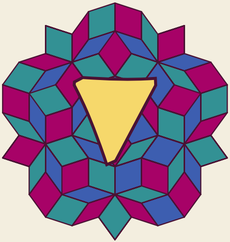
# Dense but sparse: Graphs of low complexity

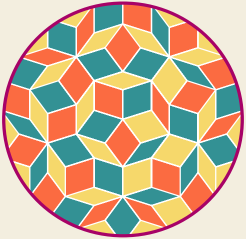


Felix Reidl
felix.reidl@gmail.com

Working Seminar on Formal Models, Discrete Structures, and Algorithms

# Part I

# Sparse classes



# The sparse class hierarchy

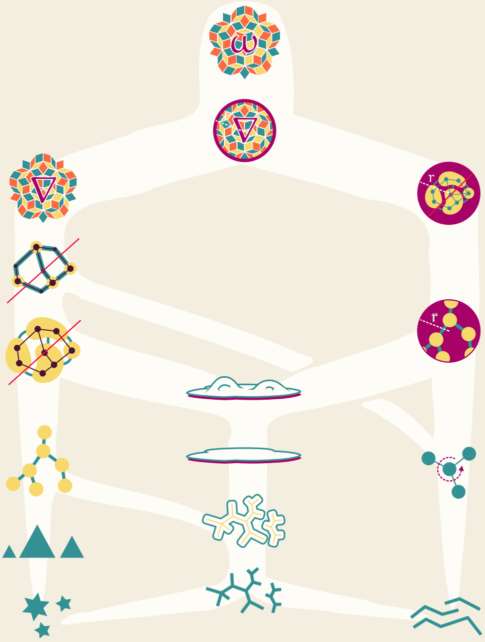
Larger classes

**Less**

Structure

**More**

Algorithmic tractability



# Parameterised graph measures

A **graph measure** is an isomorphism invariant function that maps graphs to $\mathbb{R}^+$.

e.g. density, average degree, clique number, degeneracy treewidth, etc.

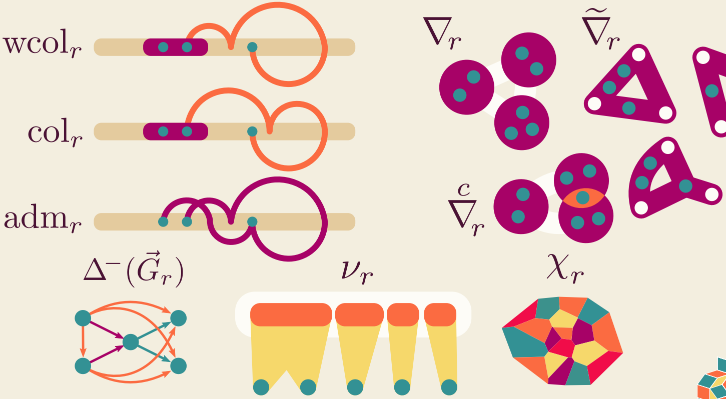
A **parameterised graph measure** is a family of graph measures $(f_r)_{r \in \mathbb{N}_0}$.

A graph class $\mathcal{G}$ is $f_r$-**bounded** if there exists $g$ s.t.

$$f_r(\mathcal{G}) = \sup_{G \in \mathcal{G}} f_r(G) \leqslant g(r) \text{ for all } r.$$

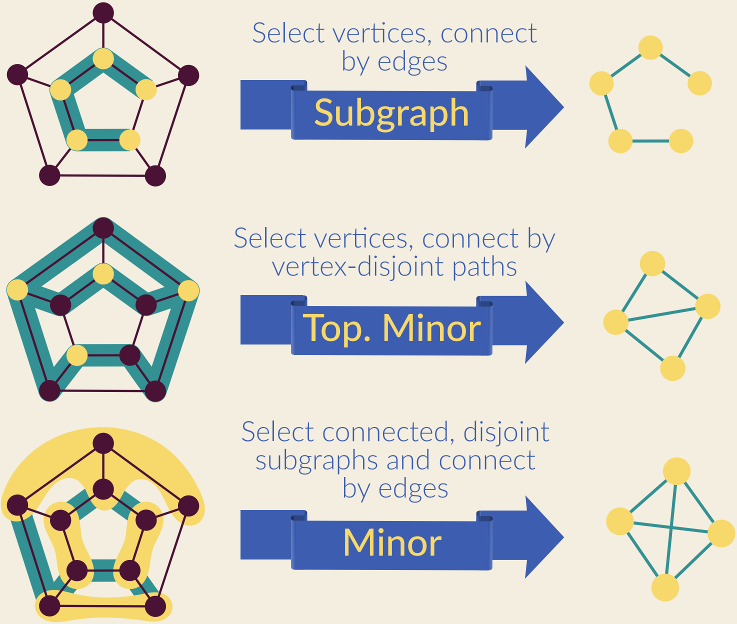
# Bounded expansion

Nešetřil & Ossona de Mendez:
Many notions of $f_r$-boundedness are equivalent!




Nešetřil J, Ossona de Mendez P. **Sparsity**.
Algorithms and Combinatorics. 2012;28.

# Substructures



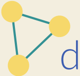
# Forbidden Substructures

$H$ does not appear as a **subgraph.**

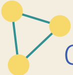
$H$ does not appear as a **topological minor**.

$H$ does not appear as a **minor**.

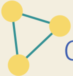
# Forbidden Substructures

 does not appear as a **subgraph.**

= Triangle-free graphs

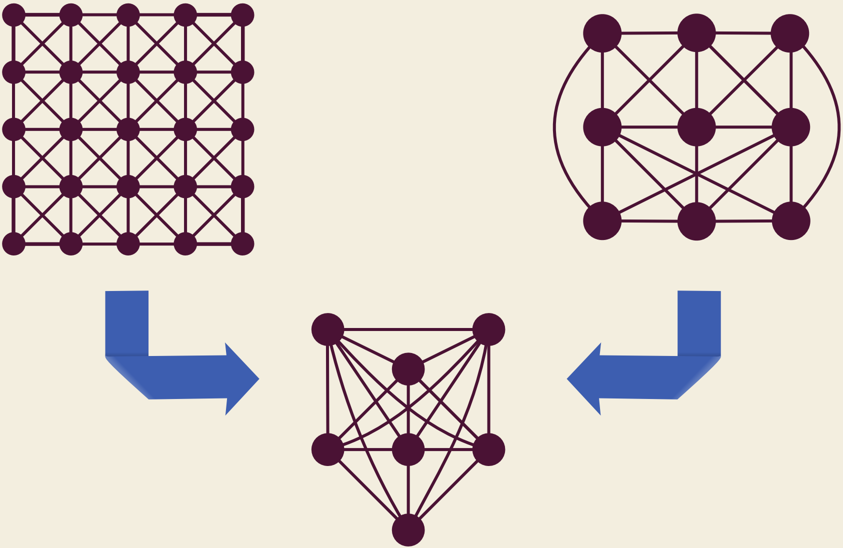
 does not appear as a **topological minor**.

= Forests

 does not appear as a **minor**.

= Forests

# Not all minors are equal!



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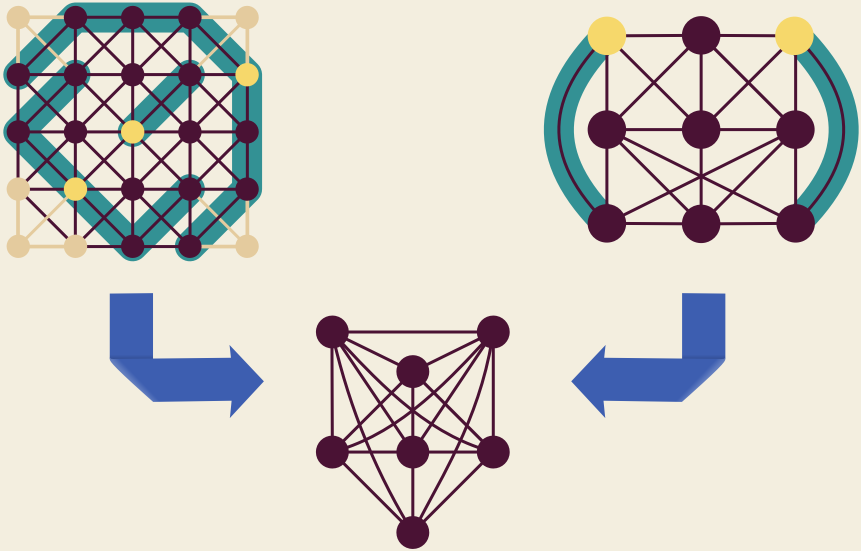

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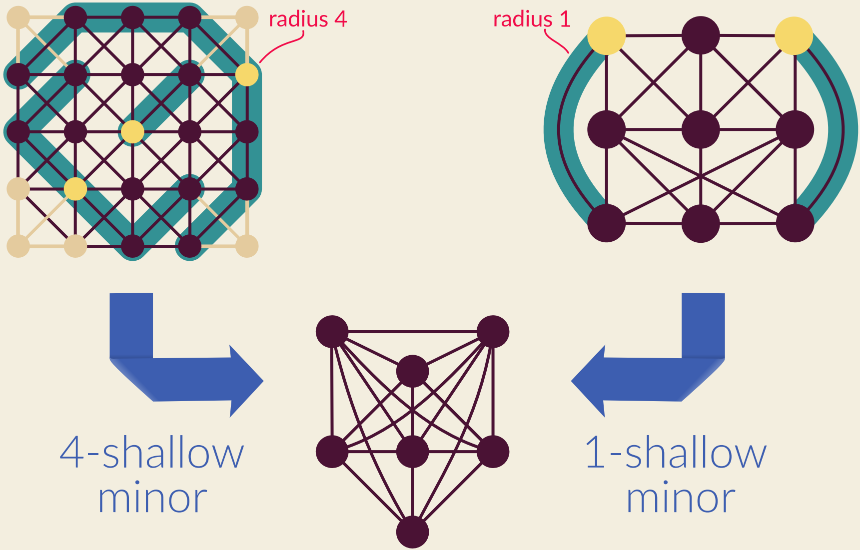
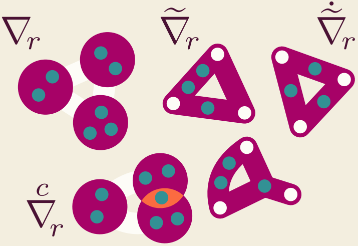


# Shallow minors & bounded expansion

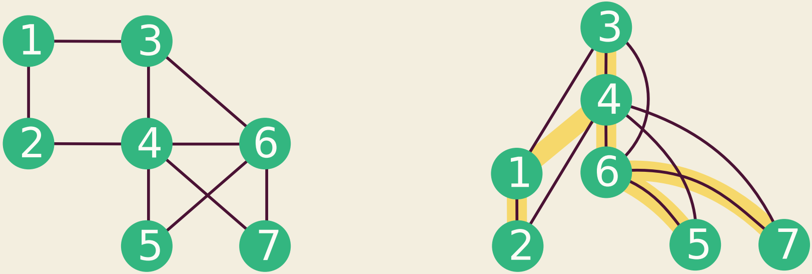
Define $\nabla_r$ to be the maximum density of all r-shallow minors that are contained in a graph.

A graph class has bounded expansion iff it is $\nabla_r$-bounded.




**Bounded expansion:**
Density of minors bounded by a function of their depth

# Treedepth




Problems expressible in **monadic second order logic** can be solved in linear time with **elementary dependence on the problem description** on graphs of bounded treedepth.

Gajarský J, Hliněný P. **Deciding Graph MSO Properties: Has it all been told already?** 2012 Apr 25.

Hlineny P, Gajarsky J. **Kernelizing MSO Properties of Trees of Fixed Height, and Some Consequences.** Logical Methods in Computer Science. 2015 Apr 1;11.

# Low treedepth colourings

A vertex colouring is an **r-treedepth colouring** if every set of $i < r$ colours induce a subgraph of treedepth $i$.

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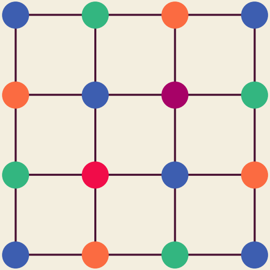

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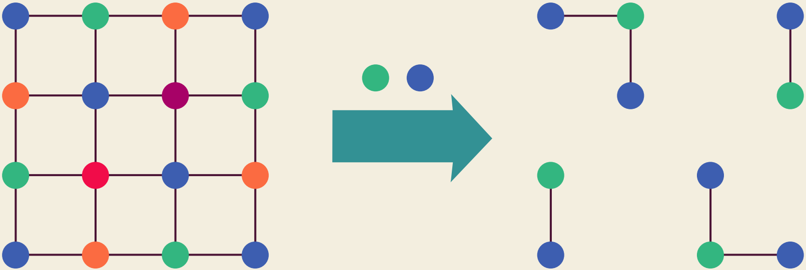

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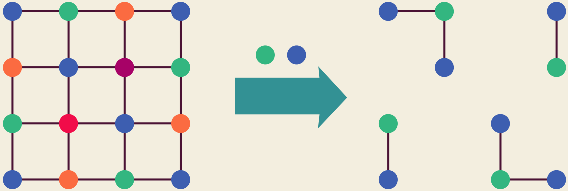

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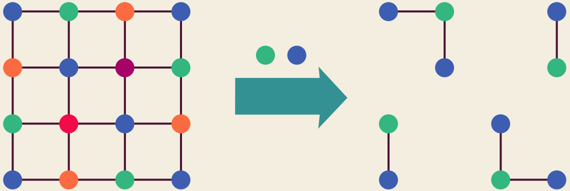

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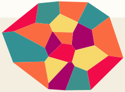
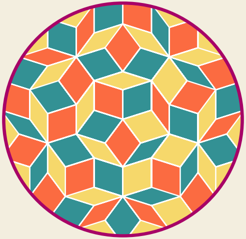
$\chi_r$

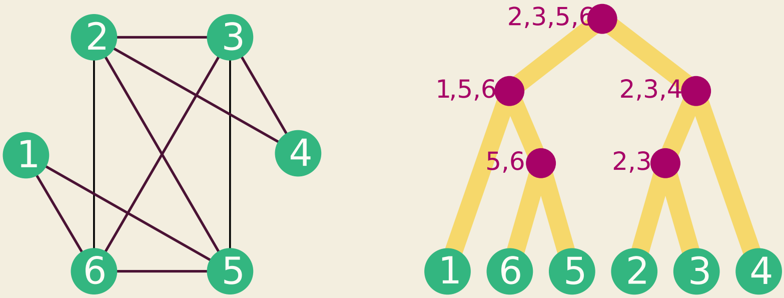


# Part II

# Dense classes



# SC-depth




Problems expressible in guarded **monadic second order logic** can be solved in linear time with **elementary dependence on the problem description** on graphs of bounded SC-depth.

Ganian R, Hliněný P, Nešetřil J, Obdržálek J, de Mendez PO, Ramadurai R.
**When trees grow low: Shrubs and fast MSO1.**
MFCS 2012 Aug 27 (pp. 419-430). Springer, Berlin, Heidelberg.

Ganian R, Hliněný P, Nešetřil J, Obdržálek J, de Mendez PO.
**Shrub-depth: Capturing Height of Dense Graphs.**
arXiv preprint arXiv:1707.00359. 2017 Jul 2.

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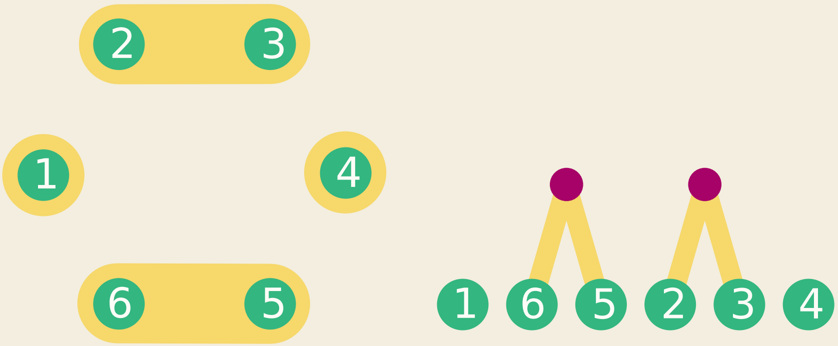

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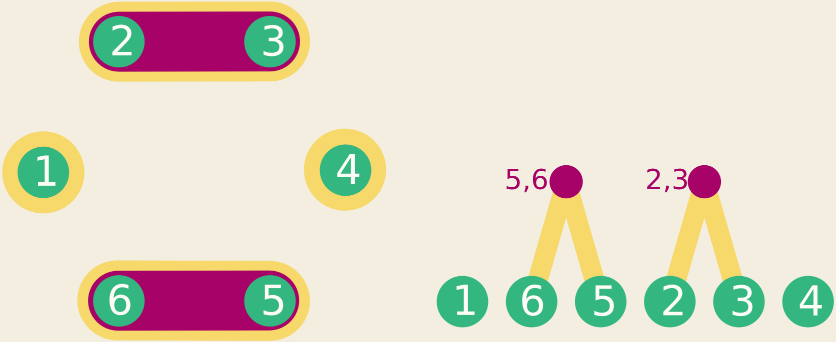

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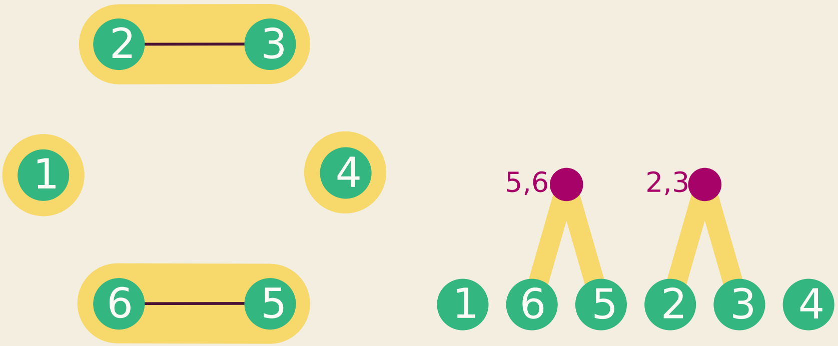

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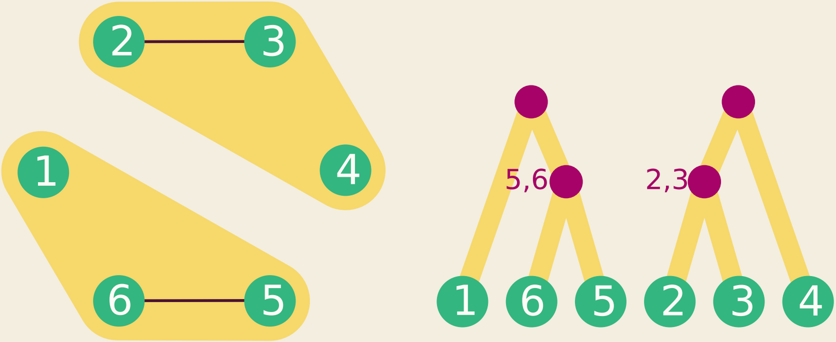

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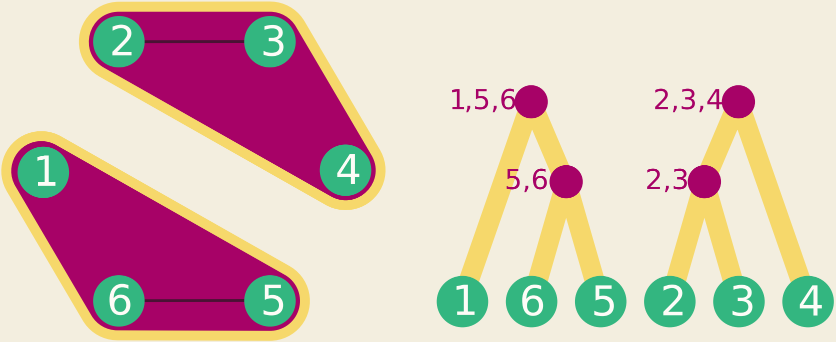

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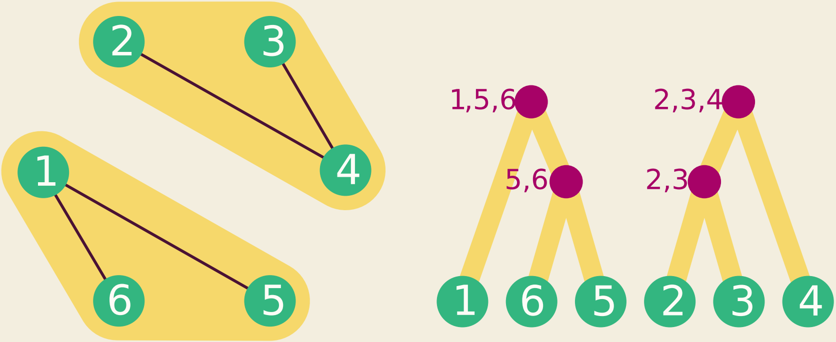

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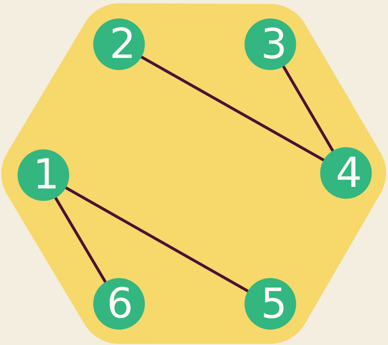


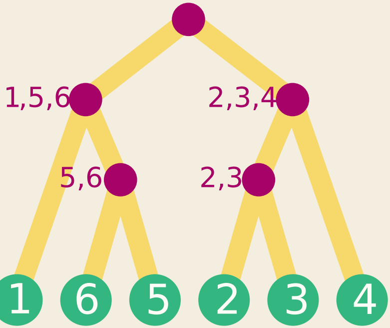

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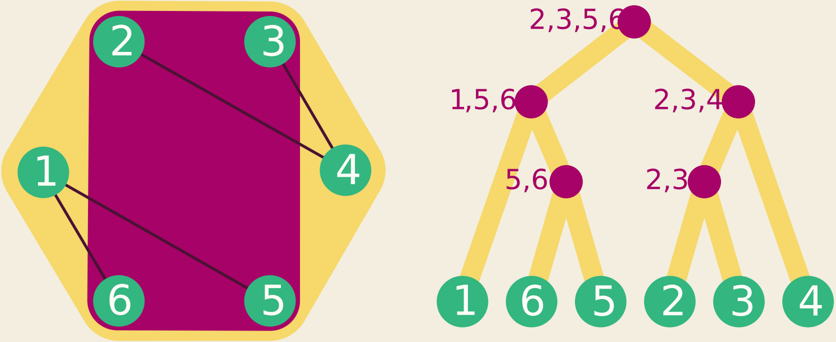

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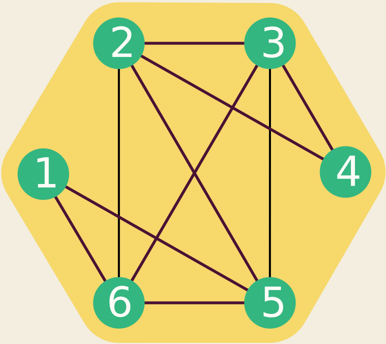


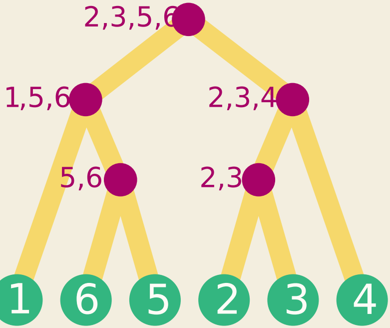

# Low SC-depth colourings

A graph class admits **low-SC-depth colourings** if for every integer $p$ we can colour every member of the class with at most $f(p)$ colours, such that $i < p$ colour classes together induce a subgraph of SC-depth $g(i)$.

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A graph class admits low-SC-depth colourings if and only if it is an FO-interpretation of a bounded expansion class.

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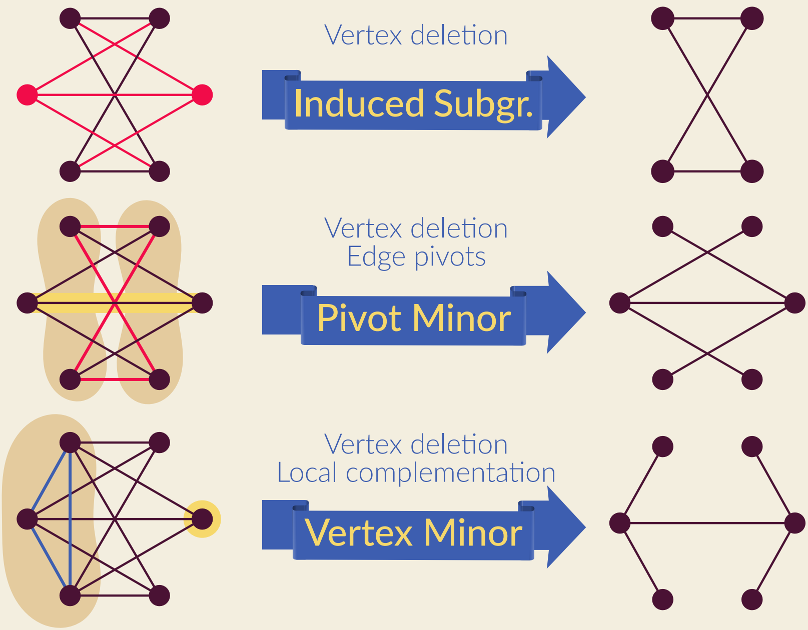
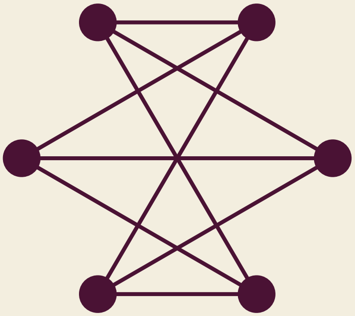
*But how do they **look like**???*

# Substructures for dense classes



# Vertex minors



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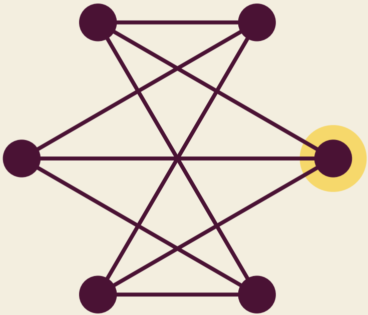

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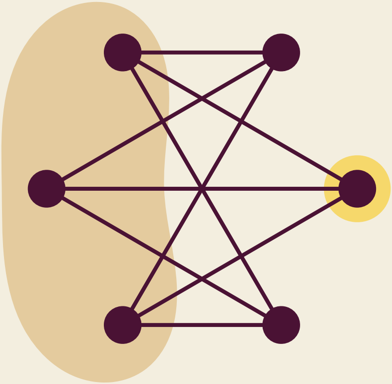

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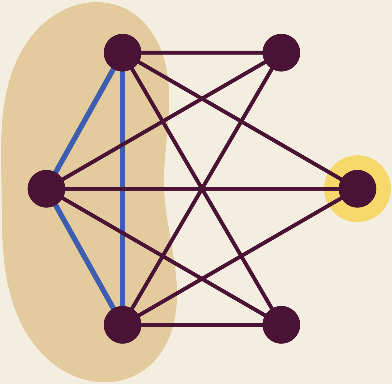

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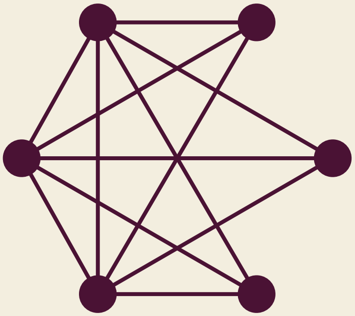

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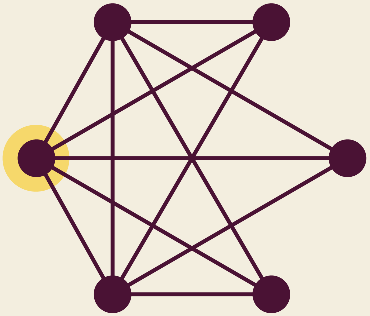

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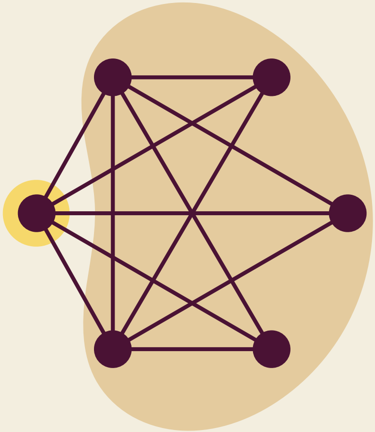

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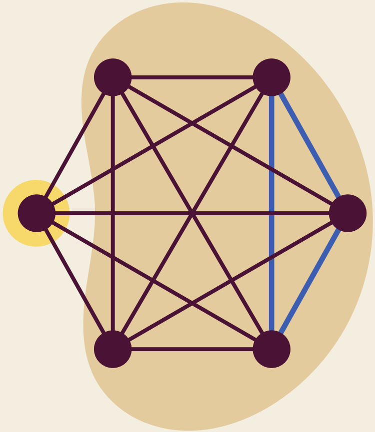

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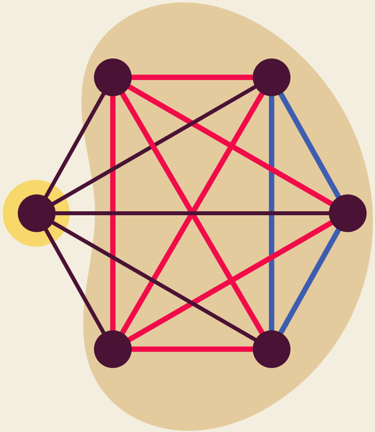

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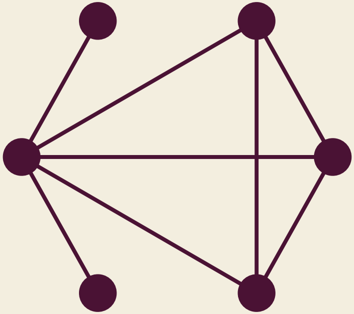

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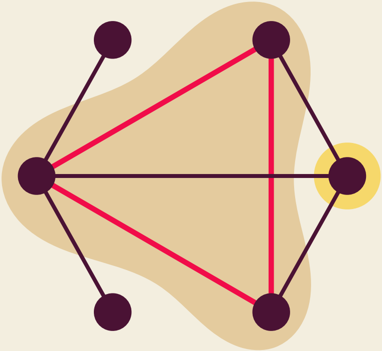

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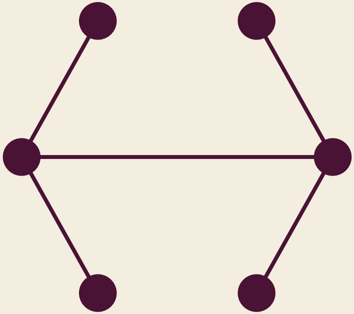
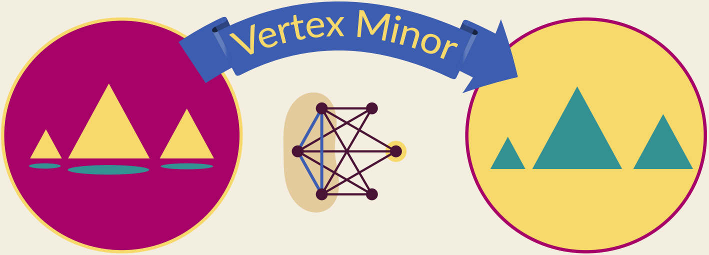
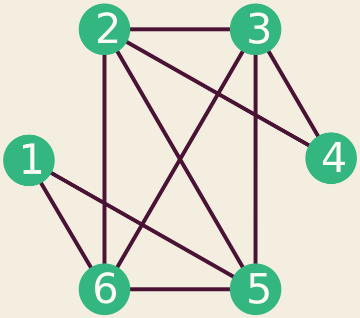


# From SC-depth to treedepth



# Skeleton of low SC-depth graphs



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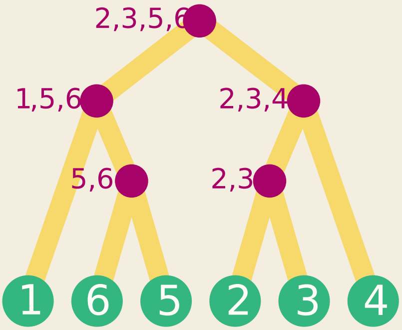


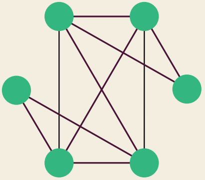

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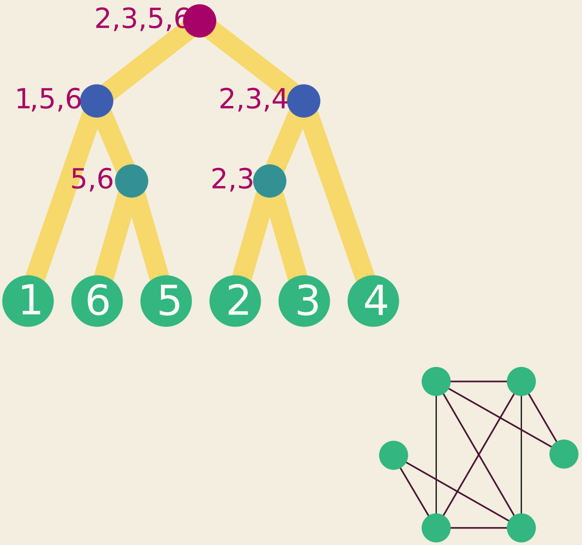

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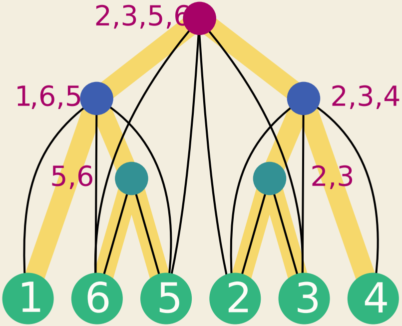


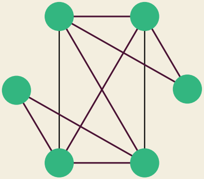

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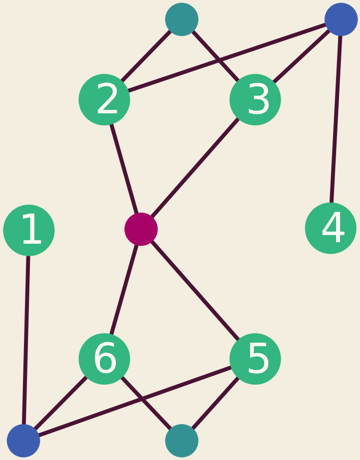


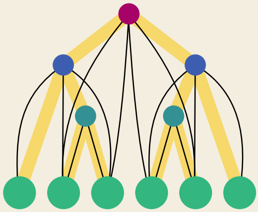

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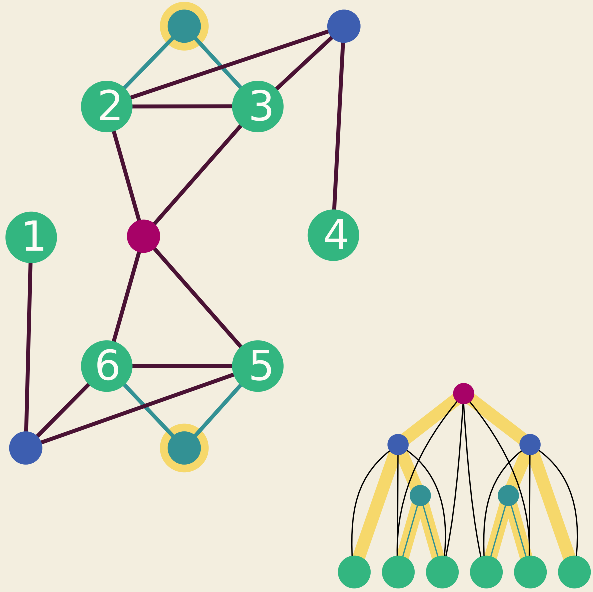

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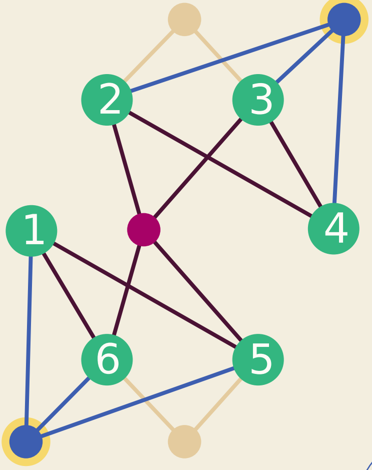


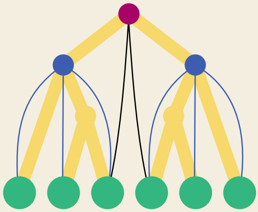

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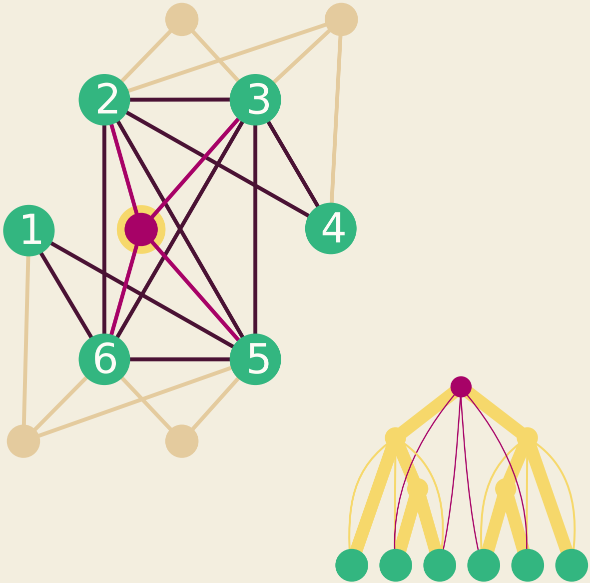

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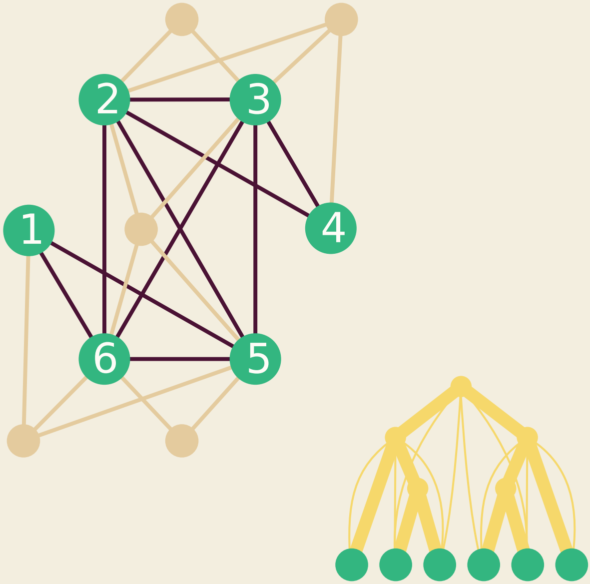

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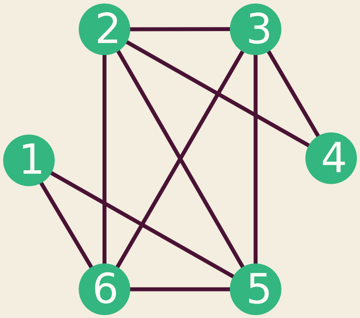


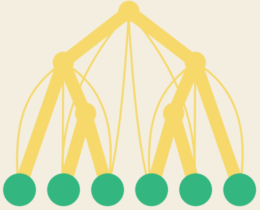

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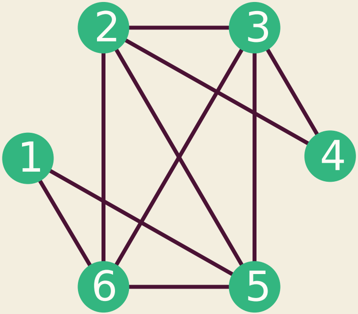


**Theorem** (Ganian et al.)
Every graph of SC-depth d is a vertex minor of a graph with treedepth d+1.

Ganian R, Hliněný P, Nešetřil J, Obdržálek J, de Mendez PO.
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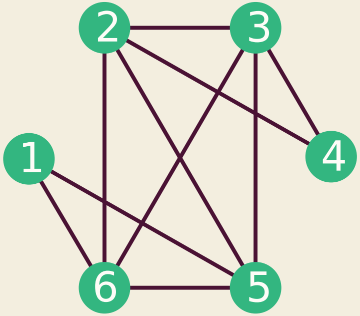


**Theorem** (Ganian et al.)
Every graph of SC-depth d is a shallow vertex minor of a graph with treedepth d+1.

Ganian R, Hliněný P, Nešetřil J, Obdržálek J, de Mendez PO.
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# Shallowness, revisited

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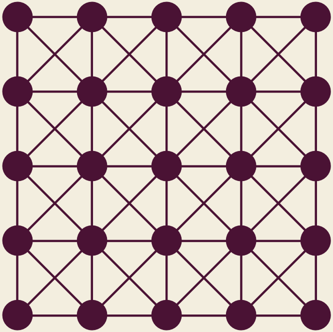

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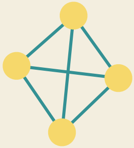

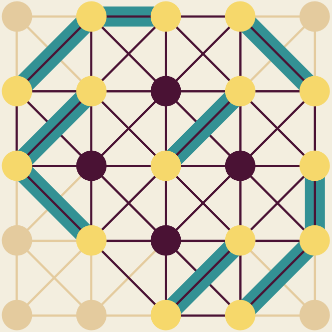

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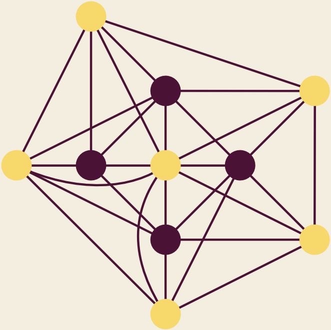

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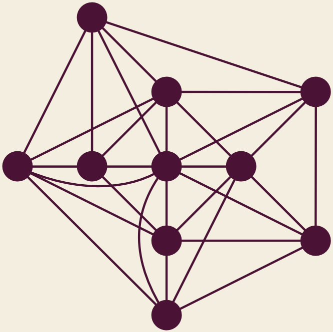

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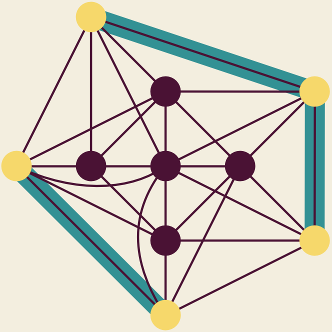

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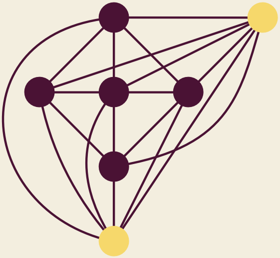

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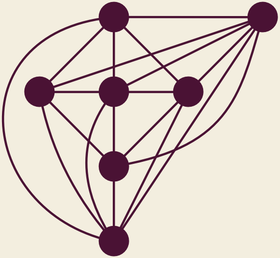

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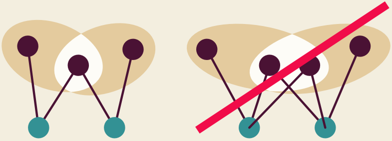
**Detached**: independent +
neighbourhoods intersect
in at most one vertex



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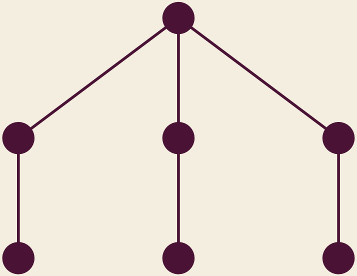

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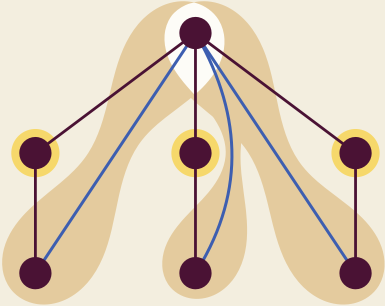

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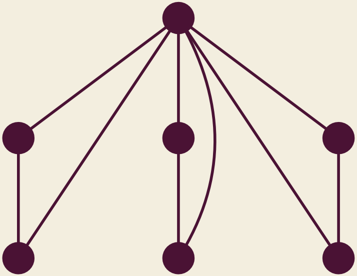

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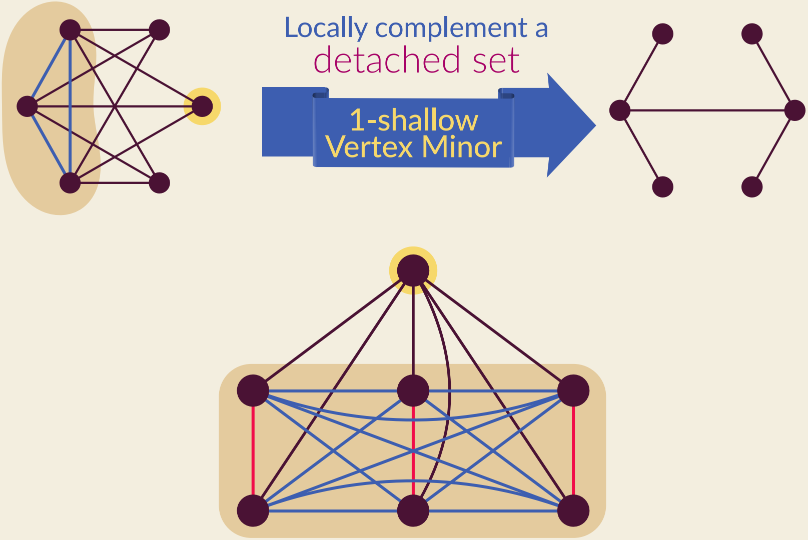

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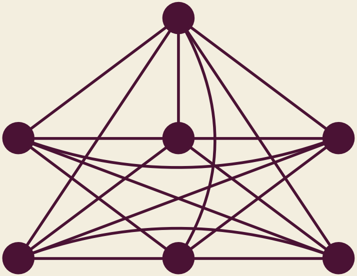

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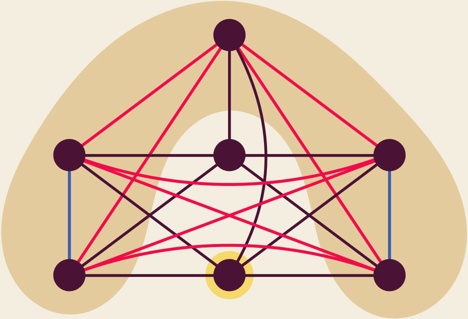

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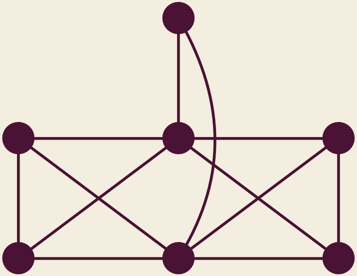

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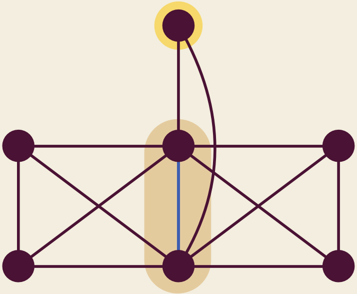

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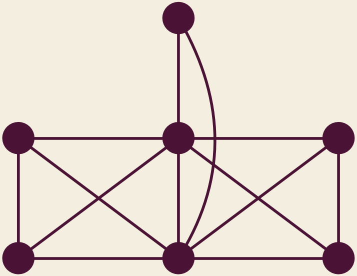

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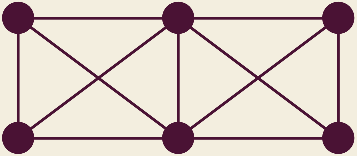

# Low SC-depth colourings

**Theorem** (Ossona de Mendez et al., unpublished)

A graph class admits low-SC-depth colourings if and only if it is an FO-interpretation of a bounded expansion class.

**Corollary**

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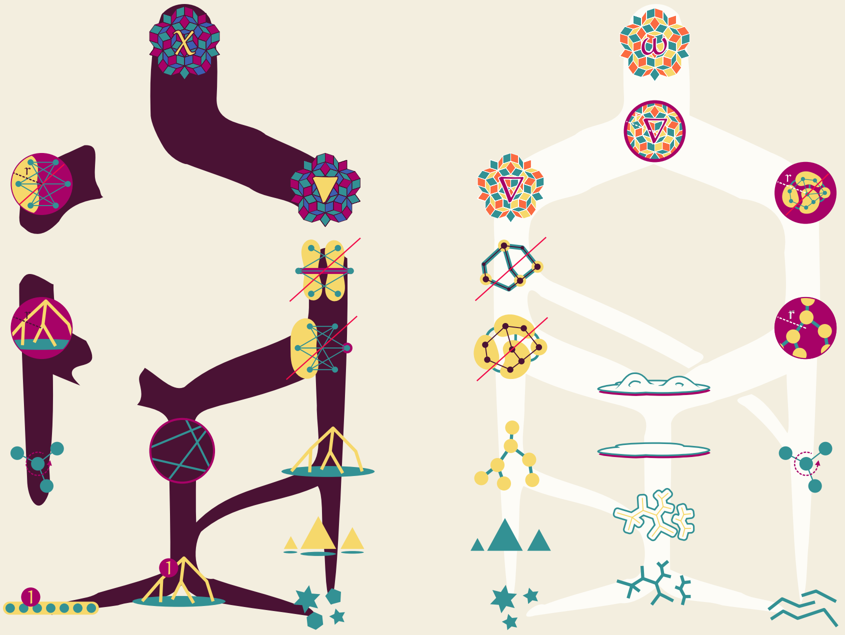
**Corollary**

Every class obtained from taking shallow vertex minors of a bounded expansion class admits low SC-depth colourings.

**Claim** (Work in progress)

Every class that admits low SC-depth colourings can be derived by taking shallow vertex minors from a bounded expansion class.

# Fragments of a dense hierarchy



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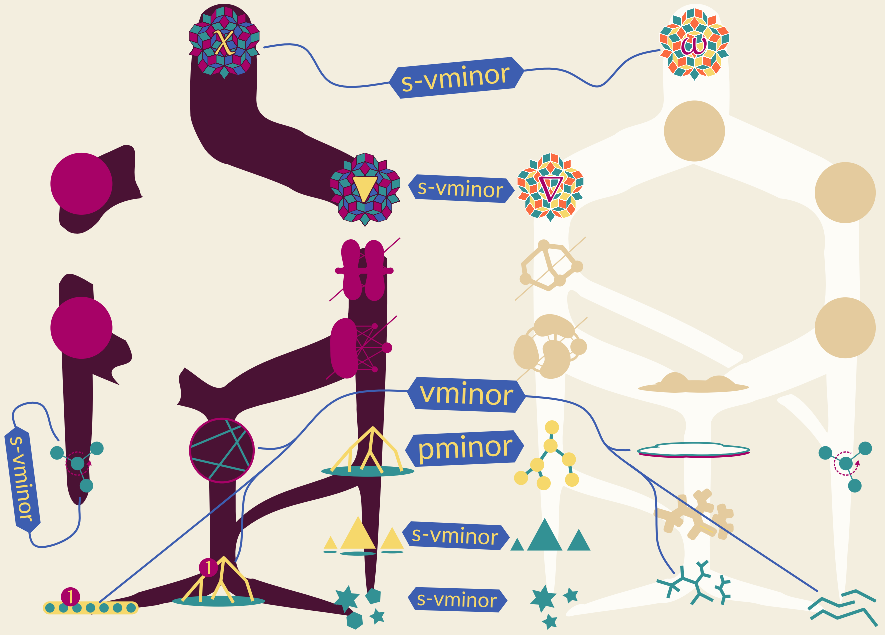


# Expansion vs Complexity

Monotone

Low treedepth col.

shallow minors

shallow top. minors

lexicographic product
with small graph

bounded chromatic
number

density / degeneracy

wcol / col / adm

neighbourhood complexity

Hereditary

Low SC-depth col.

shallow vminors

shallow pminors

lexicographic product
with low SC-depth graph

$\chi$-bounded

???

# Open questions (some of them)

Pruning sequence?

Is hereditary neighbourhood complexity bounded?

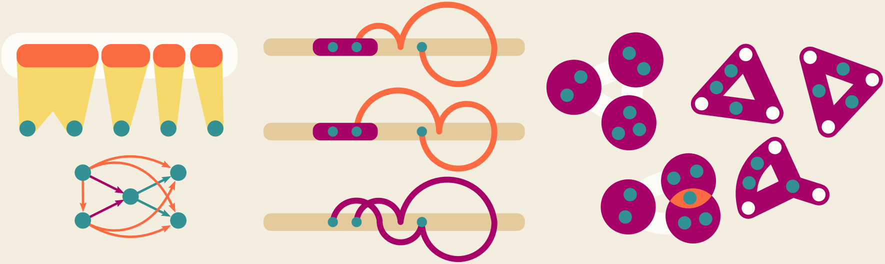
DOMSET kernel? SC-depth ‘modulator’ kernel?

Exact definition of nowhere complex classes?

How to compute low SC-depth colourings?

Shallow vminors/pminors for matroids?

Dense analogue to all nice things in the sparse world?



# THANKS!

*Questions?*

A big thanks to my collaborators:

Blair D. Sullivan
Kyle Kloster
Patrice Ossona de Mendez
Sebastian Siebertz