Approximate Evaluation of First-Order Counting Queries

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Algorithmic Meta-Theorems

“All problems expressible in a certain logic can be solved efficiently on certain graphs”

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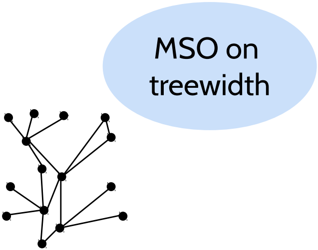

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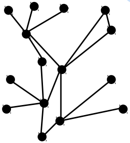
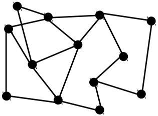
MSO on treewidth

FO on sparse graphs





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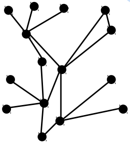

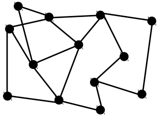

FO({>0}) for ***approximation*** on sparse graphs

Logic

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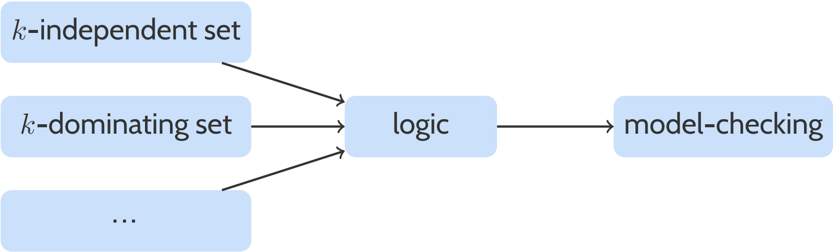
Best algorithms on general graphs: $n^{O(k)}$

Model-Checking



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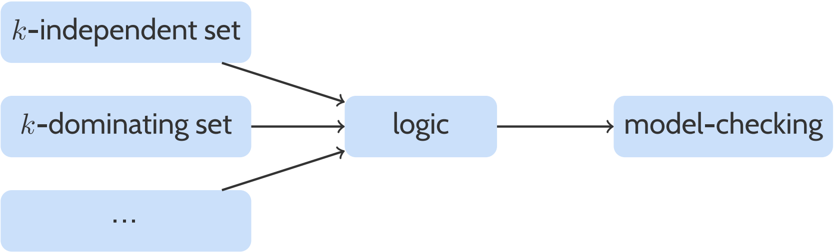

MC($\mathcal{G}$, L)

Input:	A graph $G \in \mathcal{G}$ and a sentence $\varphi \in \mathrm{L}$
Parameter:	$\lvert\varphi\rvert$
Problem:	Is φ true in G?

Goal: linear FPT run time $f(|\varphi|)n$

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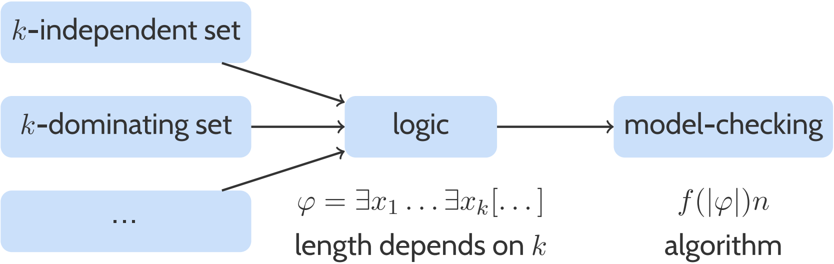

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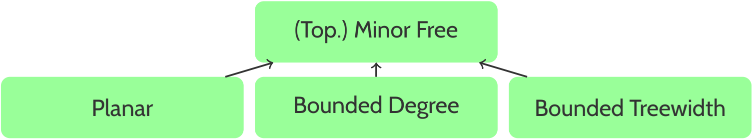
Some Sparse Graph Classes

Planar | Bounded Degree | Bounded Treewidth

If $\mathcal{G}$ has bounded treewidth then MC($\mathcal{G}$, MSO) $\in$ FPT.

[Courcelle 1990]

Some Sparse Graph Classes



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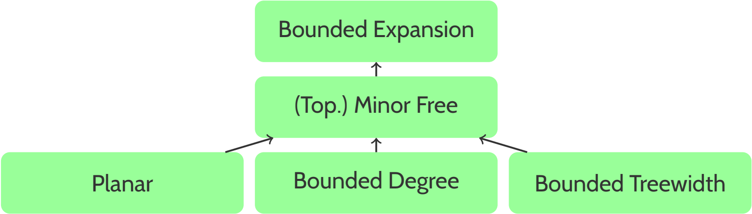

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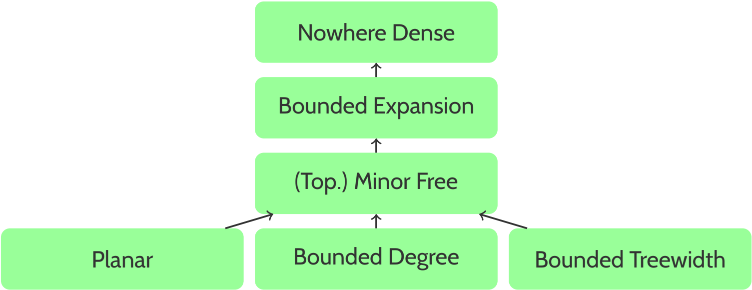

If $\mathcal{G}$ has bounded treewidth then MC($\mathcal{G}$, MSO) $\in$ FPT.

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If $\mathcal{G}$ is nowhere dense then MC($\mathcal{G}$, FO) $\in$ FPT.

[Grohe, Kreutzer, Sieberz 2011]

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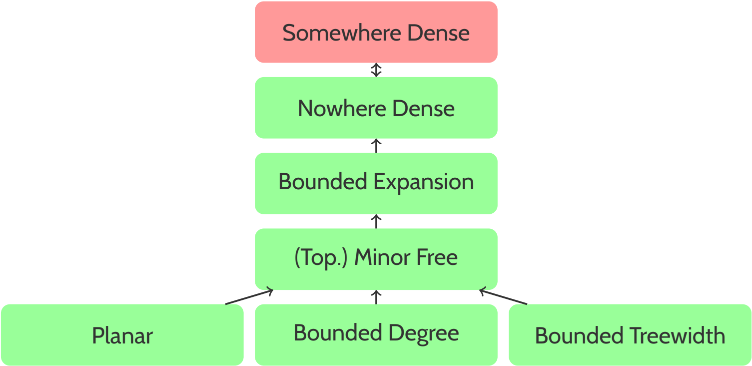

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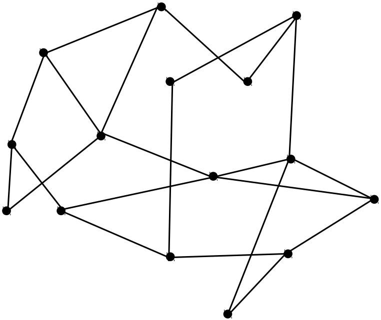
Bounded Expansion



$$\frac{|E|}{|V|} \leq c$$

for every graph
in the graph class.

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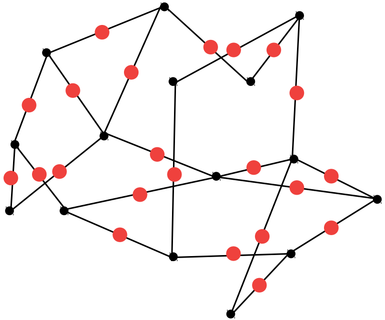

$$\frac{|E|}{|V|} \leq 2$$

for every graph
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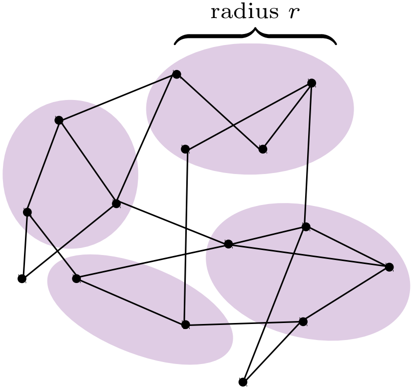

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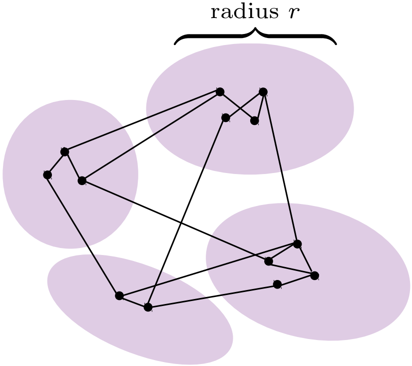

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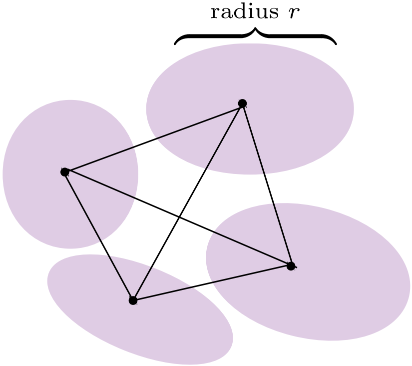

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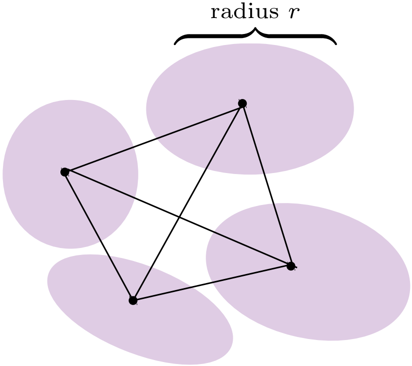

$$\frac{|E|}{|V|} \leq f(r)$$

for every r-shallow minor of every graph in the graph class.

Partial Dominating Set

Input:	A graph G and $k, m \in \mathbf{N}$
Parameter:	k
Problem:	Are there k vertices dominating m vertices?

Counting Problems

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Can be solved on H-minor free graphs in time $(g(H)k)^k n^{O(1)}$.

[Amini, Fomin, Saurabh, 2008]

Can be solved on apex-minor-free graphs in time $2^{\sqrt{k}} n^{O(1)}$.

[Fomin, Lokshtanov, Raman, Saurabh, 2011]

Is W[1]-hard for 2-degenerate graphs.

[Golovach, Villanger 2008]

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Length of formula depends only on k (and not on m)

Counting Logic

Definition of FO($\{>0\}$)

built recursively using

- the rules of FO
- $\#y\ \varphi \geq m$ for every $m \in \mathbf{N}$ and FO($\{>0\}$) formula φ

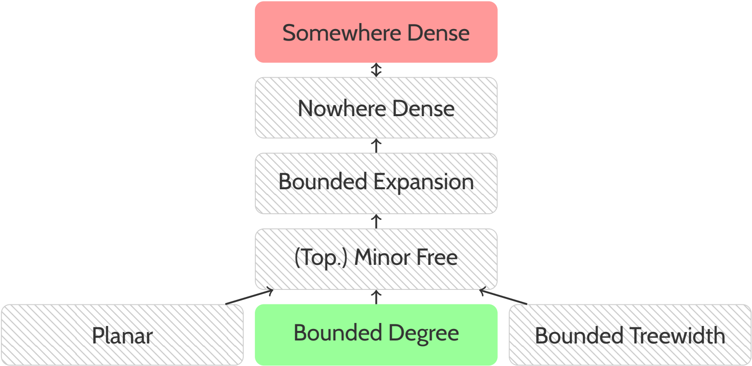
Example 1: Partial Dominating Set

$$\exists x_1 \ldots \exists x_k\, \#y\, \Big(\bigvee_i y \sim x_i \land y = x_i\Big) \geq m$$

Example 2: h-Index

$$\#\text{mypaper}\ \Big(\#\text{otherpaper cite}(\text{otherpaper}, \text{mypaper}) \geq h\Big) \geq h$$

Good News



If $\mathcal{G}$ has bounded degree then MC($\mathcal{G}$, FOC) $\in$ FPT.

[Kuske, Schweikardt 2017]

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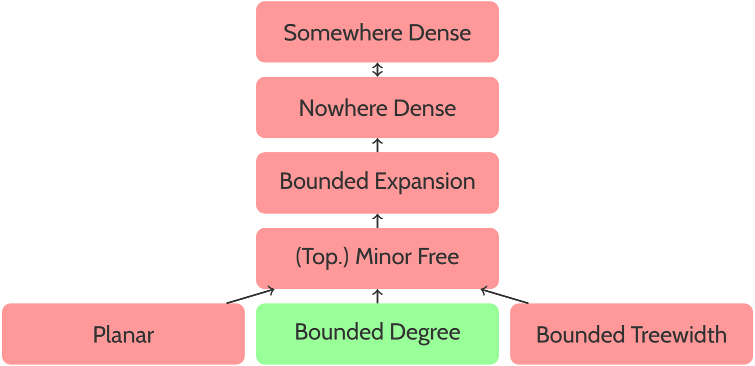

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MC($\mathcal{G}$, FO($\{>0\}$)) is AW[$*$]-hard on trees.

similar to [Grohe, Schweikardt 2018]

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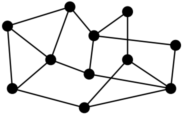

contains k-clique

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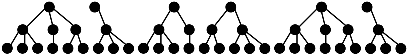

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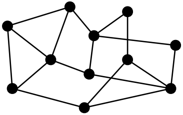

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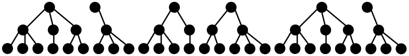

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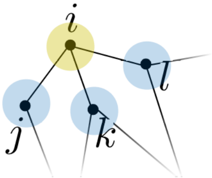

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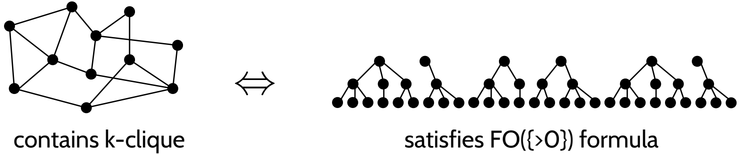

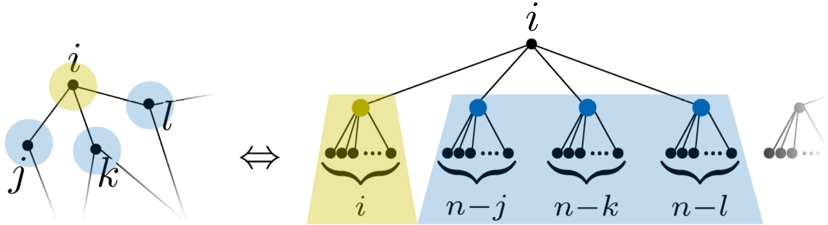

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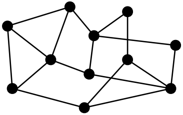

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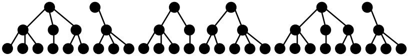

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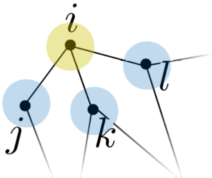

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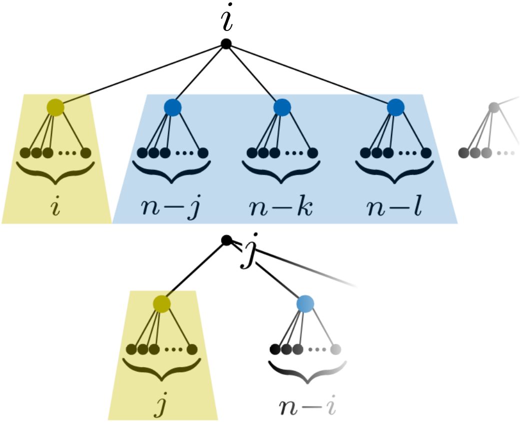
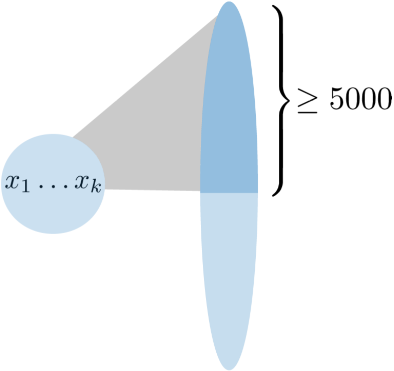


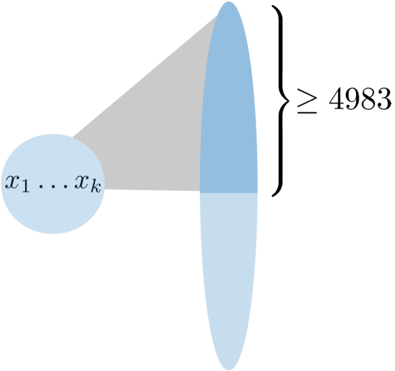
Stability

Are there k vertices dominating at least $m = 5000$ vertices?



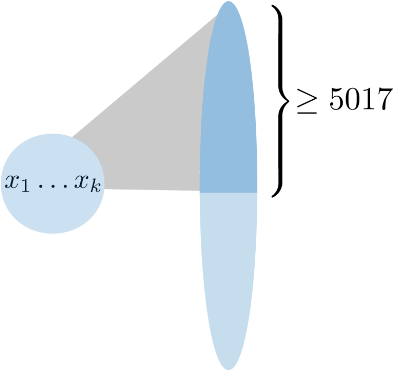
Stability

Are there k vertices dominating at least $m = 4983$ vertices?



Stability

Are there k vertices dominating at least $m = 5017$ vertices?



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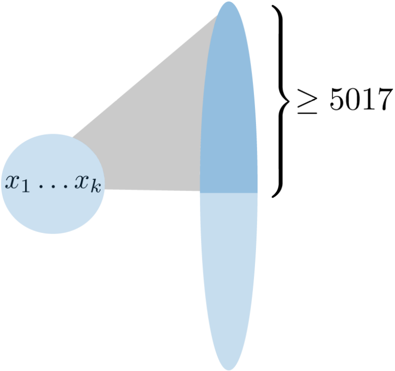

Let $\varepsilon > 0$. A formula φ is ε-*unstable* on a graph G if scaling the counting literals by $(1 \pm \varepsilon)$ changes whether φ is true in G.

Approximate Model-Checking

Theorem

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- *If*  *then φ is false on G.*
- *If*  *then φ is ε-unstable on G.*

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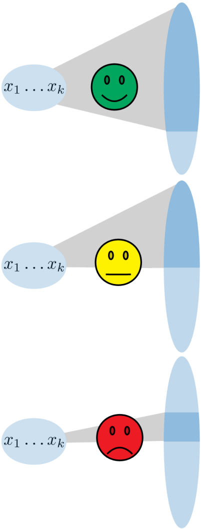
PARTIAL DOMINATING SET: $\exists x_1 \ldots \exists x_k \, \#y \left(\bigvee_i y \sim x_i \wedge y = x_i \right) \geq m$



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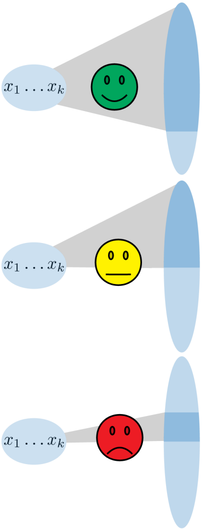

There exists a set dominating $\geq (1+\varepsilon)m$ vertices.

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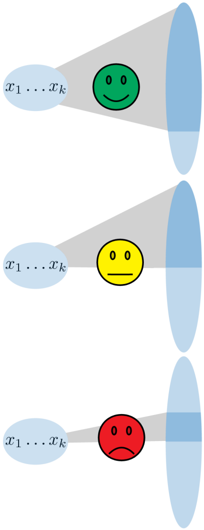

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All sets dominate $< (1 - \varepsilon)m$ vertices.

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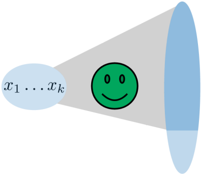

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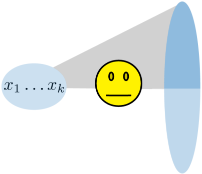
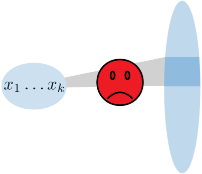


All sets dominate $< (1+\varepsilon)m$ vertices and there exists a set dominating $\geq (1-\varepsilon)m$ vertices.



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This holds for all problems of the form

$$\exists x_1 \ldots \exists x_k \# y \ \underbrace{\varphi(y x_1 \ldots x_k)}_{\in \mathrm{FO}}.$$

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FO($\{>0\}$) allows comparing $\#y$ and $m \in \mathbf{N}$.

Theorem

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- counting tuples $\#yz$ (e.g., $\#yz\ \varphi > m$)
- multiplying of counting terms (e.g., $\#y\ \varphi \cdot \#z\ \psi > m$)

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- comparing $\#y$ and $\#z$ (e.g., $\#y\ \varphi > \#z\ \psi$)
- counting tuples $\#yz$ (e.g., $\#yz\ \varphi > m$)
- multiplying of counting terms (e.g., $\#y\ \varphi \cdot \#z\ \psi > m$)
- subtraction of counting terms (e.g., $\#y\ \varphi - \#z\ \psi > m$)

Summary

FO($\{>0\}$) is

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Slight extensions of FO($\{>0\}$) are

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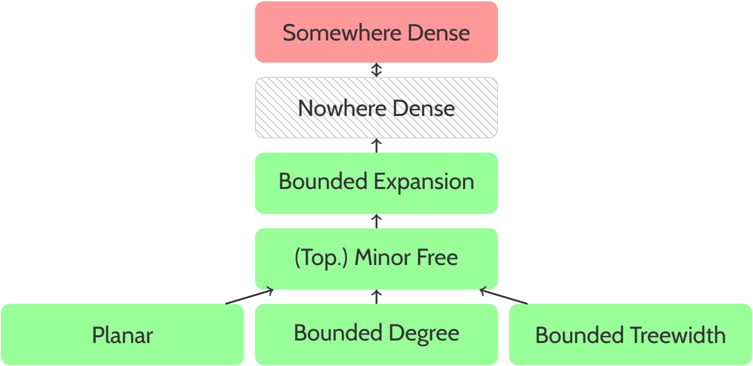
$\Rightarrow$ FO($\{>0\}$) seems like "the right logic" for approximation on sparse graphs

Big Question



Can we generalize our results to nowhere dense graph classes?

Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

$$m_1 \leq \#x_1 \left(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right)$$

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$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \left(m_3 \leq \#x_3 \ \overbrace{\varphi(x_1 x_2 x_3)}^{\text{quantifer-free FO}} \right)\right)$$

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Proof Sketch – Quantifier Elimination

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$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \quad \overbrace{\varphi'(x_1 x_2)}^{\text{quantifier-free FO}} \quad\right)$$

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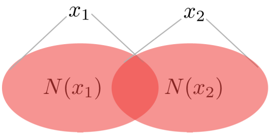
Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

$$\overbrace{\varphi'''}^{\text{quantifier-free FO}}$$

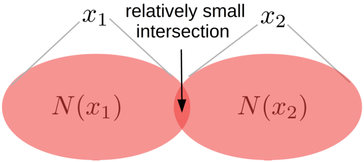
Proof Sketch – Domination

$$\underbrace{\#x_3\ (x_3 \sim x_1 \vee x_3 \sim x_2) \geq m}_{\text{replace with quantifier-free FO}}$$



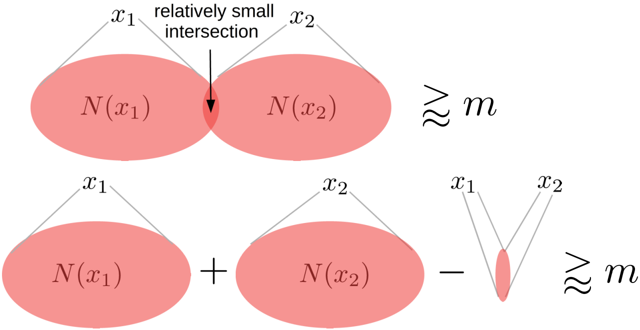
$\geq m$

Proof Sketch – Small Intersection

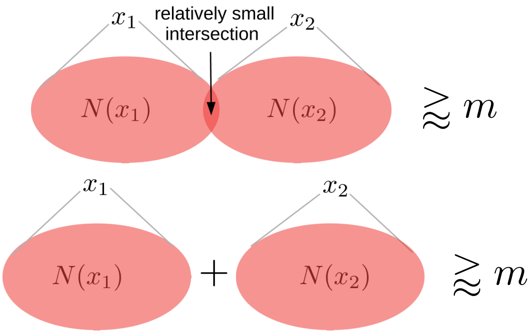


$\gtrapprox m$

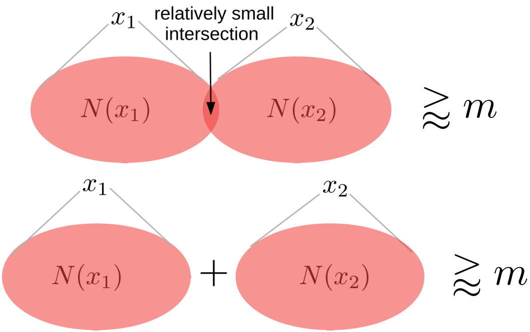
Proof Sketch – Small Intersection



Proof Sketch – Small Intersection

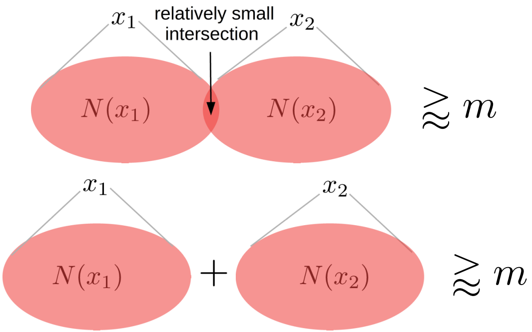


Proof Sketch – Small Intersection



$R_{\geq i}(x)$ true
iff $|N(x)| \geq i$

Proof Sketch – Small Intersection



$$\bigvee_{i \in \{0,1,2,\ldots,m\}} R_{\geq i}(x_1) \wedge R_{\geq m-i}(x_2)$$

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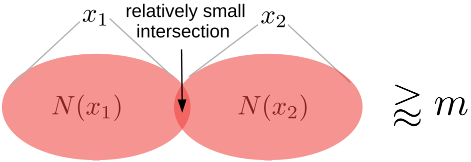
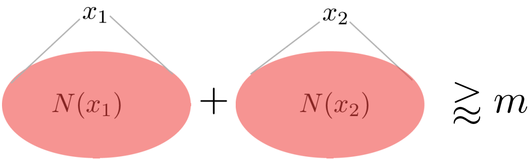
Proof Sketch – Small Intersection



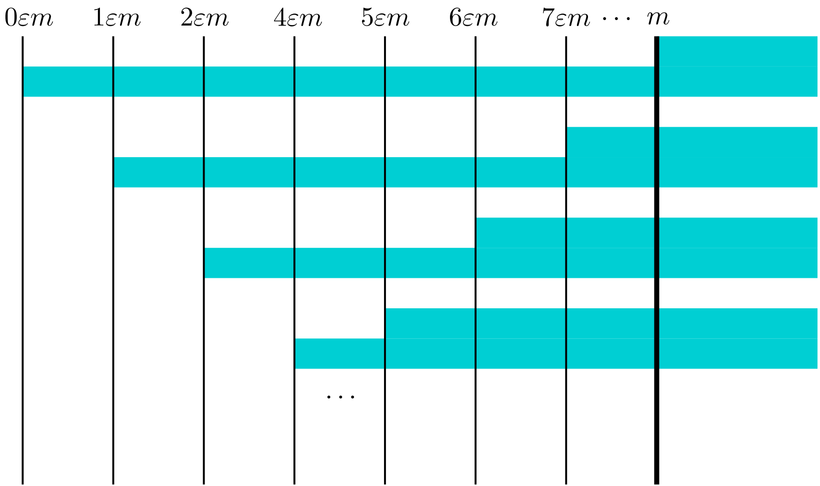


$$\bigvee_{i \in \{0\varepsilon m, 1\varepsilon m, 2\varepsilon m, \dots, m\}} R_{\geq i}(x_1) \wedge R_{\geq m-i}(x_2)$$

$R_{\geq i}(x)$ true iff $|N(x)| \geq i$

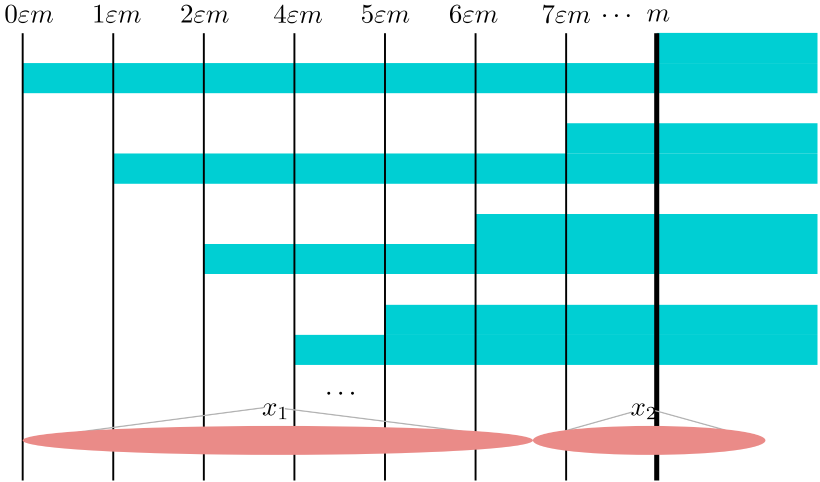
Proof Sketch – Small Intersection

$$\bigvee_{i=0}^{1/\varepsilon} R_{\geq \varepsilon m i}(x_1) \wedge R_{\geq m - \varepsilon m i}(x_2)$$



Proof Sketch – Small Intersection

$$\bigvee_{i=0}^{1/\varepsilon} R_{\geq \varepsilon m i}(x_1) \wedge R_{\geq m-\varepsilon m i}(x_2)$$



Proof Sketch – Small Intersection

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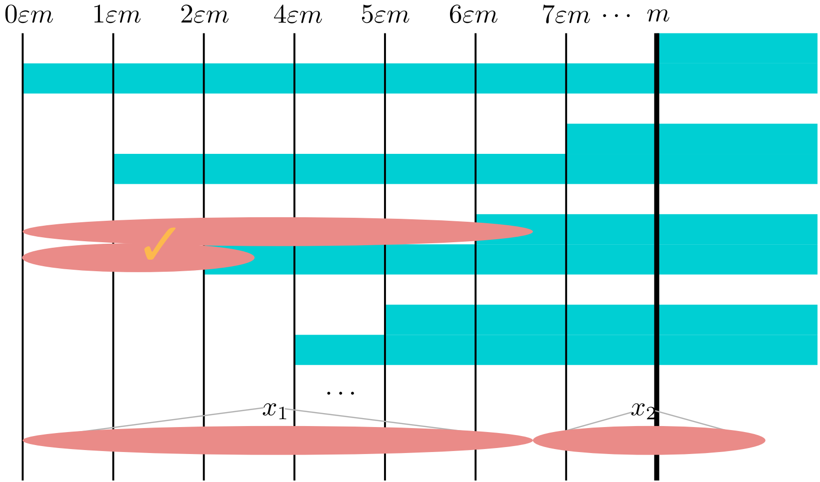
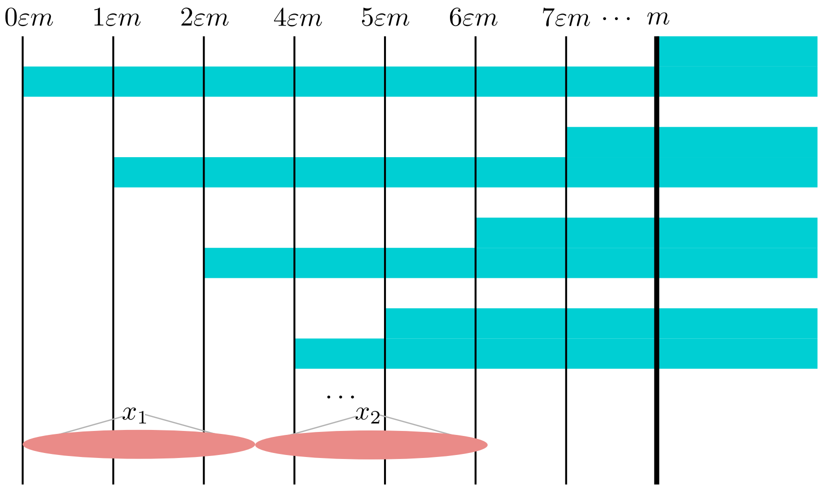


Proof Sketch – Small Intersection

$$\bigvee_{i=0}^{1/\varepsilon} R_{\geq \varepsilon m i}(x_1) \wedge R_{\geq m - \varepsilon m i}(x_2)$$



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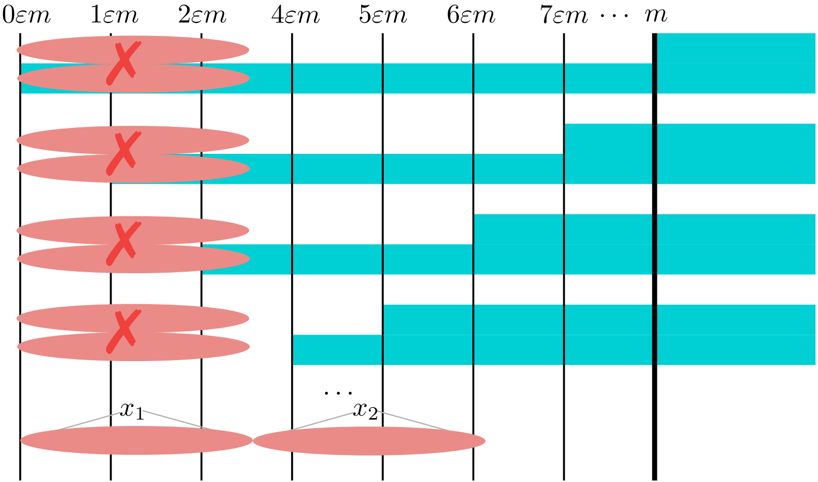

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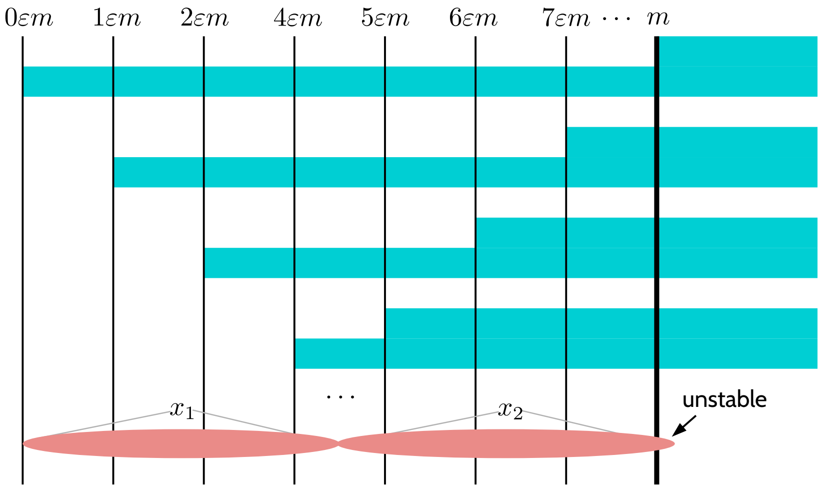

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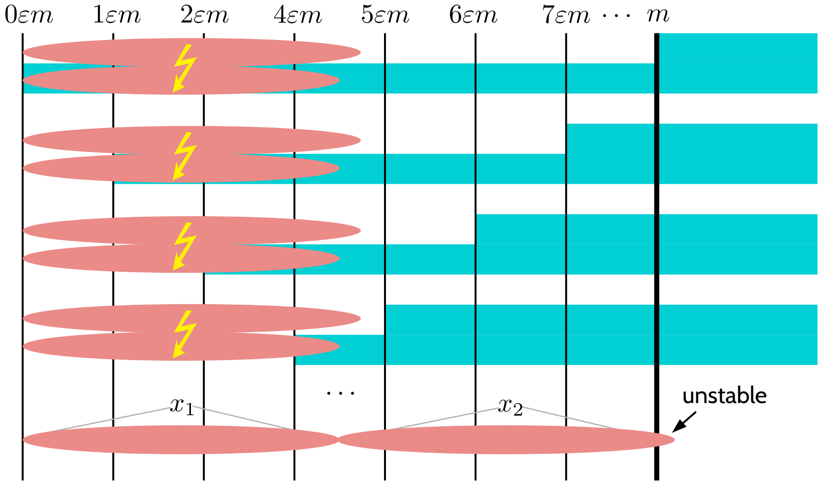
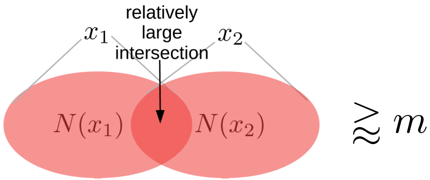
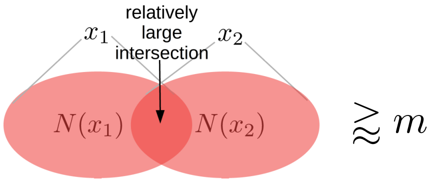
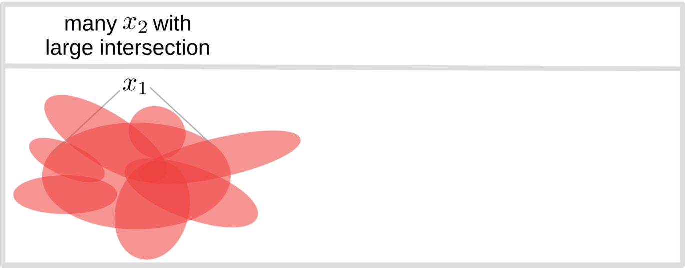


Proof Sketch – Large Intersection



Proof Sketch – Large Intersection





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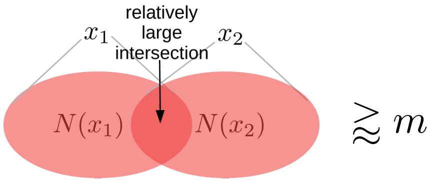
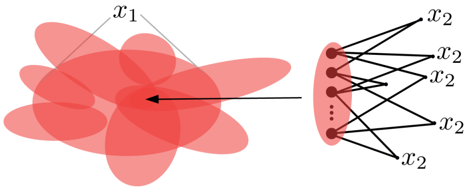
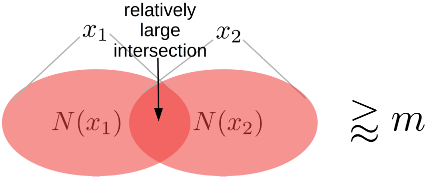


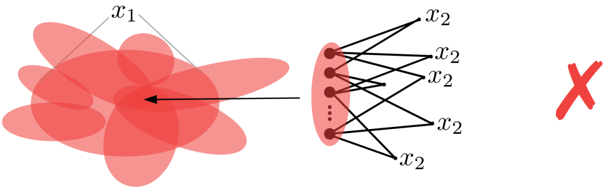
many x_2 with large intersection $\implies$ large 1-subdivided clique



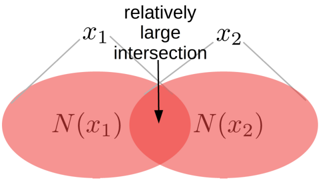
Proof Sketch – Large Intersection



many x_2 with large intersection $\implies$ large 1-subdivided clique $\implies$ not bounded expansion

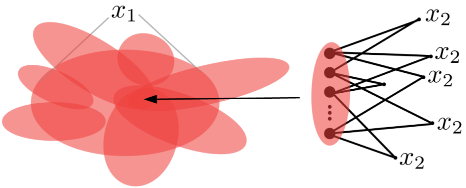


Proof Sketch – Large Intersection



$\gtrsim m$

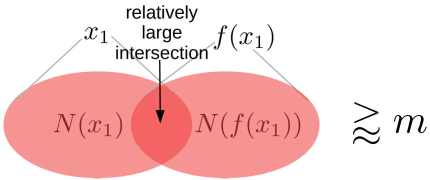
many x_2 with large intersection	$\Longrightarrow$	large 1-subdivided clique	$\Longrightarrow$	not bounded expansion



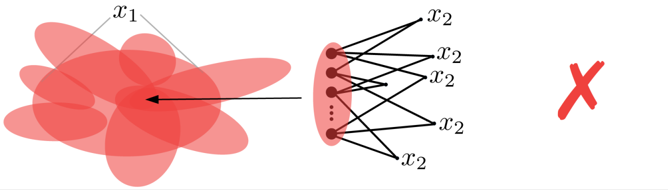
✗

We assume (for simplicity) x_1 has only one x_2
with a large intersection.

Proof Sketch – Large Intersection

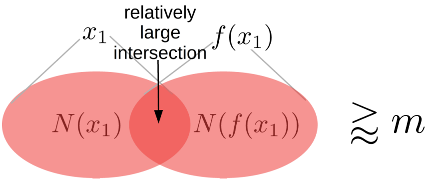


many x_2 with large intersection $\Longrightarrow$ large 1-subdivided clique $\Longrightarrow$ not bounded expansion



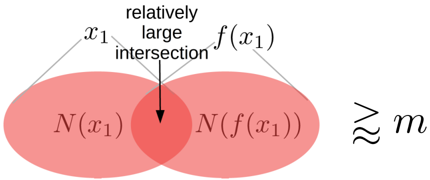
We assume (for simplicity) x_1 has only one x_2 with a large intersection.
We call it $f(x_1)$.

Proof Sketch – Large Intersection



$Q_f(x)$ true
iff $|N(x) \cup N(f(x))| \geq m$

Proof Sketch – Large Intersection



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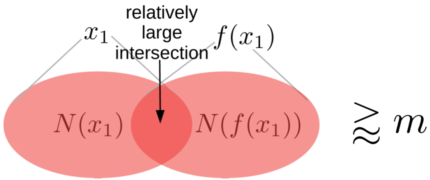
Final Formula:

$$\Big(x_2 = f(x_1) \wedge Q_f(x_1)\Big)$$

Proof Sketch – Large Intersection



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Final Formula:

$$\Big(x_2 = f(x_1) \wedge Q_f(x_1)\Big) \vee$$
$$\Big(x_2 \neq f(x_1) \wedge \varphi_{\text{small}}(x_1, x_2)\Big)$$

Proof Sketch – Quantifier Elimination

Gradually simplify formula.

$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \left(\underbrace{m_3 \leq \#x_3 \quad \overbrace{\varphi(x_1 x_2 x_3)}^{\text{quantifer-free FO}}}_{\text{replace with quantifier-free FO}} \right) \right)$$

Proof Sketch – Quantifier Elimination

Gradually simplify formula.

$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \quad \overbrace{\varphi'(x_1 x_2)}^{\text{quantifier-free FO}} \quad \right)$$

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Proof Sketch – Quantifier Elimination

Gradually simplify formula.

$$\overbrace{\varphi'''}^{\text{quantifier-free FO}}$$

Proof Sketch – Quantifier Elimination

Gradually simplify formula.

Questions or Feedback?

dreier@cs.rwth-aachen.de