Maximum Shallow Clique Minors in Preferential Attachment Graphs have Polylogarithmic Size

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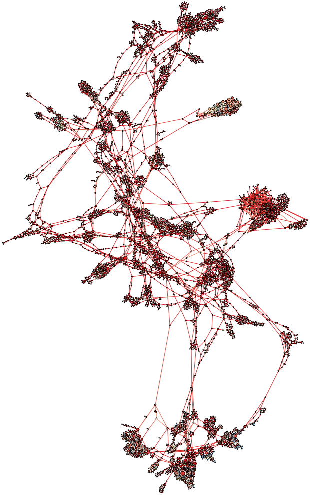
Jan Dreier, Philipp Kuinke, Peter Rossmanith

RWTH Aachen University, Germany

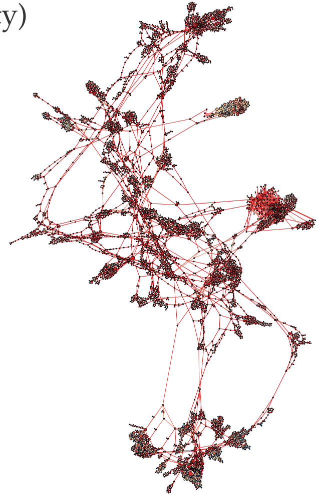
RANDOM 2020

Properties of Complex Networks



Properties of Complex Networks

- low diameter (small world property)



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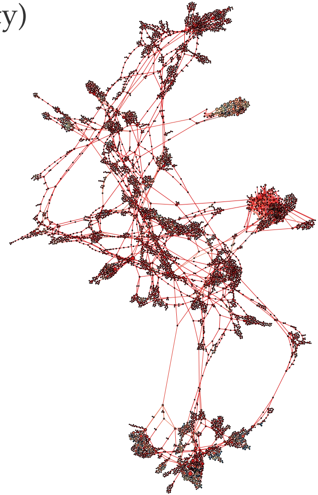
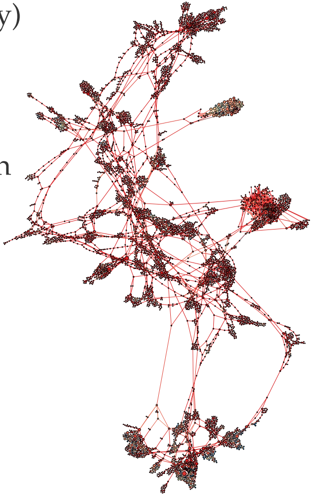


Properties of Complex Networks

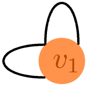
- low diameter (small world property)
- clustering
- inhomogeneous degree distribution
- . . .



Preferential Attachment

- G_m^n graph with vertices $v_1, \ldots, v_n$ and m edges per vertex

- G_2^1:



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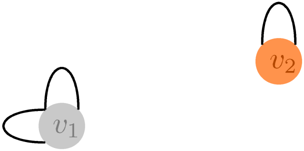

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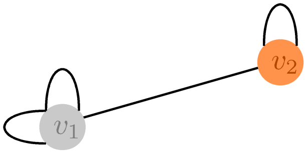

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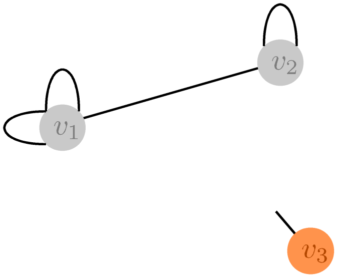

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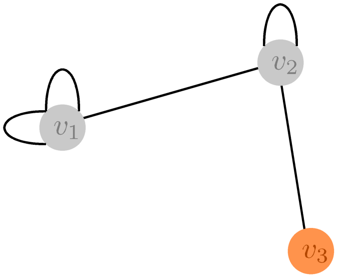

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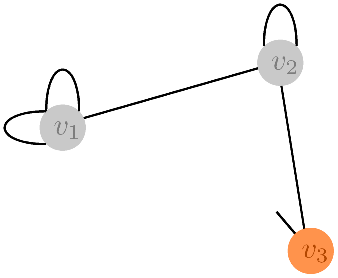

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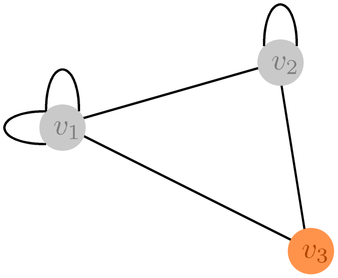

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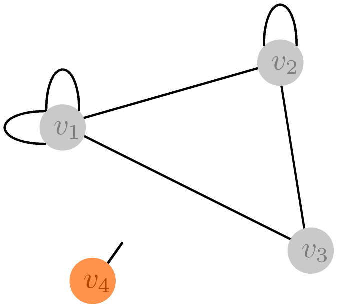

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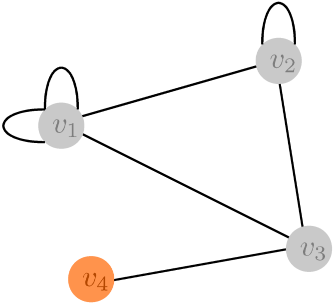

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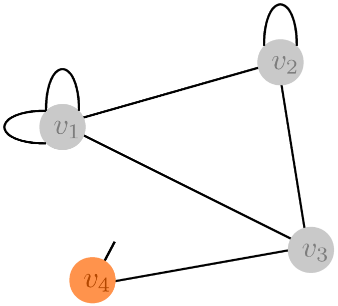
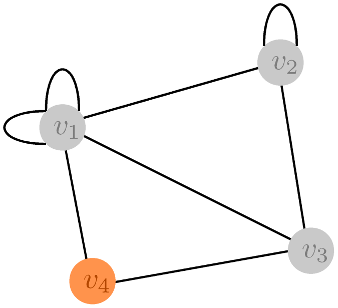


Preferential Attachment

- G_m^n graph with vertices $v_1, \ldots, v_n$ and m edges per vertex
- G_2^4:



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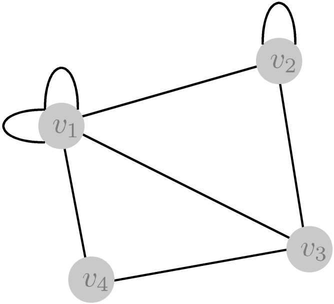

Preferential Attachment

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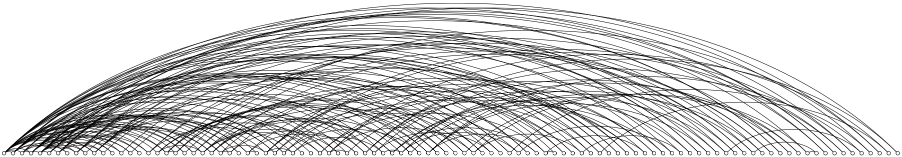
- G_2^{100}:



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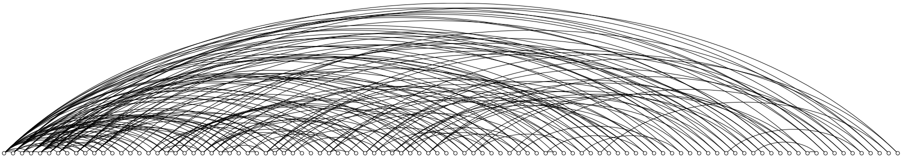
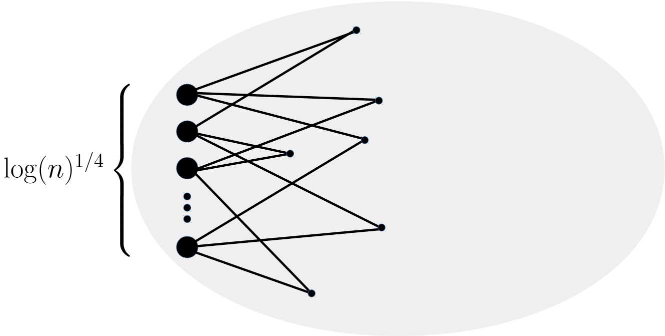


$$\text{Expected degree of } v_i \text{ in } G_m^n \approx \sqrt{\frac{n}{i}}$$

Results

1-subdivided clique



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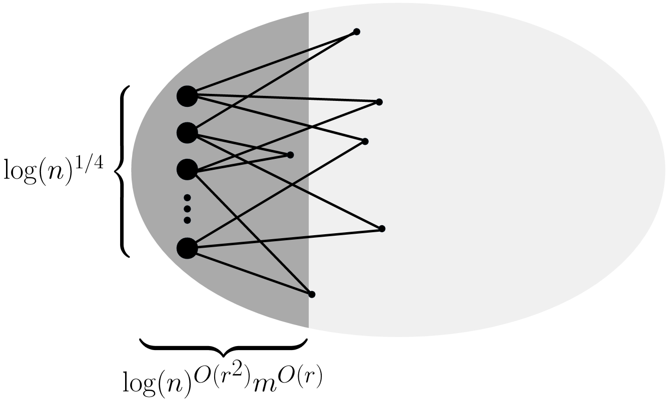
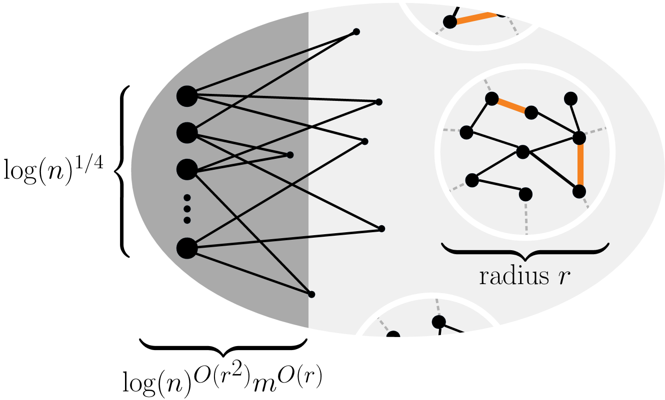


Results

1-subdivided clique

r-neighborhoods trees with at most 2 extra edges



Sparsity

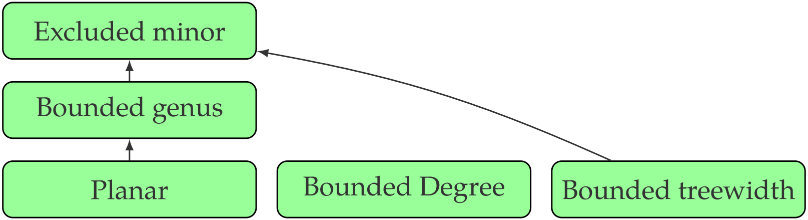
Planar

Bounded Degree

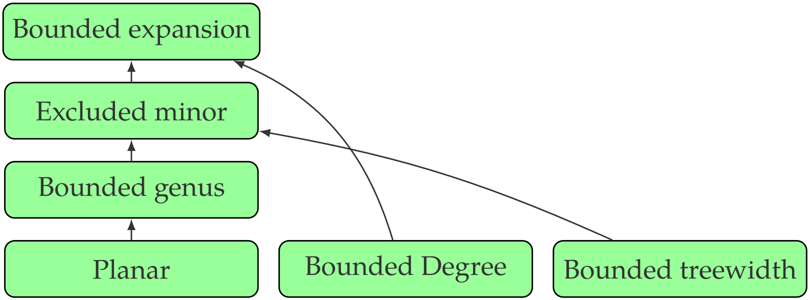
Bounded treewidth

Sparsity

Sparsity


Excluded minor
Bounded genus
Planar
Bounded Degree
Bounded treewidth

Sparsity


Bounded expansion
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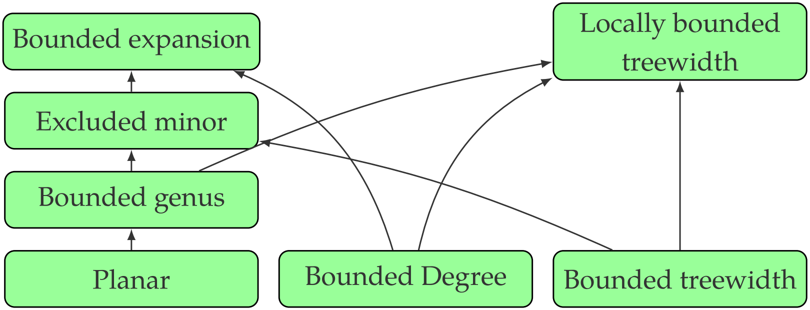
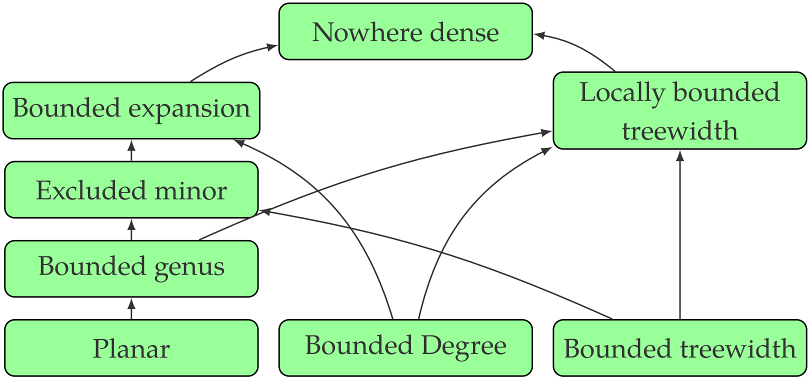
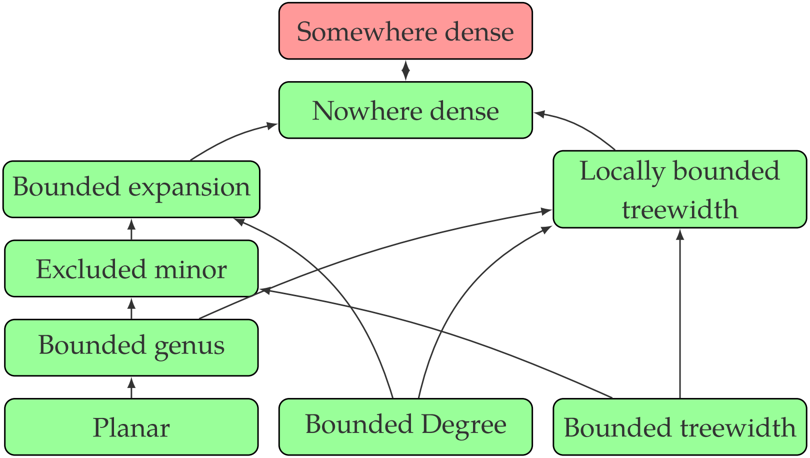

Bounded expansion
Locally bounded treewidth
Excluded minor
Bounded genus
Planar
Bounded Degree
Bounded treewidth

Sparsity



Nowhere dense
Bounded expansion
Locally bounded treewidth
Excluded minor
Bounded genus
Planar
Bounded Degree
Bounded treewidth

Sparsity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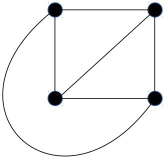
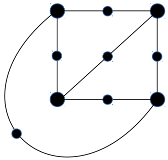
Sparsity

clique of size 4



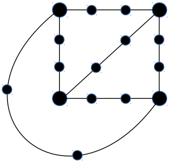
Sparsity

1-subdivided clique of size 4



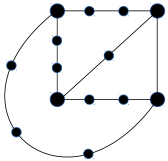
Sparsity

2-subdivided clique of size 4



Sparsity

$\leq$3-subdivided clique of size 4



Sparsity

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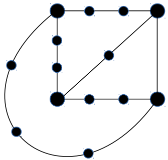

Let $\mathcal{G}$ be a graph class.
How large is the largest $\leq r$-subdivided clique contained as a subgraph of any graph in $\mathcal{G}$?

Sparsity

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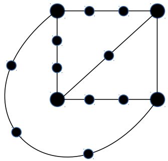

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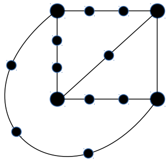

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example: planar graphs

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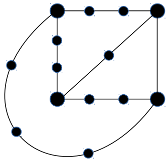

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example: planar graphs	example: complete graphs

Subdivided Cliques

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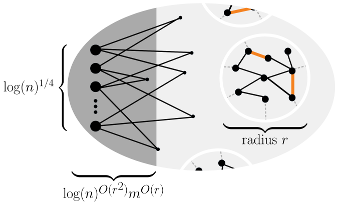

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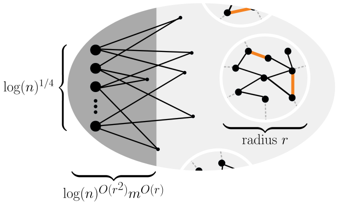

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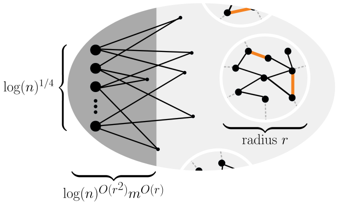

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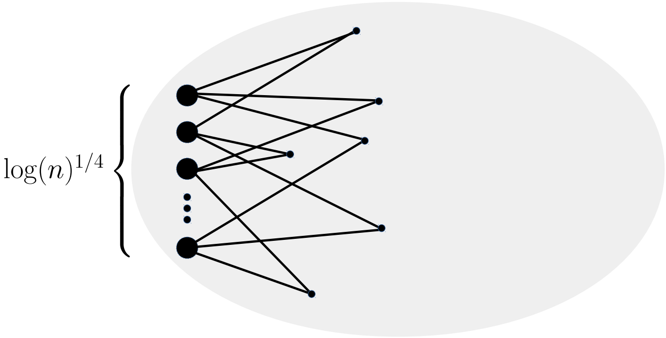
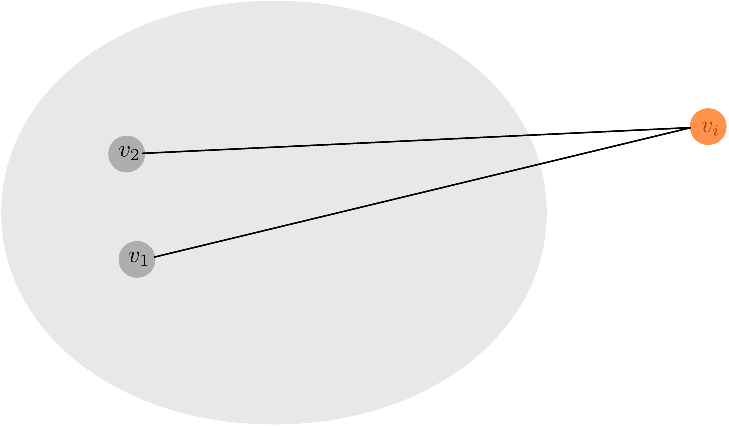
Preferential attachment graphs are asymptotically somewhere dense.

Lower Bounds

1-subdivided clique



Joining First Two Vertices



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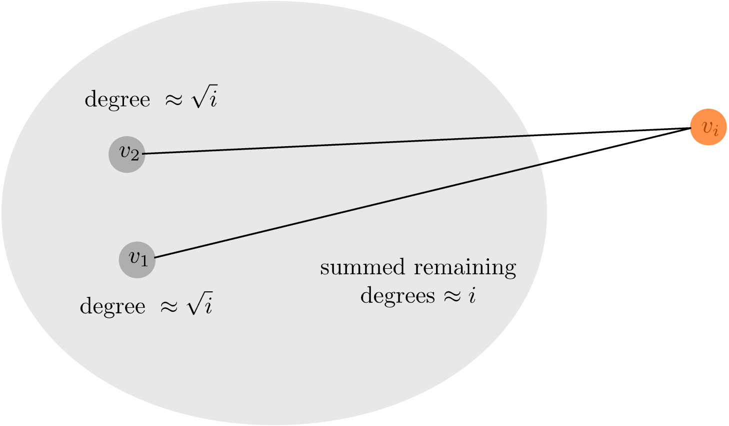

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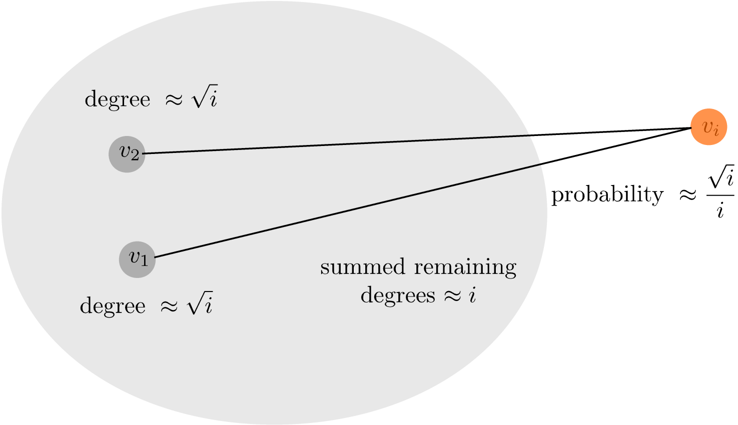

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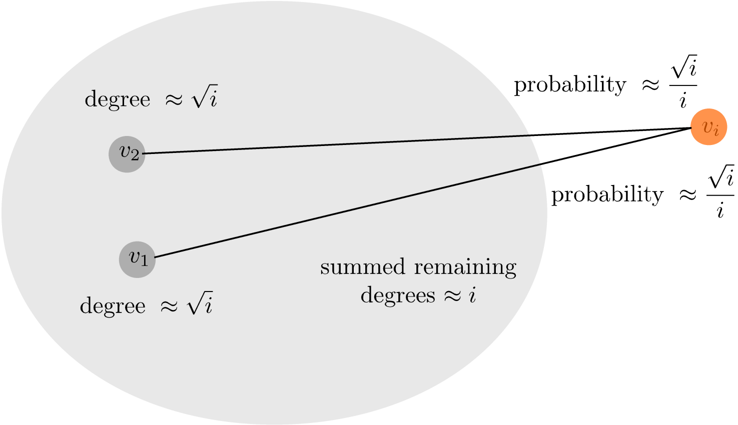

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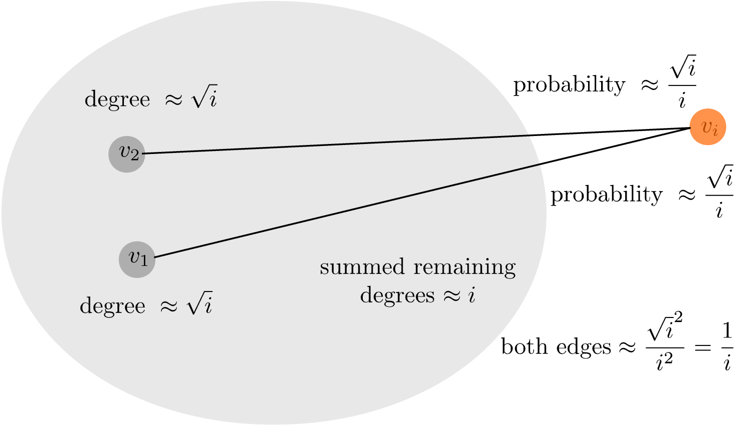

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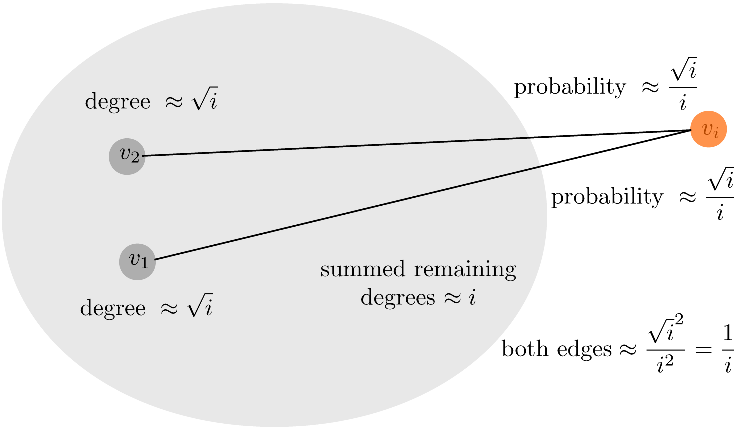

no connection after n vertices $\approx \prod_{i=O(1)}^{n} \left(1 - \frac{1}{i}\right)$

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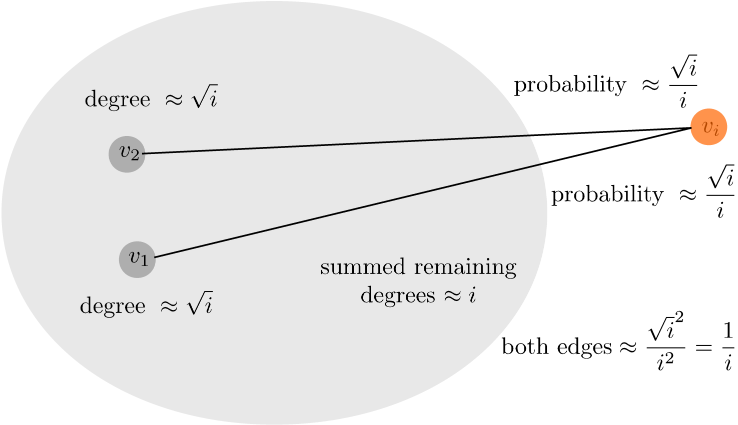

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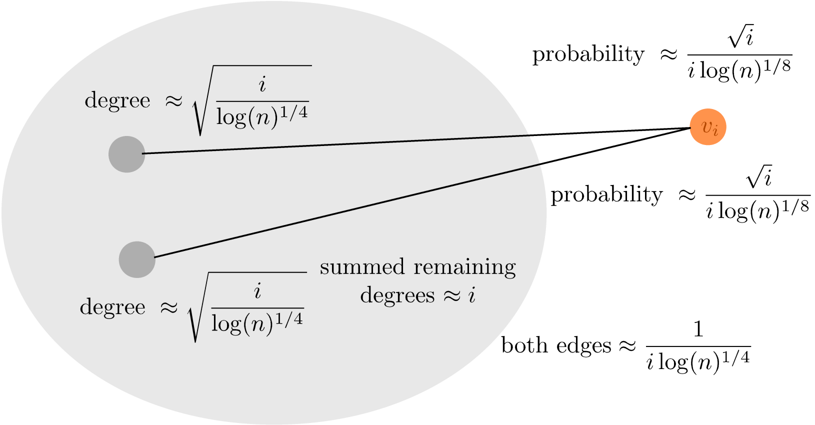
Joining First Later Vertices

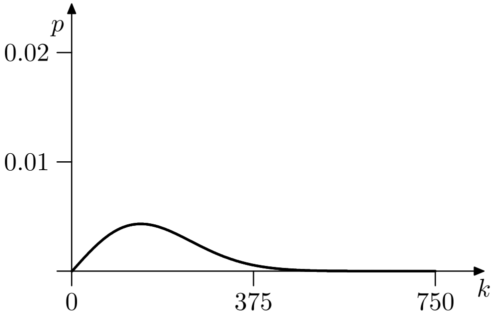


no connection after n vertices $\approx \prod_{i=O(1)}^{n} \left(1 - \frac{1}{i \log(n)^{1/4}}\right)$ converges to zero

Concentration

First vertex

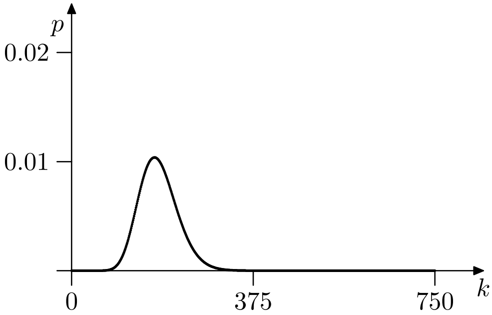
Degree after 10000 steps



Concentration

Any set of vertices with degree 18 after 100 steps

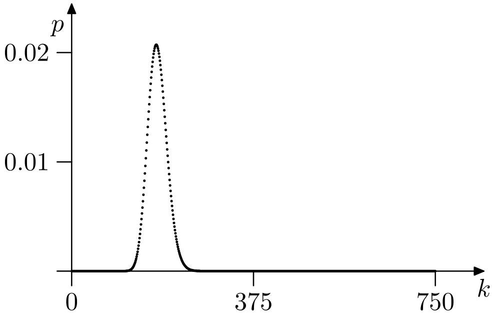
Degree after 10000 steps



Concentration

Any set of vertices with degree 56 after 1000 steps

Degree after 10000 steps



Lower Bounds

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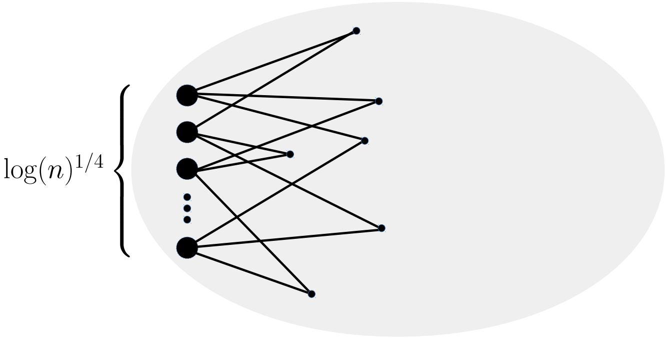

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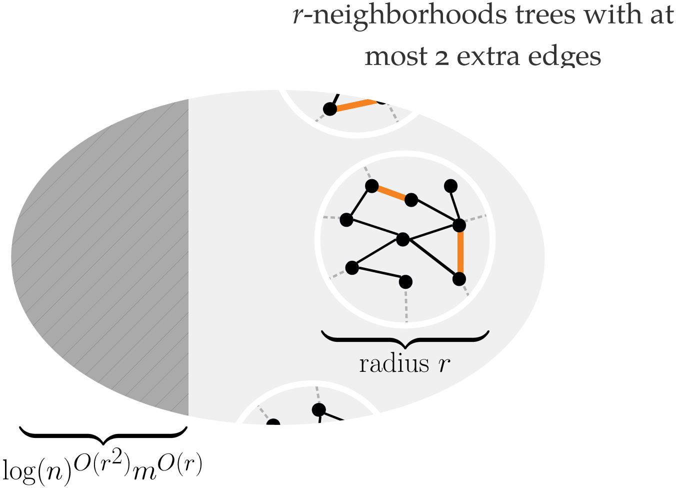
- Concentration bound $\Rightarrow$ Nails have in round i with high probability degree at least $\approx \frac{\sqrt{i}}{\log(n)^{1/4}}$

Upper Bounds



Upper Bounds

probability of 4-clique after removal of few vertices

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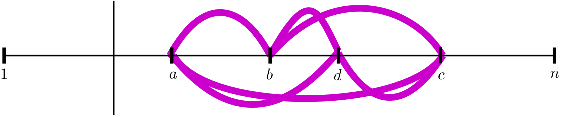
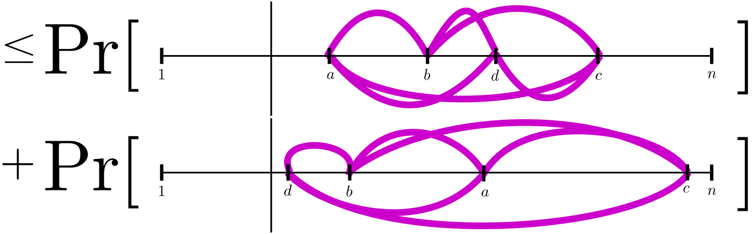
$\leq \Pr[$  $]$

Upper Bounds

probability of 4-clique after removal of few vertices



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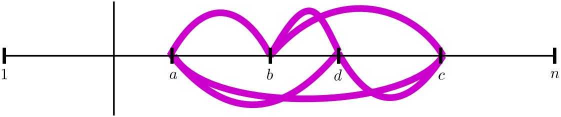
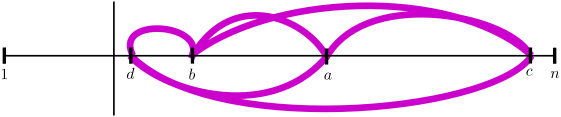
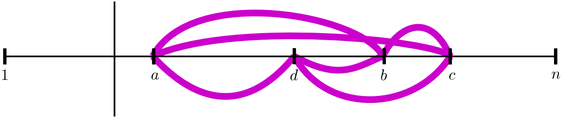
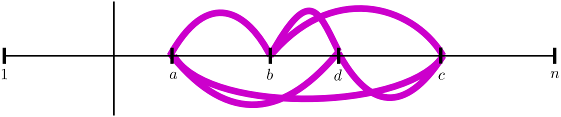
probability of 4-clique after removal of few vertices

$\leq \Pr[$  $]$

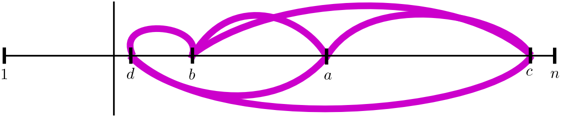
$+ \Pr[$  $]$

$+ \Pr[$  $]$

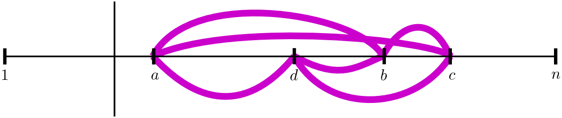
probability of 4-clique after removal of few vertices

$\leq \Pr[$ 

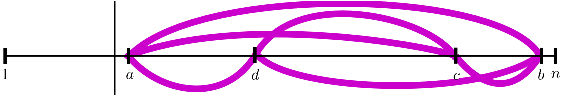
$]$

$+ \Pr[$ 

$]$

$+ \Pr[$ 

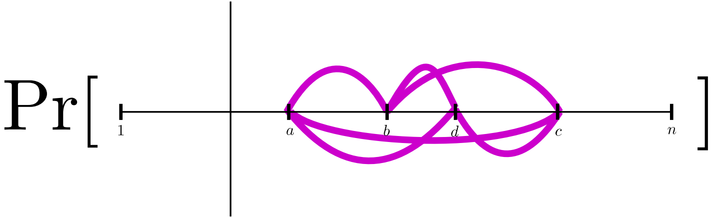
$]$

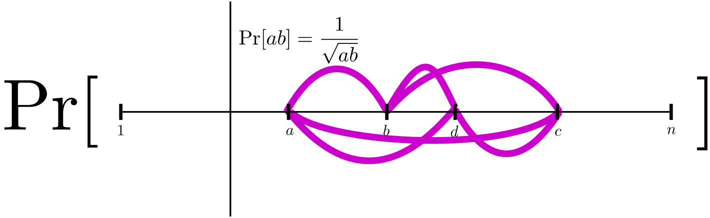
$+ \Pr[$ 

$]$

$+ \cdots$

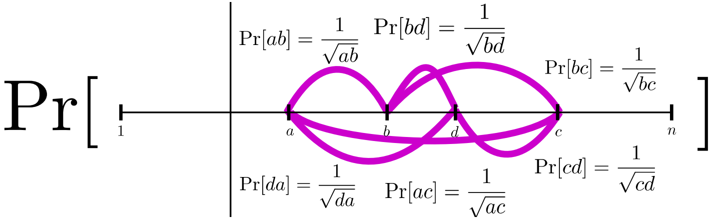
Upper Bounds



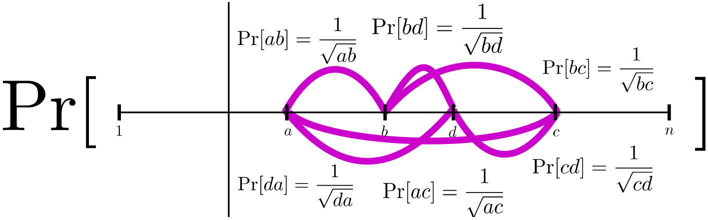


Pr[
$\Pr[ab] = \frac{1}{\sqrt{ab}}$
1
a
b
d
c
n
]

Upper Bounds



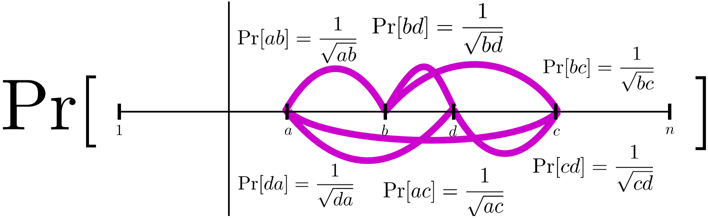
Upper Bounds



$$\leq \Pr[ab] \cdot \Pr[bc] \cdot \Pr[cd] \cdot \Pr[da] \cdot \Pr[bd] \cdot \Pr[ac]$$

$$\leq \frac{1}{(abcd)^{3/2}}$$

Upper Bounds



$$\leq \log(n)^k \cdot \Pr[ab] \cdot \Pr[bc] \cdot \Pr[cd] \cdot \Pr[da] \cdot \Pr[bd] \cdot \Pr[ac]$$

$$\leq \frac{\log(n)^k}{(abcd)^{3/2}}$$

Upper Bounds

probability of 4-clique after removing $\log(n)^k$ vertices

Upper Bounds

probability of 4-clique after removing $\log(n)^k$ vertices

$$\leq \sum_{a=\log(n)^k}^{n}$$

Upper Bounds

probability of 4-clique after removing $\log(n)^k$ vertices

$$\leq \sum_{a=\log(n)^k}^{n} \sum_{b=\log(n)^k}^{n}$$

Upper Bounds

probability of 4-clique after removing $\log(n)^k$ vertices

$$\leq \sum_{a=\log(n)^k}^{n} \sum_{b=\log(n)^k}^{n} \sum_{c=\log(n)^k}^{n}$$

Upper Bounds

probability of 4-clique after removing $\log(n)^k$ vertices

$$\leq \sum_{a=\log(n)^k}^{n} \sum_{b=\log(n)^k}^{n} \sum_{c=\log(n)^k}^{n} \sum_{d=\log(n)^k}^{n}$$

Upper Bounds

probability of 4-clique after removing $\log(n)^k$ vertices

$$\leq \sum_{a=\log(n)^k}^{n} \sum_{b=\log(n)^k}^{n} \sum_{c=\log(n)^k}^{n} \sum_{d=\log(n)^k}^{n} \frac{\log(n)^k}{(abcd)^{3/2}}$$

Upper Bounds

probability of 4-clique after removing $\log(n)^k$ vertices

$$\leq \sum_{a=\log(n)^k}^{n} \sum_{b=\log(n)^k}^{n} \sum_{c=\log(n)^k}^{n} \sum_{d=\log(n)^k}^{n} \frac{\log(n)^k}{(abcd)^{3/2}}$$

$$= \log(n)^k \sum_{a=\log(n)^k}^{n} \frac{1}{a^{3/2}} \sum_{b=\log(n)^k}^{n} \frac{1}{b^{3/2}} \sum_{c=\log(n)^k}^{n} \frac{1}{c^{3/2}} \sum_{d=\log(n)^k}^{n} \frac{1}{d^{3/2}}$$

Upper Bounds

probability of 4-clique after removing $\log(n)^k$ vertices

$$\leq \sum_{a=\log(n)^k}^{n} \sum_{b=\log(n)^k}^{n} \sum_{c=\log(n)^k}^{n} \sum_{d=\log(n)^k}^{n} \frac{\log(n)^k}{(abcd)^{3/2}}$$

$$= \log(n)^k \sum_{a=\log(n)^k}^{n} \frac{1}{a^{3/2}} \sum_{b=\log(n)^k}^{n} \frac{1}{b^{3/2}} \sum_{c=\log(n)^k}^{n} \frac{1}{c^{3/2}} \sum_{d=\log(n)^k}^{n} \frac{1}{d^{3/2}}$$

converges to zero

Summary

1-subdivided clique

r-neighborhoods trees with at most 2 extra edges