Kernelization using structural parameters on sparse graph classes

(or: Structural Parameters—a necessary evil)

Jakub Gajarský[1] Petr Hliněný[1] Jan Obdržálek[1]
Sebastian Ordyniak[1] **Felix Reidl**[2] Peter Rossmanith[2]
Fernando Sánchez Villaamil[2] Somnath Sikdar[2]

[1]Faculty of Informatics [2]Theoretical Computer Science

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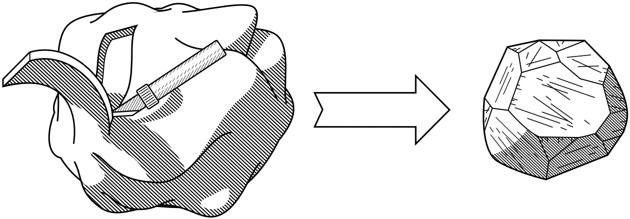
The story so far

Kernelization



- Problem is fixed-parameter tractable iff it has a kernelization algorithm
- Goal: to obtain *polynomial* or even *linear* kernels.

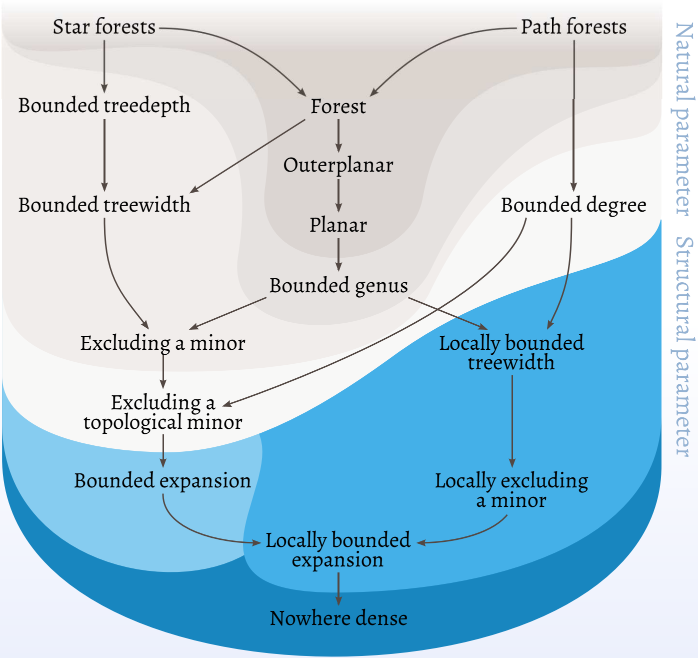
Basic technique of kernelization:
Devise *reduction rules* that preserve equivalence of instances; apply exhaustively, prove kernel size.

Algorithmic meta-results: nail down as many problems as possible

Previous work

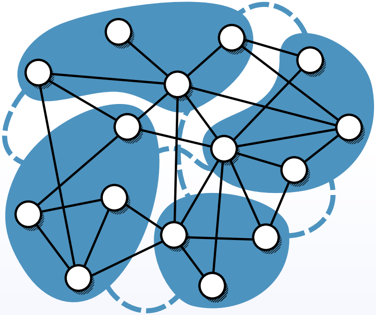
- Framework for planar graphs
  Guo and Niedermeier: *Linear problem kernels for NP-hard problems on planar graphs*
- Meta-result for graphs of bounded genus
  Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh and Thilikos: *(Meta) Kernelization*
- Meta-result for graphs excluding a fixed graph as a minor
  Fomin, Lokshtanov, Saurabh and Thilikos: *Bidimensionality and kernels*
- Meta-result for graphs excluding a fixed graph as a topological minor
  Kim, Langer, Paul, R., Rossmanith, Sau and Sikdar: *Linear kernels and single-exponential algorithms via protrusion decompositions*
- *Our contribution*: Meta-result for graphs of bounded expansion, local bounded expansion and nowhere-dense graphs using *structural parameterization*

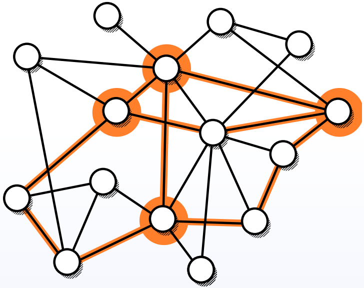
The big picture



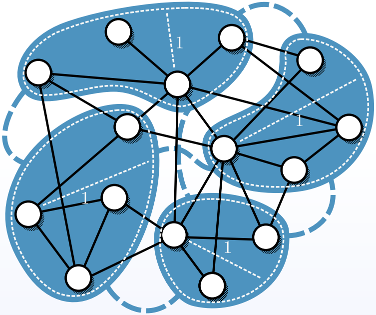
Beyond exluded minors

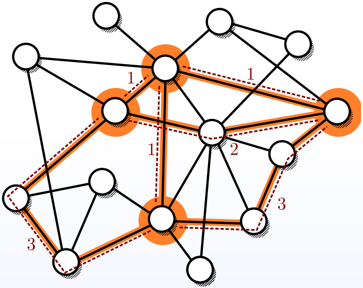
Minors, top-minors





Shallow minors, top-minors





Bounded expansion

For a graph G we denote by $G \triangledown r$ the set of its r-shallow minors.

Definition (Grad, Expansion)

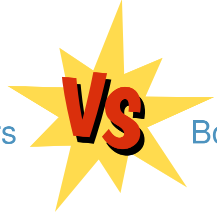
For a graph G, the *greatest reduced average density* is defined as

$$\nabla_r(G) = \max_{H \in G \triangledown r} \frac{|E(H)|}{|V(H)|}$$

For a graph class $\mathcal{G}$ the *expansion* of $\mathcal{G}$ is defined as

$$\nabla_r(\mathcal{G}) = \sup_{G \in \mathcal{G}} \nabla_r(G)$$

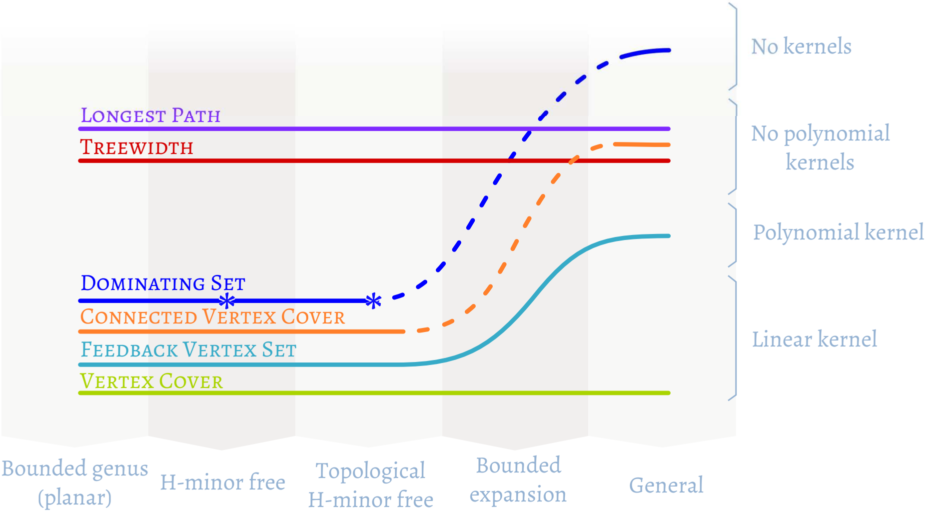
A graph class $\mathcal{G}$ has *bounded expansion* if there exists a function f such that $\nabla_r(\mathcal{G}) \leq f(r)$ for all $r \in \mathbf{N}$.



Excluded minors	Bounded expansion
d-degenerate (depening on excluded minor)	$f(0)$-degenerate (depening on expansion)
Linear number of edges	Linear number of edges
No large cliques	No large cliques
No large clique-minors	Can contain large clique minors
Closed under taking minors	“Closed” under taking shallow minors
Degeneracy of every minor is d	Degeneracy of minors depends on its “size”

Techniques from result on H-topological-minor-free graphs stop working because they use large (non-shallow) topological minors.

Why we must run into trouble



The exemplary obstacle:
Treewidth-t-Deletion

The problem

TREEWIDTH-t DELETION

Input: A graph G, an integer k

Problem: Is there a set $X \subseteq V(G)$ of size at most k such that $\mathbf{tw}(G - X) \leq t$?

- TREEWIDTH-1 DELETION = FEEDBACK VERTEX SET
- Model problem for previous results
- $k^{f(t)}$-kernel on general graphs

$\Rightarrow$ Probably none of size $O(f(t)k^c)$ (c independent of t)

Kernel on bounded expansion graphs implies same kernel on general graphs

From general to sparse

1. Treewidth closed under subdivision of edges

$\Rightarrow$ Treewidth-modulator closed under subdivision of edges

$\Rightarrow$ Instances of $\textsc{Treewidth-}t\ \textsc{Deletion}$ closed under subdivision of edges

2. Subdividing each edge of a graph $|G|$ yields a graph of *bounded expansion*

General kernel from sparse kernel:
Reduce (G, k) to $(\tilde{G}, k)$ by subdividing every edge $|G|$ times, output kernel of $(\tilde{G}, k)$.

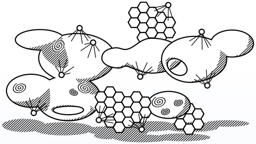
If we want a kernel, we need a parameter that is not closed under edge subdivision

Structural parameterization to the rescue

The natural view

Bounded Expansion — ?

⊂



H-Topological-Minor-Free — *Treewidth-bounding*

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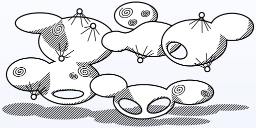

H-Minor-Free — *Bidimensional* *+separation property*

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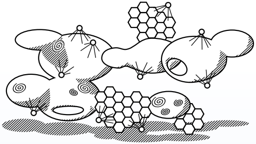
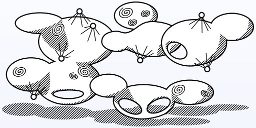
Bounded Genus — *Quasi-compact*

The structural view

	*Bounded Expansion*	?
	∪	
	*H-Topological-Minor-Free*	*Treewidth-t Modulator*
	∪	
	*H-Minor-Free*	*Treewidth-t Modulator*
	∪	
	*Bounded Genus*	*Distance-c Treewidth-t Modulator*

The structural view

Bounded Expansion *Treedepth-d Modulator*

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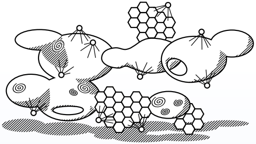

H-Topological-Minor-Free *Treewidth-t Modulator*

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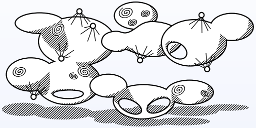
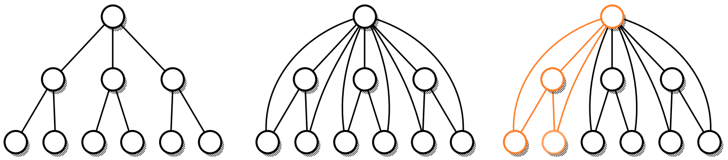


H-Minor-Free *Treewidth-t Modulator*

∪

Bounded Genus *Distance-c Treewidth-t Modulator*

Tree*depth*?

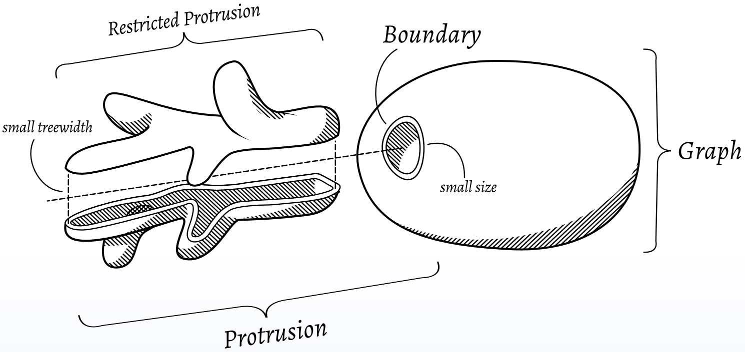


For a graph G with **td**$(G) \leq d$:

- G embeddable in closure of tree (forest) of depth d
- Graph does not contain path of length 2^d
- **tw**$(G) \leq$ **pw**$(G) \leq d - 1$

If X is a treedepth-d-modulator, $G - X$ does not contain long paths

Protrusion anatomy

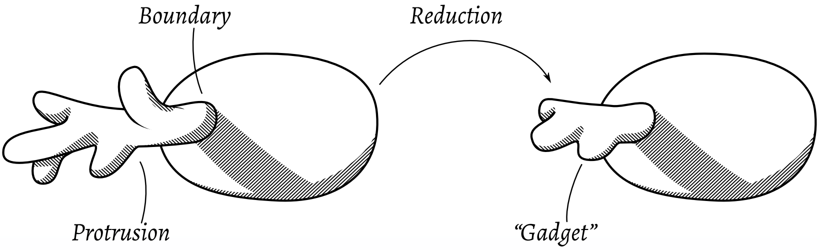


Definition

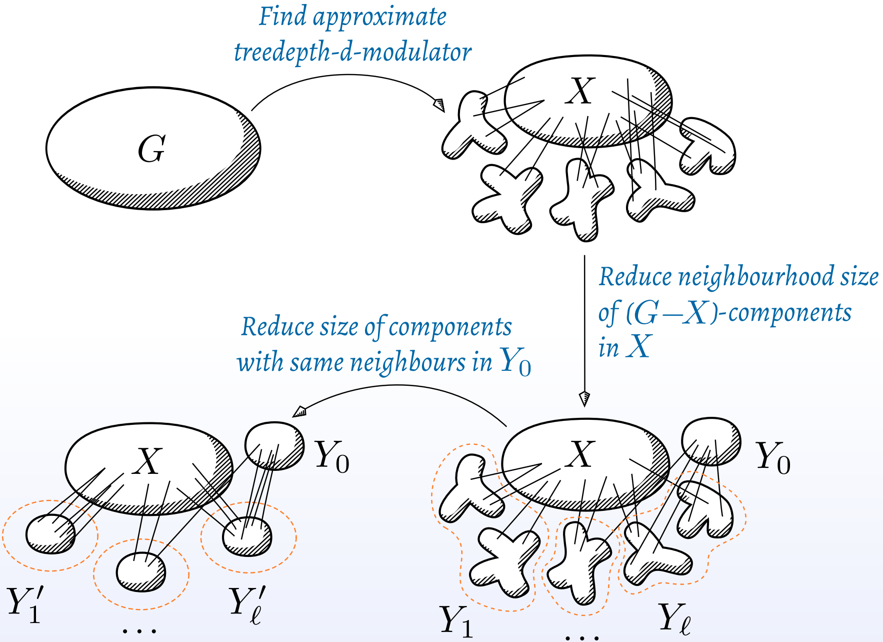
$X \subseteq V(G)$ is a t-*protrusion* if

1. $|\partial(X)| = |N(X) \setminus X| \leq t$ (small boundary)
2. $\mathbf{tw}(G[X]) \leq t$ (small treewidth)

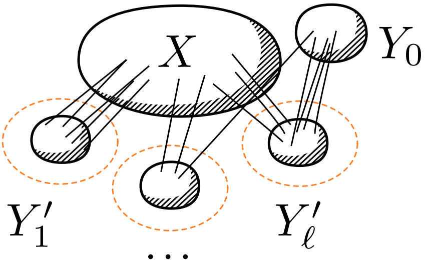
The magic reduction rule



- We want to replace a large protrusion by something smaller
- Possible if problem has *finite integer index*
- Recursive structure of graphs of small treewidth (i.e. protrusion) helps
- Lots of technicalities omitted. . .


Find approximate
treedepth-d-modulator
G
X
Reduce neighbourhood size
of (G−X)-components
in X
Reduce size of components
with same neighbours in Y0
X
Y0
Y'1
…
Y'ℓ
X
Y0
Y1
…
Yℓ

Using sparseness



- $Y_i, 1 \leq i \leq \ell$ have constant size after protrusion reduction
- $|Y_0| = O(|X|)$ (follows from degeneracy of 2^d-shallow minors)
- $\ell = O(|Y_0|) = O(|X|)$ (ditto)
- Hidden constants depend on expansion $\nabla_{2^d}(\mathcal{G}) \leq f(2^d)$

The result

Theorem

Any graph-theoretic problem that has finite integer index on graphs of constant treedepth admits linear kernels on graphs of bounded expansion if parameterized by a modulator to constant treedepth.*

- Kernelization possible in linear time

* Structural parameter enables us to relax the FII condition

⇒ Kernels for problems like TREEWIDTH and LONGEST PATH

- Structural parameter helps to include decision problems like 3-COLORABILITY and HAMILTIONIAN PATH
- Quadratic kernels on graphs of locally bounded expansion
- Polynomial kernels on nowhere dense graphs

Consequences

The problems. . .

Dominating Set, Connected Dominating Set, r-Dominating Set, Efficient Dominating Set, Connected Vertex Cover, (Connected) Vertex Cover, Hamiltonian Path/Cycle, 3-Colorability, Independent Set, Feedback Vertex Set, Edge Dominating Set, Induced Matching, Chordal Vertex Deletion, Interval Vertex Deletion, Odd Cycle Transversal, Induced d-Degree Subgraph, Min Leaf Spanning Tree, Max Full Degree Spanning Tree, Longest Path/Cycle, Exact s,t-Path, Exact Cycle, Treewidth, Pathwidth

. . . parameterized by a treedepth-modulator have . . .

- . . . linear kernels on graphs of bounded expansion
- . . . quadratic kernels on graphs of locally bounded expansion
- . . . polynomial kernels on nowhere-dense graphs

Conclusion

Our interpretation:

- Larger graph classes need stronger parameters
- Transition to structural parameters opens up a lot of possibilities
- Treedepth-modulator is a useful parameter (also works well on general graphs as a relaxation of vertex cover)

Open questions:

- Problem categories: closed under subdivision vs. not closed. Weaker parameterization for latter?
- Linear kernels for graphs with locally bounded treewidth?
- Lower bounds!

Thanks!