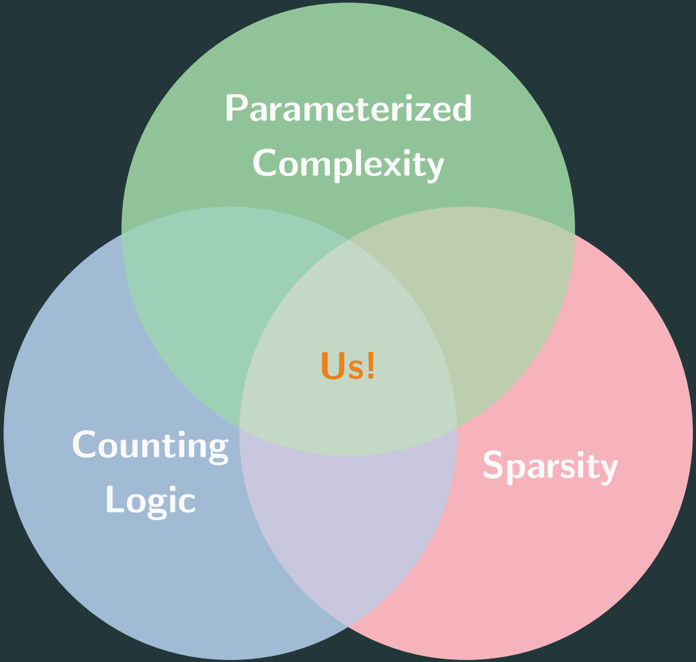
Parameterized Complexity
Us!
Counting Logic
Sparsity

Evaluating Restricted First-Order Counting Properties on Nowhere Dense Classes and Beyond

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joint work with Jan Dreier and Peter Rossmanith

ESA 2023, September 5, 2023

Theoretical Computer Science
RWTH AACHEN UNIVERSITY

Model-Checking

- Graph class $\mathcal{C}$
- Logic L

$MC(\mathcal{C}, L)$

Input:	A graph $G \in \mathcal{C}$ and a sentence $\varphi \in L$
Problem:	Is φ true in G? ($G \models \varphi$?)
Parameter:	$\lvert\varphi\rvert$

Case: First-Order Logic

Complexity on all graphs: $n^{O(|\varphi|)}$

Goal: Find classes $\mathcal{C}$ where FO-MC is FPT (time complexity of $f_{\mathcal{C}}(|\varphi|) \cdot n^d$)

Model-Checking of FO

Theorem (Grohe, Kreutzer, Siebertz '11)

FO-model checking on nowhere dense in almost linear FPT time, i.e., $f(|\varphi|, \varepsilon)n^{1+\varepsilon}$ for every $\varepsilon > 0$.

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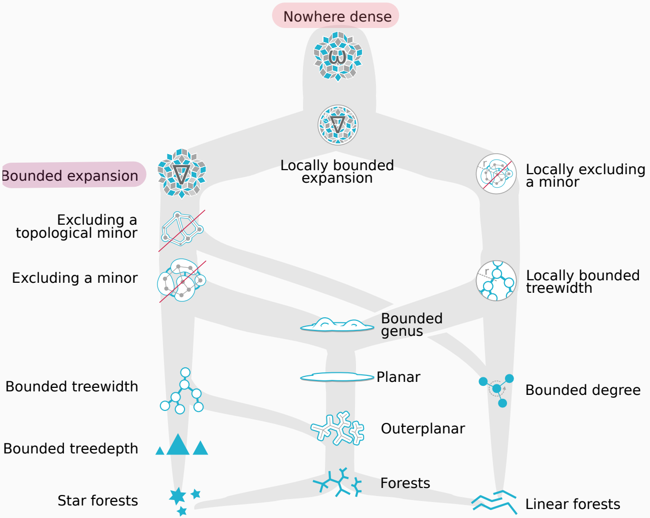
Dominating set of size k in FO:

$$\exists x_1 \ldots \exists x_k \forall y (\bigvee_{1 \leq i \leq k} E(y, x_i) \vee y = x_i)$$

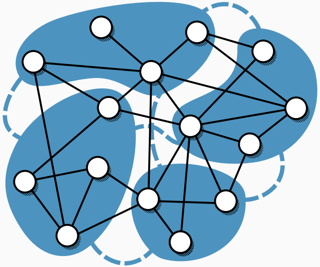
$\implies$ DominatingSet is FPT on nowhere dense, when parameterized by k

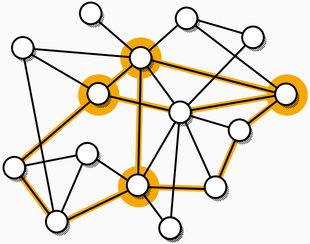
Sparse Graph Classes



Nowhere Dense Classes

A class is nowhere dense
if for every depth r, an $f(r)$-clique is forbidden as r-shallow minor.





Many different characterizations:

splitter game, weak coloring numbers, low neighborhood complexity, treedepth colorings, neighborhood covers, uniform quasi-widenes, …

Monotone + not nowhere dense ⇒ dense

Robust under:

taking subgraphs+shallow minors, adding apex, lexproduct with small graphs, …

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- Partial Dominating Set? Let's look!

Partial Dominating Set

PartDomSet

Input: A graph G and $k, t \in \mathbb{N}$

Problem: Are there k vertices dominating $\geq t$ vertices?

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- Can be solved on classes $\mathcal{C}$ of bounded expansion in time $f_{\mathcal{C}}(k)n$

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Open for nowhere dense!

First-Order with Some Counting: $\mathrm{FO}(\{>0\})$

Definition of $\mathrm{FO}(\{>0\})$ (Dreier, Rossmanith '21)

Built recursively using

- the rules of FO
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h-index:

$$\#\text{mypaper} \left(\#\text{otherpaper}\, \text{cite}(\text{mypaper}, \text{otherpaper}) \geq h \right) \geq h$$

Known Results for $\mathbf{FO}(\{>0\})$

Exact evaluation is *hard* on forest of depth 4

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Exact evaluation is *hard* on forest of depth 4

Theorem (Dreier, Rossmanith '21)

On classes of bounded expansion, in linear FPT time

1. $(1+\varepsilon)$-*approximation of* FO($\{>0\}$),
2. *Exact evaluation of formulas* $\exists x_1 \ldots \exists x_k \# y \underbrace{\varphi(y, x_1, \ldots, x_k)}_{\text{FO w/o } \#} \geq m$

$\Rightarrow$ PartDomSet in time $f(k)n$ on bounded expansion

Our Result A

Theorem

From nowhere dense class

$$\overbrace{G}\models \exists x_1 \dots x_k \#y \; \underbrace{\varphi(y, x_1, \dots, x_k)}_{\text{quantifier-free}} \geq m$$

evaluate in time $f(|\varphi|, \varepsilon)n^{1+\varepsilon}$ *for every* $\varepsilon > 0$.

$\Rightarrow$ PartDomSet in almost linear FPT time on nowhere dense classes

Our Result B

Almost nowhere dense

- generalizes nowhere dense,
- generalizes graphs of *low degree*
- is somewhere dense.

Theorem

From almost nowhere dense class

$$\overbrace{G} \models \exists x_1 \dots x_k \# y \underbrace{\varphi(y, x_1, \dots, x_k)}_{\text{quantifier-free}} \geq m$$

evaluate in time $f(|\varphi|, \varepsilon)n^{1+\varepsilon}$ *with an additive error of* n^ε *for every* $\varepsilon > 0$.

$\Rightarrow$ Approximate PartDomSet on almost nowhere dense classes with additive error n^ε in almost linear FPT time.

Conclusion

Our results: Model-checking in time $f(|\varphi|, \varepsilon)n^{1+\varepsilon}$ of formulas

$$G \models \exists x_1 \dots x_k \# y \underbrace{\varphi(y, x_1, \dots, x_k)}_{\text{quantifier-free}} \geq m$$

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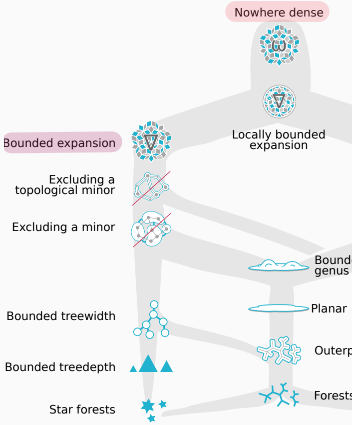
Outlook:

- Extend above to full FO-formulas φ
- $(1+\varepsilon)$-approximation for FO($\{>0\}$) on nowhere dense
- Structurally sparse classes



Thank you!

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