Solving a Family of Multivariate Optimization and Decision Problems on Classes of Bounded Expansion

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Joint work with Peter Rossmanith

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Theoretical Computer Science | RWTH AACHEN UNIVERSITY

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(*Dramatization)

First-Order with Some Counting: $\mathrm{FO}(\{>0\})$

Definition of $\mathrm{FO}(\{>0\})$ (Dreier, Rossmanith, '21)

Built recursively using

- the rules of FO
- $\#y\,\varphi(y, x_1, \ldots, x_k) \geq m$ for every $m \in \mathbb{N}$ and $\mathrm{FO}(\{>0\})$ formula φ

Fragment of $\mathrm{FO}(\mathbb{P})$ and $\mathrm{FOC}(\mathbb{P})$ (Kuske, Schweikardt '17)

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PartDomSet:

$$\exists x_1 \ldots \exists x_k \#y \Big(\bigvee_{1 \leq i \leq k} E(y, x_i) \vee y = x_i \Big) \geq t$$

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h-index:

$$\#\text{mypaper}\left(\#\text{otherpaper}\,\text{cite}(\text{mypaper}, \text{otherpaper}) \geq h\right) \geq h$$

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1. $(1+\varepsilon)$*-approximation of* $\mathsf{FO}(\{>0\})$,
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Theorem (Dreier, M., Rossmanith '23)

On nowhere dense classes, in almost linear FPT time:

Exact evaluation of formulas $\exists x_1 \ldots \exists x_k \# y \underbrace{\varphi(y, x_1, \ldots, x_k)}_{\text{quantifier-free}} \geq m$

$\Rightarrow$ PartDomSet in (almost) linear FPT time on bounded expansion & nowhere dense

Other variants

Red Blue Partial Dominating Set

Input: A graph G and $k, t_{\text{red}}, t_{\text{blue}} \in \mathbb{N}$

Problem: Are there k vertices dominating $\geq t_{\text{red}}$ red and $\geq t_{\text{blue}}$ blue vertices?

Parameter: k

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$$G \models \exists x_1 \ldots \exists x_k (\#y\, Red(y) \wedge dom(y, \bar{x})) \geq t_{\text{red}} \wedge (\#y\, Blue(y) \wedge dom(y, \bar{x})) \geq t_{\text{blue}}$$

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Problem: Are there k vertices dominating *exactly* t vertices?

$$G \models \exists x_1 \dots \exists x_k \#y\, dom(y, \bar{x}) = t$$

Cannot be expressed as $\exists x_1 \dots \exists x_k \#y \underbrace{\varphi(y, x_1, \dots, x_k)}_{\text{quantifier-free}} \geq m$

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Our goal: Lift result to $\exists x_1 \ldots \exists x_k \underbrace{\bigvee \bigwedge (\#y\, \varphi_i(y\bar{x}) \geq m_i)}_{\text{boolean combination of } \ell \text{ counting terms}}$

Algorithmic Result

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Theorem (Our positive result)

On classes of bounded expansion, we can decide in time $f(k,\ell)n^{\ell+1}$ whether

$$G \models \exists x_1 \ldots x_k P(\#y \underbrace{\varphi_1(y\bar{x})}_{\textit{first-order}}, \ldots, \#y\, \varphi_\ell(y\bar{x}))$$

where P is some efficiently computable predicate over $\mathbb{N}^\ell$.

Moreover, we can count the number of solutions.

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Moreover, we can count the number of solutions.

$\implies$ Exact Partial Dominating Set in time $f(k)n^2$ on bounded expansion.

$\implies$ Red Blue Partial Dominating Set in time $f(k)n^3$ on bounded expansion (can be improved to $f(k)n^2$).

Step 1: Reduction to a Simpler Problem

Original Problem:	$\rightarrow$	**Simpler Problem**
$G \in \mathcal{C}$ of bounded expansion	$\rightarrow$	$\vec{G} \in \mathcal{C}'$ of bounded expansion
FO-formulas $\begin{cases} \varphi_1 \\ \vdots \\ \varphi_\ell \end{cases}$	$\rightarrow$	$\left.\begin{matrix} c_1 \\ \vdots \\ c_\ell \end{matrix}\right\}$ vertex weight fcts

For all $\bar{u} = u_1 \dots u_k \in V(G)^k$:

$$G \models \begin{aligned} & \#y\, \varphi_1(y\bar{u}) \geq t_1 \\ \wedge & \quad \vdots \\ \wedge & \#y\, \varphi_\ell(y\bar{u}) \geq t_\ell \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} & c_1(u_1) + \dots + c_1(u_k) \geq t_1 \\ & \vdots \\ & c_\ell(u_1) + \dots + c_\ell(u_k) \geq t_\ell \\ & \vec{G} \models \omega(\bar{u}) \text{ (quantifier-free)} \end{aligned}$$

Courcelle with Semiring Homomorphisms

Often don't want one satisfying assignment but computing a property of the set of satisfying assignments

Example: Set of all vertex covers $\mapsto$ minimum weight VC, number of VCs, all VCs...

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Definition

A problem P is an *MSO-evaluation problem* if it can be expressed as computing $h(\mathrm{sat}(\varphi, G))$ for some homomorphism h into a semiring and MSO-formula φ.

Example: min. weight VC: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$, h maps set to sum of weights

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Theorem (Courcelle, Mosbah '93)

An MSO-evaluation problem P can be solved in time $f_P(tw)nt$ on graphs of treewidth tw where t is the time complexity of the semiring operations.

Result

Maintain a table: for every $(t_1, \dots, t_\ell)$ the number of $\bar{u} \in V(G)^{\bar{x}}$ with

- $\vec{G} \models \omega(\bar{u})$
- $c_1(u_1) + \cdots + c_1(u_k) = t_1$
- $\vdots$
- $c_\ell(u_1) + \cdots + c_\ell(u_k) = t_\ell$

Show that this forms a semiring with appropriate operations!

Complexity of operations is $\sim n^{\ell+\varepsilon}$

Theorem (Our positive result)

On classes of bounded expansion, we can decide in time $f(k,\ell)n^{\ell+1}$ whether

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Lower Bounds

kSum

k-Sum Problem: given m numbers $x_1, \ldots, x_m$; target T

Find k numbers that add up to exactly T

Algorithms known for k-Sum: • $\widetilde{O}(Tm)$ • $O(m^{\lceil k/2 \rceil})$

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Theorem (Abboud et al. '21)

For every $\varepsilon > 0$, k-Sum is not in time $T^{1-\varepsilon} m^{o(k)}$ (under SETH).

Reduction

Example: $T = 100; k = 3$

$x_1 = 32,$ $x_2 = 42,$ $x_3 = 53,$ $x_4 = 15.$

Reduction to model-checking of our fragment (on star forests):

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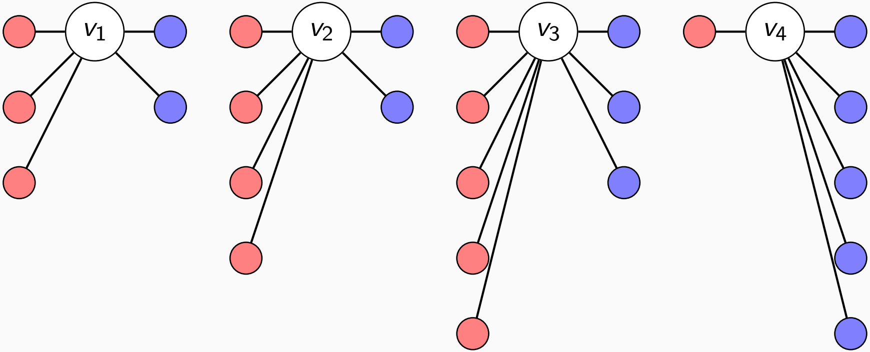

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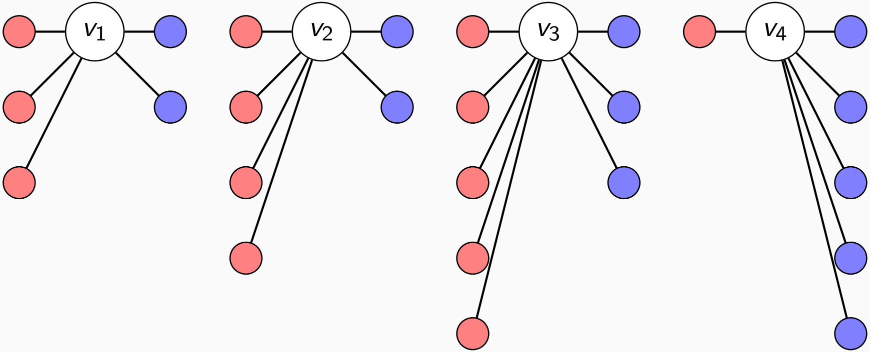

$G \models \exists x_1 \exists x_2 \exists x_3 \#y\, Red(y) \wedge dom(y, \bar{x}) = 9 \wedge \#y\, Blue(y) \wedge dom(y, \bar{x}) = 10$

Reduction: Parameters

Theorem (Reminder)

For every $\varepsilon > 0$, k-$\textsc{Sum}$ is not in time $T^{1-\varepsilon} m^{o(k)}$ (under SETH).

In our example: Parameter $\ell = 2$, size $|G| \leq 2\sqrt{T}m$.

$\implies$ quadratic lower bound for model-checking

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In general: ℓ freely choosable $\implies |G| = O(\sqrt[\ell]{T}\ell m)$

Have to "guess carry-overs": only $f(k, \ell)$ many choices

Theorem (Our Lower Bound)

On star forests, for formulas of the form

$\exists x_1 \dots \exists x_k (\#y\, \varphi_1(y, x_1 \dots x_k) = t_1 \wedge \dots \wedge \#y\, \varphi_\ell(y, x_1 \dots x_k) = t_\ell)$

there is no model-checking algorithm in time $f(k, \ell) n^{\ell - \varepsilon}$, for any function f or $\varepsilon > 0$.

Conclusion

Our results: Model-checking of $FO(\mathbb{P})$ formulas

$$G \models \exists x_1 \dots x_k P(\#y \underbrace{\varphi_1(y\bar{x})}_{\text{first-order}}, \dots, \#y\, \varphi_\ell(y\bar{x}))$$

on classes of bounded expansion

- in time $f(k,\ell)n^{\ell+1}$ polylog n
- not in time $f(k,\ell)n^{\ell-\varepsilon}$ for all $\varepsilon > 0$ under SETH

For the case of boolean combinations of $\#\varphi(y\bar{x}) \geq m$:

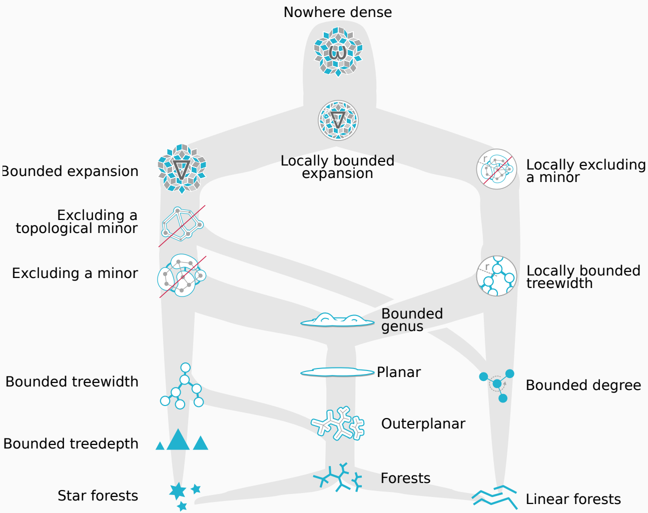
- in time $f(k,\ell)n^{\ell}$ polylog n
- not in time $f(k,\ell)n^{\ell/2-\varepsilon}$ for all $\varepsilon > 0$ under SETH

Outlook:

- Close the gaps
- Lift to (structurally) nowhere dense classes

Appendix

Sparse Graph Classes



By Felix Reidl

Partial Dominating Set

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- W[1]-hard for 2-degenerate graphs
- Can be solved on H-minor free graphs in time $(g(H)k)^k n^{O(1)}$
- Can be solved on classes $\mathcal{C}$ of bounded expansion in time $f_{\mathcal{C}}(k)n$
- Can be solved on nowhere dense classes $\mathcal{C}$ in time $f_{\mathcal{C}}(k)n^{1+\varepsilon}$