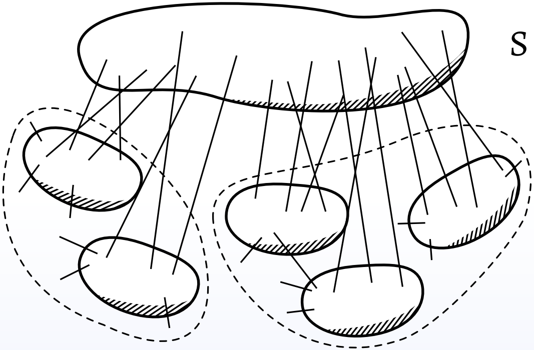
Our proof that such a decomposition exists is very technical.

A simplification of our proof appears in:

A parameterized single-exponential algorithm for hitting planar minors. Kim, Paul, and Sau.

Large-degree components

Ingredient one



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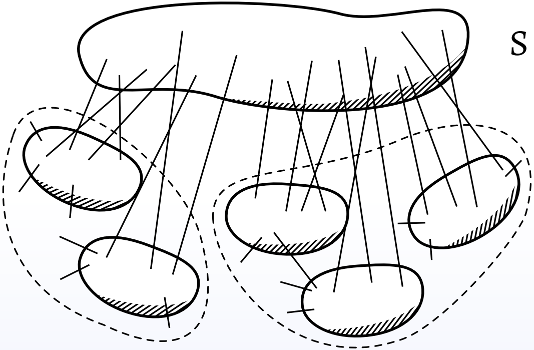

- Same idea as before: contract connected subgraphs into edges in S

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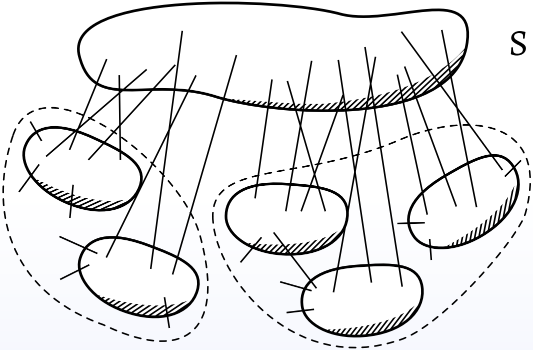


- Same idea as before: contract connected subgraphs into edges in S
- Exhaustive, else K_r as a subgraph in S and thus H as a topological minor in G

Large-degree components

Ingredient two

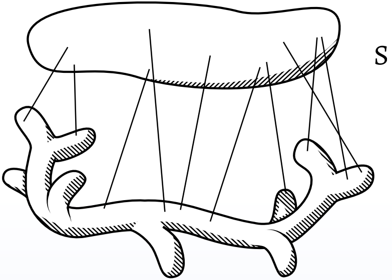
Every large-degree component can be broken into:

- path-like structures (paths in a tree-decomposition);
- star-like structures (join nodes in a tree-decomposition).

We use tree-decompositions to effect such a break-up.

Large-degree components

Ingredient two



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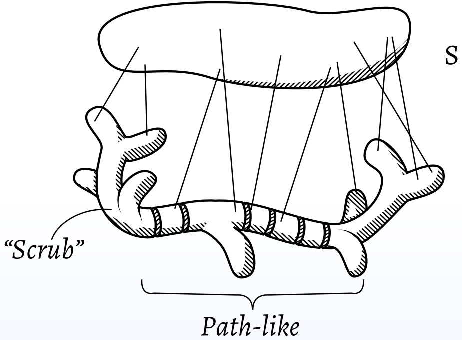

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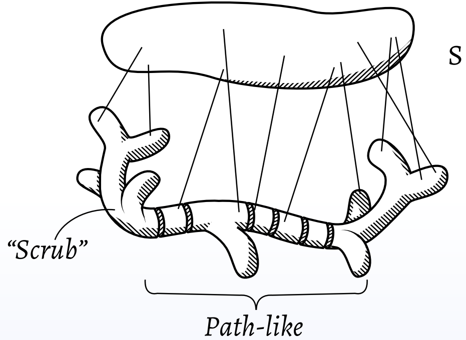

- Walk along path-decomposition.

Large-degree components

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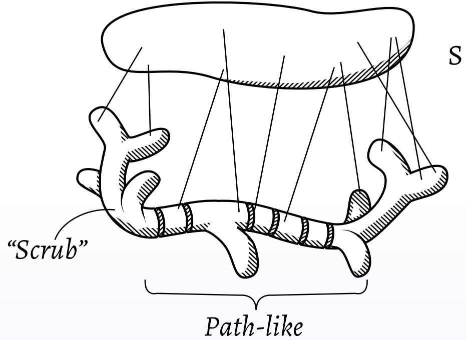
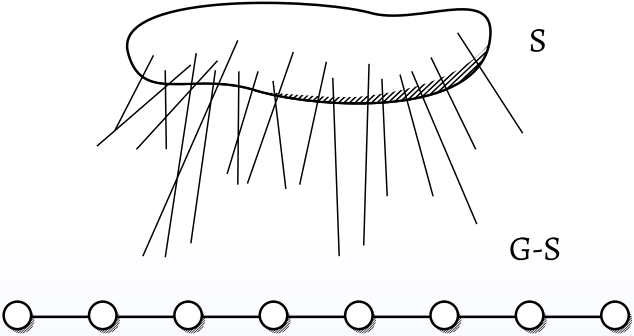


- Walk along path-decomposition.
- If more than $\omega(2t + r)$ vertices seen: subgraph has large degree wrt S.

Examples



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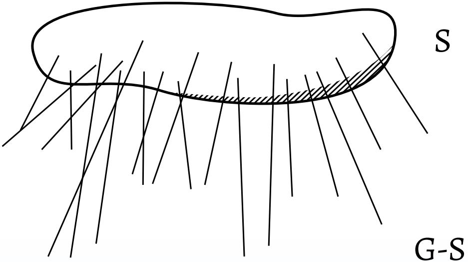

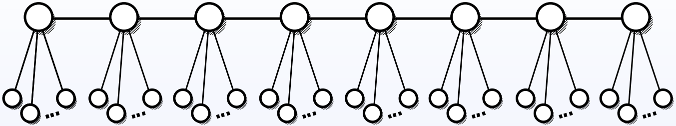

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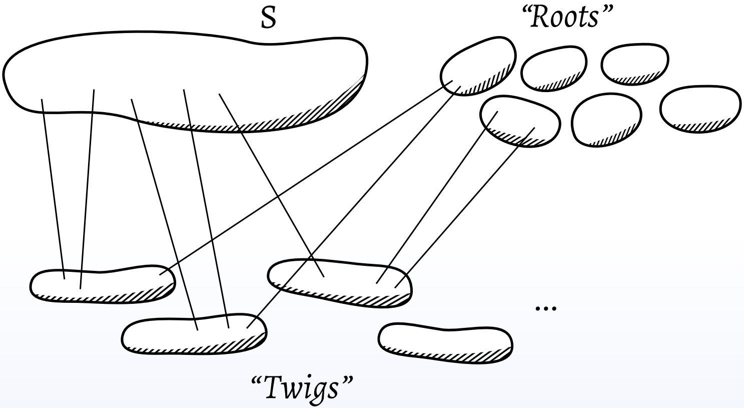


Conclusion

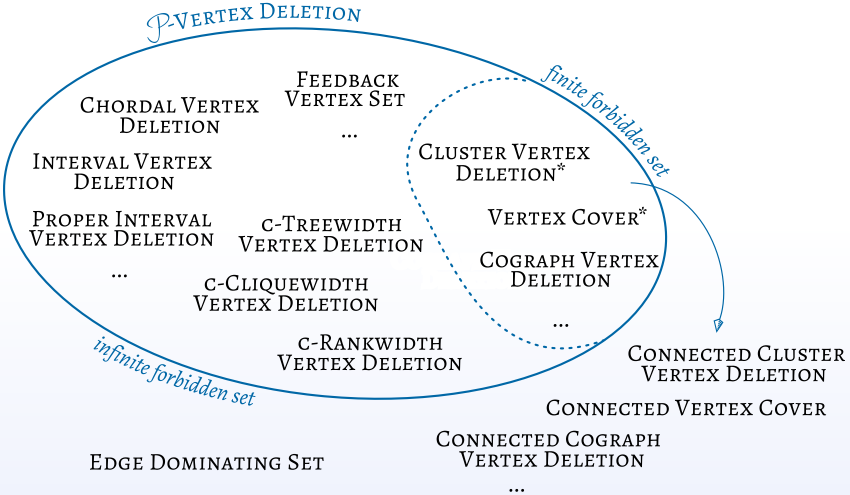
The result

Parameterized graph problems that have

1. finite integer index, and are
2. treewidth bounding,

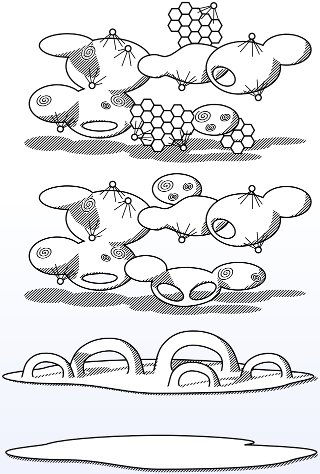
admit linear kernels on graphs excluding a fixed topological minor.

Examples



𝒫-Vertex Deletion
finite forbidden set
infinite forbidden set
Chordal Vertex Deletion
Feedback Vertex Set
...
Interval Vertex Deletion
Proper Interval Vertex Deletion
...
c-Treewidth Vertex Deletion
c-Cliquewidth Vertex Deletion
c-Rankwidth Vertex Deletion
Cluster Vertex Deletion*
Vertex Cover*
Cograph Vertex Deletion
...
Connected Cluster Vertex Deletion
Connected Vertex Cover
Connected Cograph Vertex Deletion
...
Edge Dominating Set

Trade-off: class of instances vs. problem requirements



H-Topological-Minor-Free	*Treewidth-bounding*
$\cup$	
H-Minor-Free	*Bidimensional* *+ separation property*
$\cup$	
Bounded Genus	*Quasi-compact*
$\cup$	
Planar	*"Distance-property"*

Open questions

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- Are there interesting polynomially treewidth bounding problems? (We looked at *linear* treewidth bounding).

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Thank you!