Concentration Bounds for Degrees of Sets of Vertices in Preferential Attachment Graphs

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Theoretical Computer Science
RWTH Aachen University, Germany

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Motivation

Algorithms for real complex networks

$\downarrow$

Algorithms for preferential attachment graphs

$\downarrow$

Structure theorems for preferential attachment graphs

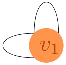
$\downarrow$

Degree bounds for preferential attachment graphs

Preferential Attachment

- G_m^n graph with vertices $v_1, \ldots, v_n$ and m edges per vertex

- G_2^1:



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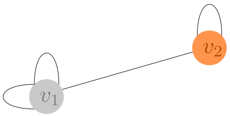

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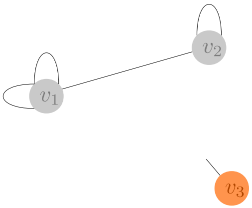

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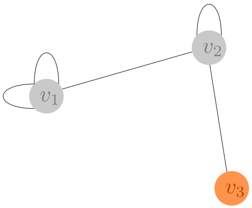

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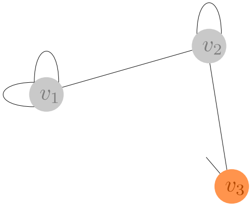

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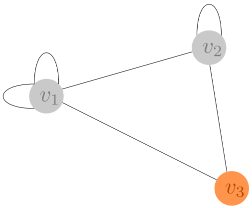

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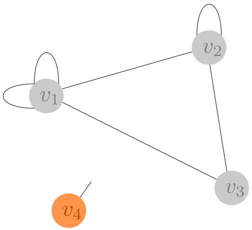

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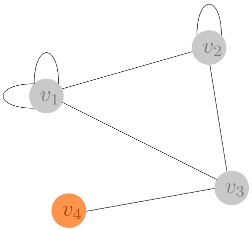

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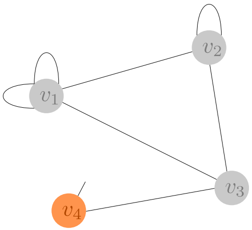

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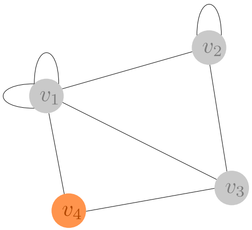
- G_m^n graph with vertices $v_1, \ldots, v_n$ and m edges per vertex

- G_2^4:



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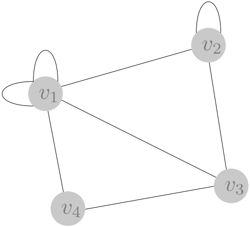
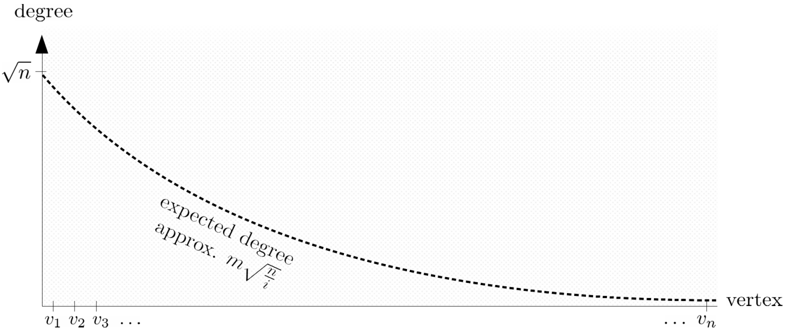


Previous Results

Distribution of $d_m^n(v_i)$ (degree of v_i in G_m^n)



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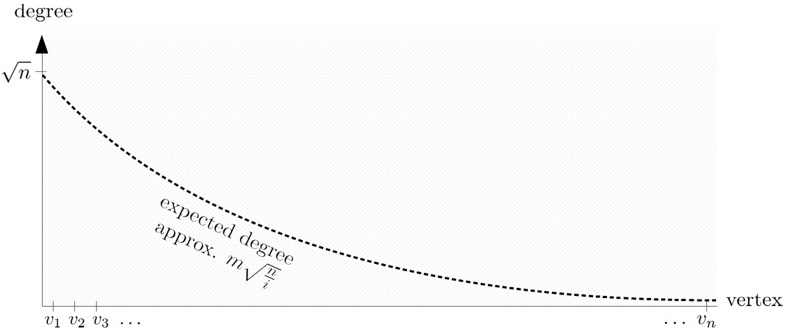

Needed: Exponentially strong concentration bounds for $d_m^n(v_i)$

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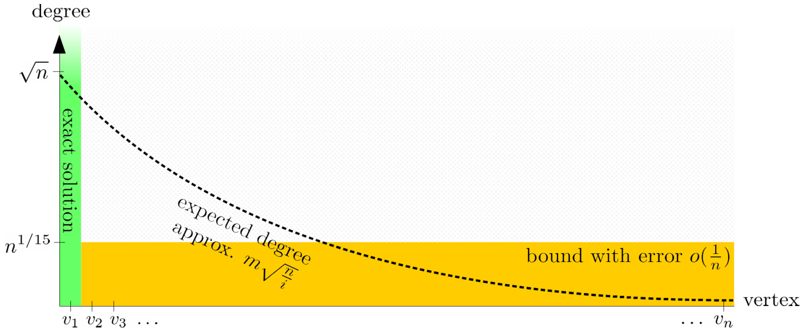

Bollobás, Riordan, Spencer, Tusnády: Exact distribution of $d_1^n(v_1)$, for $t \in \mathbf{N}$, $d \leq n^{1/15}$ approximation of $\Pr[d_m^n(v_i) = d]$ with error $o(n^{-1})$

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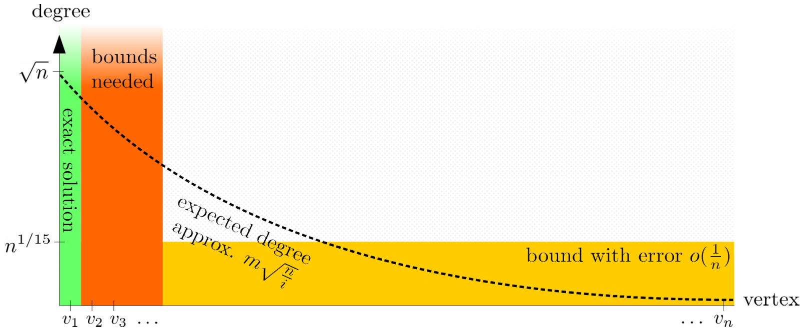

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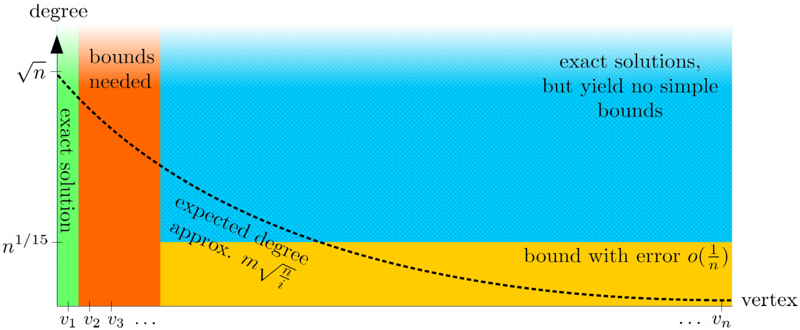

Flajolet, Dumas, Puyhaubert: Exact distribution of degrees of arbitrary vertices

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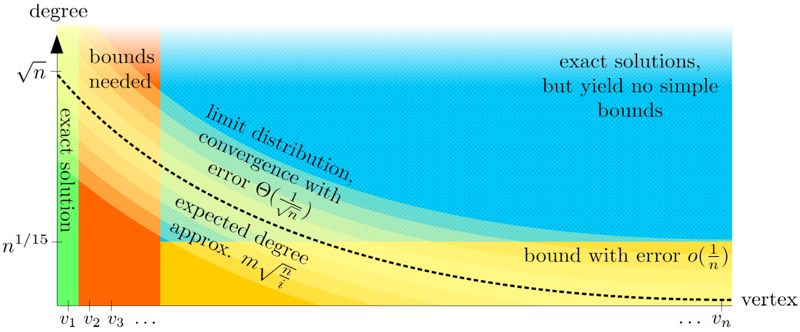
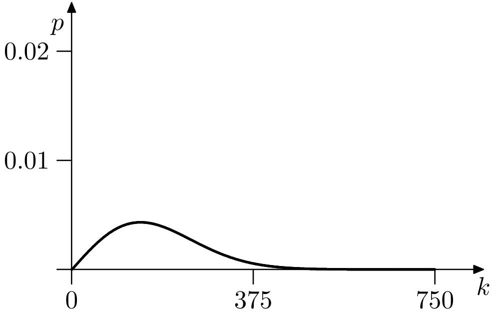


Janson: Limit theorems

Pekoz, Rollin, Ross: Rate of convergence

Concentration

$$\Pr[d_1^{10000}(v_1) = k]$$



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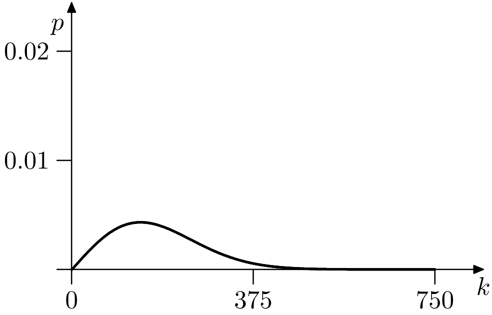
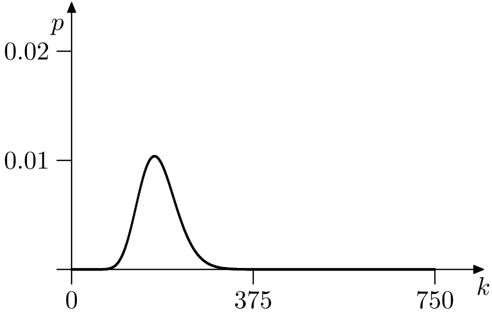


- First grows polynomially, then decreases exponentially.

Concentration

$$\Pr[d_1^{10000}(v_1) = k \mid d_1^{100}(v_1) = 18]$$



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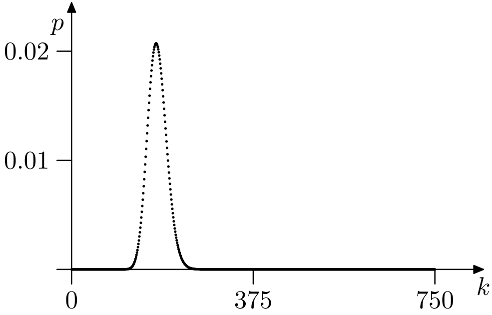
Concentration

$$\Pr[d_1^{10000}(v_1) = k \mid d_1^{1000}(v_1) = 56]$$



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Concentration

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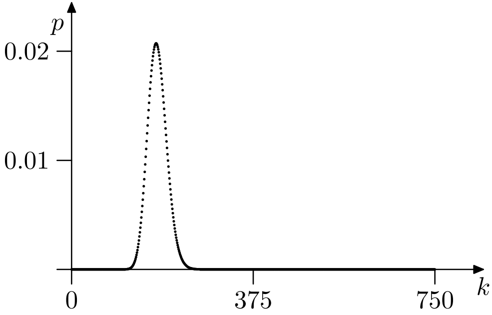


- First grows polynomially, then decreases exponentially.
- Higher initial degrees lead to stronger concentration.

Degrees of Sets

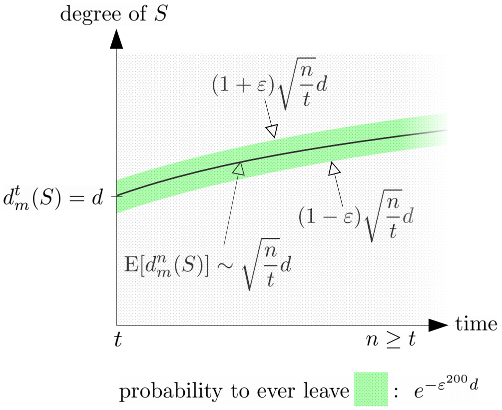
- $d_m^n(v_i)$ is degree of v_i in G_m^n
- $d_m^n(S) = \sum_{v \in S} d_m^n(v)$ is summed degree of set S in G_m^n

Goal: Exponentially strong concentration bounds for $d_m^n(S)$, assuming $d_m^t(S)$ (with $t \leq n$) is known.

Results

Let d be the degree of a set S at time t.

If d is sufficiently large, then most likely for all $n \geq t$ the degree at time n is $(1 \pm \varepsilon)\sqrt{\frac{n}{t}}d$.



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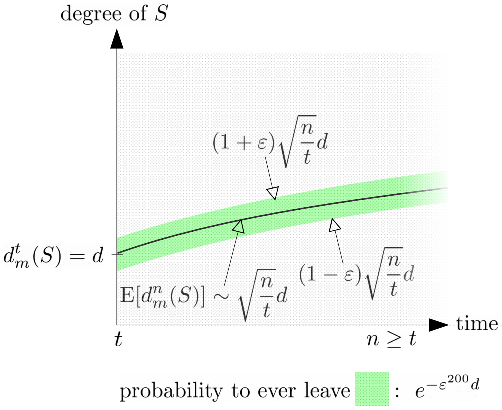
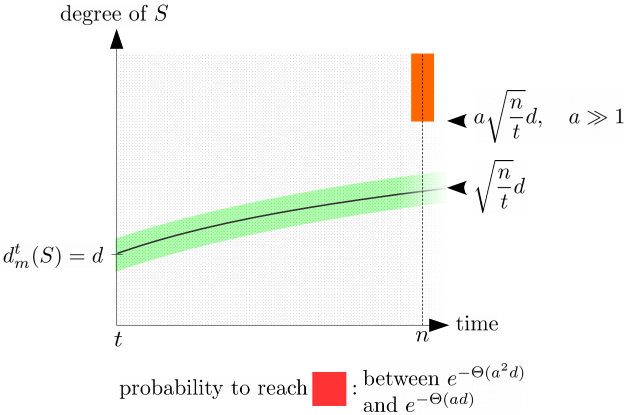


Results

Let d be the degree of a set S at time t.

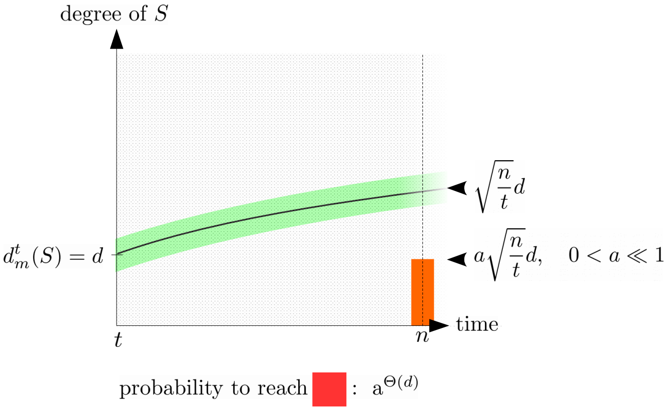
The probability that the degree is a factor $a \gg 1$ larger than expected is between $e^{-\Theta(a^2 d)}$ and $e^{-\Theta(ad)}$.



Results

Let d be the degree of a set S at time t.

The probability that the degree is a factor $0 < a \ll 1$ smaller than expected is $a^{\Theta(d)}$.



Proof Sketch

- Use Chernoff bounds to bound probability that

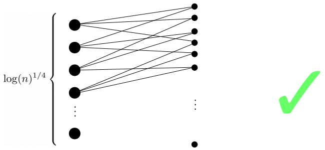
$$d_1^{(1+\delta)t}(S) = \left(1 + \delta/2 \pm 2\delta^2\right) d_1^t(S).$$

- Let $t_0 = t$ and $t_{i+1} = (1 + \delta_i) t_i$ for some sequence $\delta_1, \delta_2, \ldots$.
- Choose $\delta_1, \delta_2, \ldots$ such that short-term bounds imply long-term bounds with error $(1 \pm \varepsilon)$.
- Our choice: $\delta_i = \varepsilon / i^{2/3}$, then
  $\prod_{i=1}^{\infty}(1 + \delta_i) = \infty$,
  $\prod_{i=1}^{\infty}(1 \pm 2\delta_i^2) = (1 + O(\varepsilon))$.

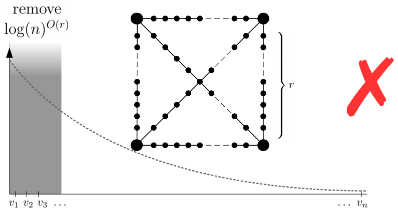
Applications

Applications: Structure

- G_m^n ($m \geq 2$) contains asymptotically almost surely a one-subdivided clique of size $\log(n)^{1/4}$.



- After removing the first $\log(n)^{O(r)}$ vertices, G_m^n contains asymptotically almost surely no r-subdivided K_4.



Applications: FPT Subgraph Counting

How often does H with k edges occur in G of size n?

- important in network analysis (motif detection)
- cannot be solved in time $f(k)n^{o(k)}$ under #ETH
- *Curticapean, Dell, Marx:* in time $k^{O(k)}n^{0.174k+o(k)}$

Theorem

Let $\varepsilon > 0$, $m \in \mathbf{N}$ and a graph H be fixed. One can count how often H occurs as a subgraph of G^n_m in time $O(n^{1+\varepsilon})$.

Applications: FPT First-Order Model Checking

First-Order Model Checking

- $\exists$ and $\forall$ quantification over vertices, equality, adjacency
- yields powerful meta-theorems
- cannot be solved in time $f(|\varphi|)n^{o(|\varphi|)}$ under ETH
- *Grohe, Kreutzer, Siebertz:* for fixed formula decidable in time $O(n^{1+\varepsilon})$ on nowhere-dense graph classes

Theorem

Let $\varepsilon > 0$, $m \in \mathbf{N}$ and $\varphi \in \mathrm{FO}$ be fixed.
One can decide whether $G_m^n \models \varphi$ in expected time $O(n^{1+\varepsilon})$.

Thank you!