Approximate First-Order Counting Queries on Sparse and Dense Graphs

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Vienna University of Technology

joint work with Peter Rossmanith

March 16, 2021

Algorithmic Meta-Theorems

*"All problems expressible in a certain logic
can be solved efficiently on certain graphs"*

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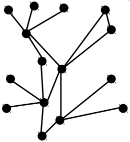
MSO on treewidth



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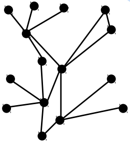
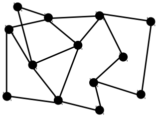
FO on sparse graphs





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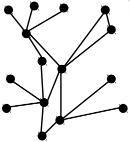

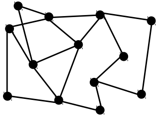

FO({>0}) for ***approximation*** on sparse graphs

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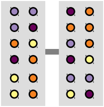

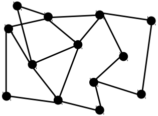

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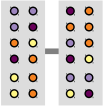

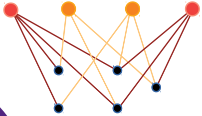

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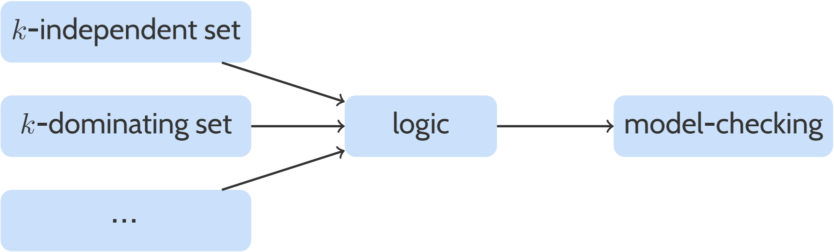
Best algorithms on general graphs: $n^{O(k)}$

Model-Checking



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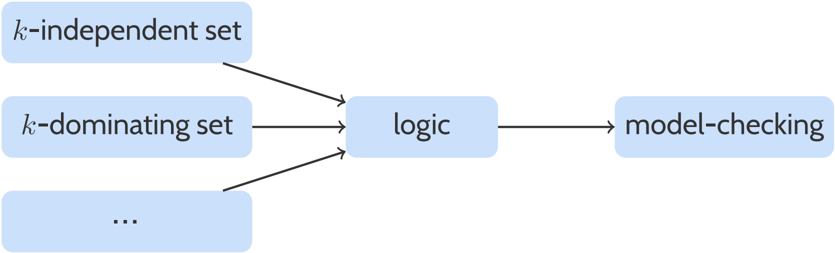

MC($\mathcal{G}$, L)

Input: A graph $G \in \mathcal{G}$ and a sentence $\varphi \in \mathrm{L}$

Parameter: $|\varphi|$

Problem: Is φ true in G?

Goal: linear FPT run time $f(|\varphi|)n$

Model-Checking

k-independent set

k-dominating set

...

$\to$ logic $\to$ model-checking

$\varphi = \exists x_1 \ldots \exists x_k [\ldots]$

length depends on k

$f(|\varphi|)n$

algorithm

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Some Sparse Graph Classes

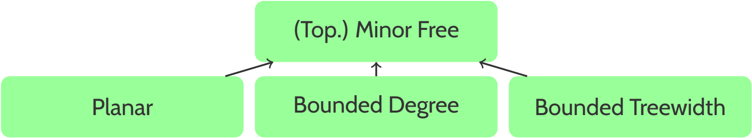
Planar

Bounded Degree

Bounded Treewidth

If $\mathcal{G}$ has bounded treewidth then MC($\mathcal{G}$, MSO) $\in$ FPT.

[Courcelle 1990]



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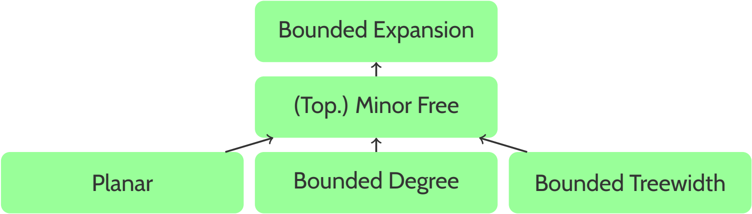

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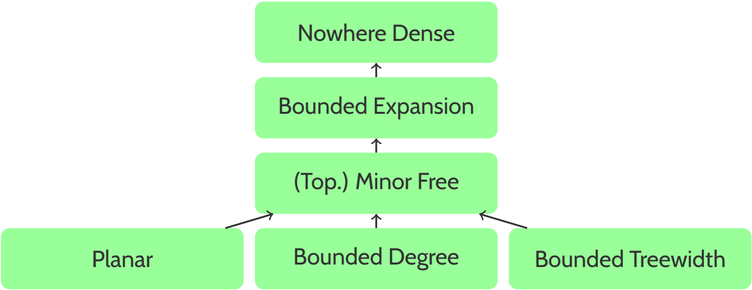

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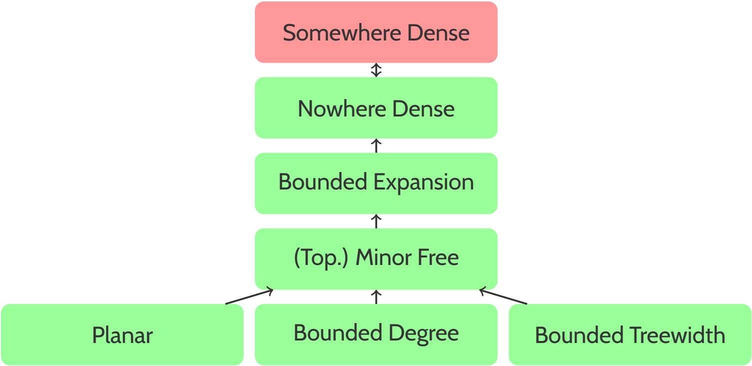

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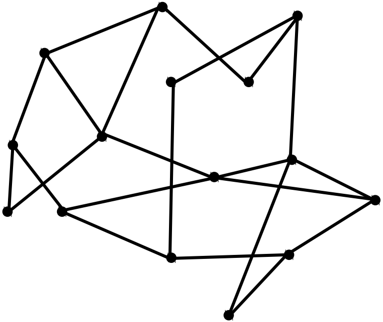
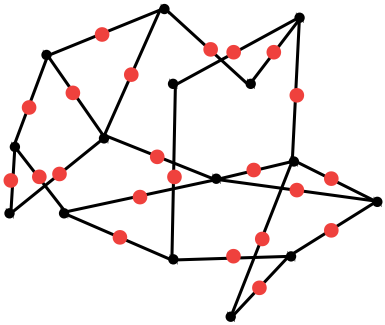
Bounded Expansion — Minors



$$\frac{|E|}{|V|} \leq c$$

for every graph
in the graph class.

Bounded Expansion — Minors



$$\frac{|E|}{|V|} \leq 2$$

for every graph
in the graph class.

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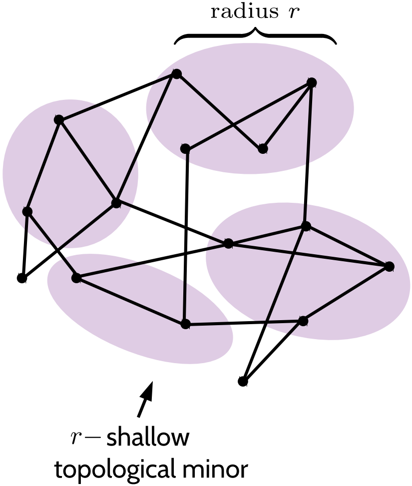

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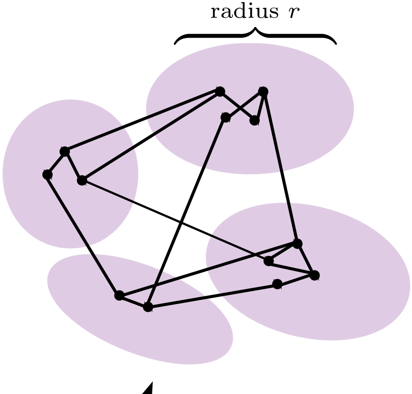

$r-$shallow
topological minor

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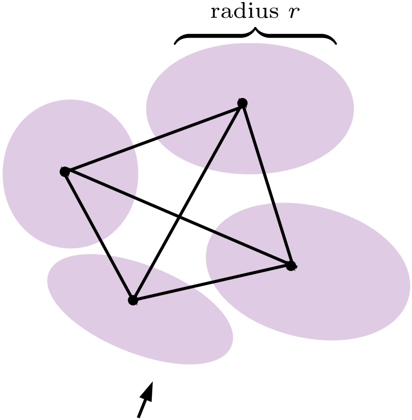

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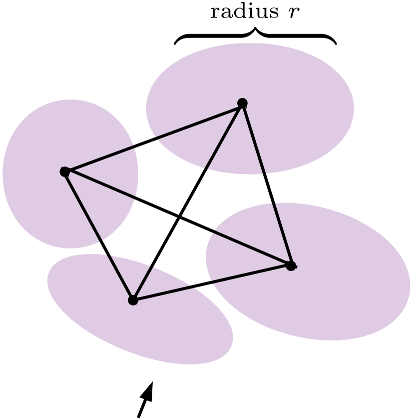


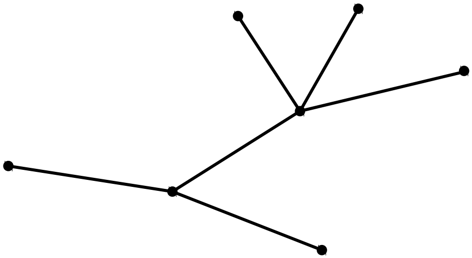
$r-$ shallow
topological minor

$$\frac{|E|}{|V|} \leq f(r)$$

for every r-shallow minor of every graph in the graph class.

Bounded Expansion — Augmentations

Possible to do these augmentations such that after r augmentations, outdegree at most $f(r)$.



orient edges

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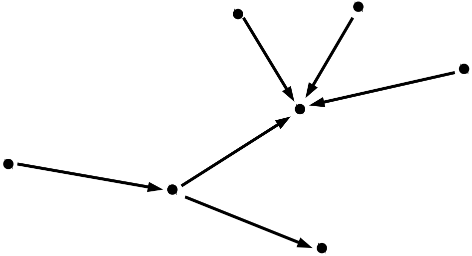
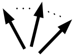
0$^{\text{th}}$ Augmentation



orient edges

max outdegree  = 2

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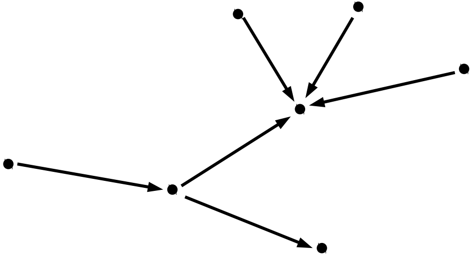
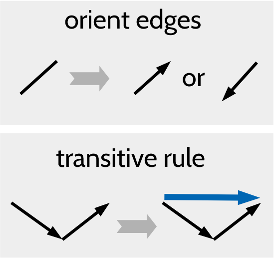
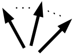
1st Augmentation





max outdegree  = 2

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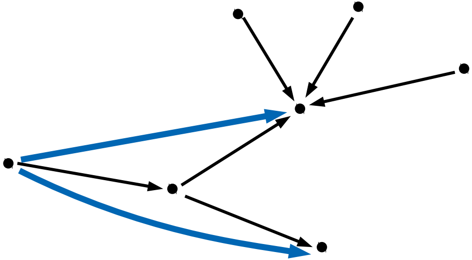
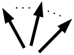


orient edges

transitive rule

max outdegree  = 3

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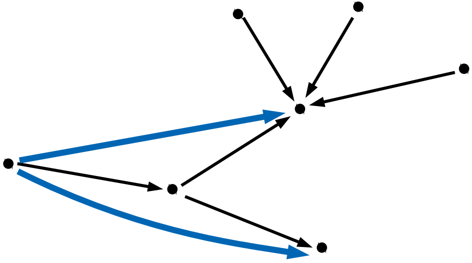
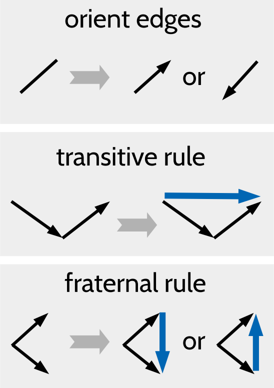
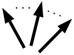




max outdegree  = 3

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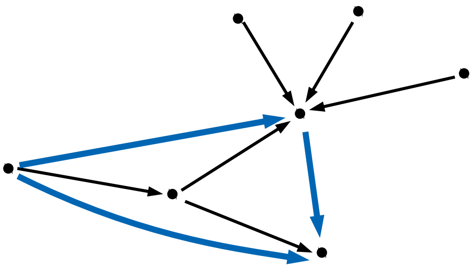
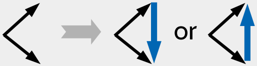
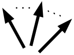


orient edges

transitive rule

fraternal rule



max outdegree  = 3

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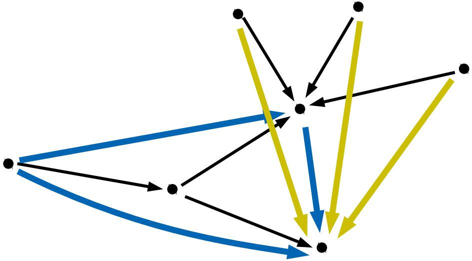
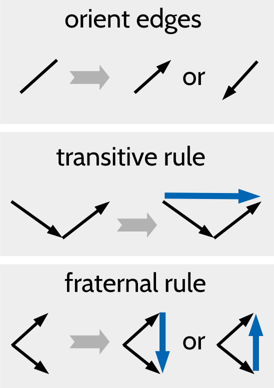
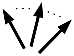




max outdegree  = 3

Approximate Counting Queries

Counting Problems

Partial Dominating Set

Input: A graph G and $k, m \in \mathbf{N}$

Parameter: k

Problem: Are there k vertices dominating m vertices?

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Can be solved on H-minor free graphs in time $(g(H)k)^k n^{O(1)}$.

[Amini, Fomin, Saurabh, 2008]

Can be solved on apex-minor-free graphs in time $2^{\sqrt{k}} n^{O(1)}$.

[Fomin, Lokshtanov, Raman, Saurabh, 2011]

Is W[1]-hard for 2-degenerate graphs.

[Golovach, Villanger 2008]

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Length of formula depends only on k (and not on m)

Counting Logic

Definition of FO$(\{>0\})$

built recursively using

- the rules of FO
- $\#y\ \varphi \geq m$ for every $m \in \mathbf{N}$ and FO$(\{>0\})$ formula φ

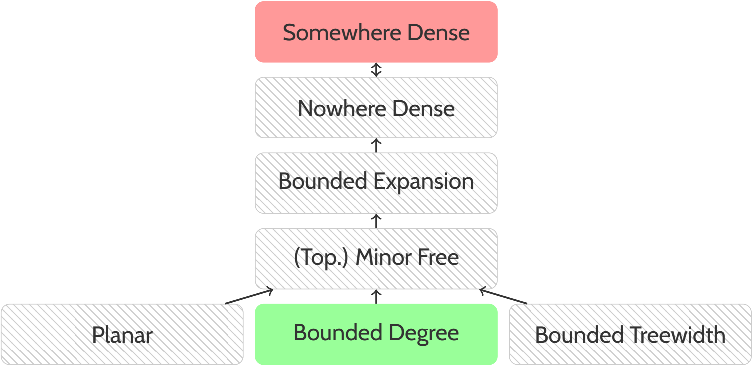
Example 1: Partial Dominating Set

$$\exists x_1 \ldots \exists x_k\, \#y\ \Big(\bigvee_i y \sim x_i \wedge y = x_i\Big) \geq m$$

Example 2: h-Index

$$\#\text{mypaper}\ \Big(\#\text{otherpaper cite}(\text{otherpaper}, \text{mypaper}) \geq h\Big) \geq h$$

Good News



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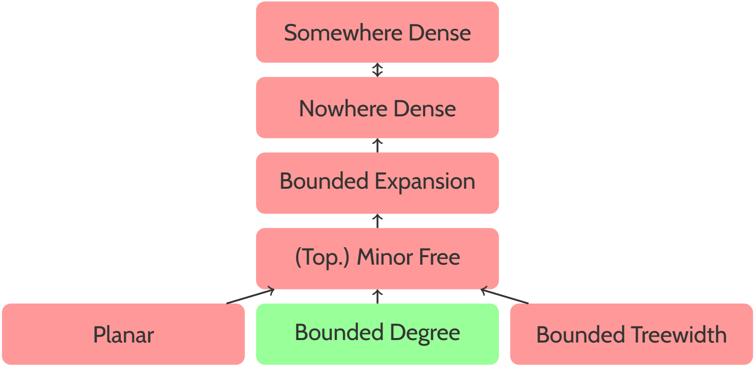
[Kuske, Schweikardt 2017]

Bad News



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MC($\mathcal{G}$, FO($\{>0\}$)) is AW[$*$]-hard on trees.

similar to [Grohe, Schweikardt 2018]

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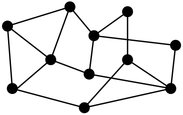
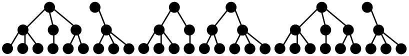


contains k-clique

$\Leftrightarrow$



satisfies FO({>0}) formula

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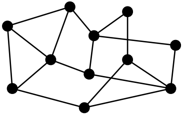

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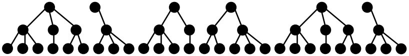

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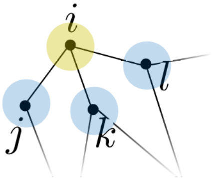

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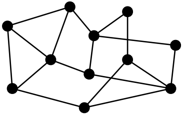

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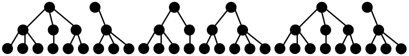

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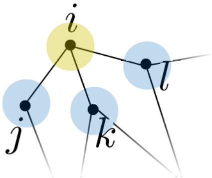

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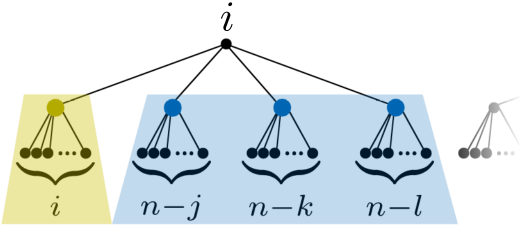

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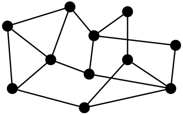
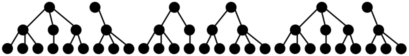
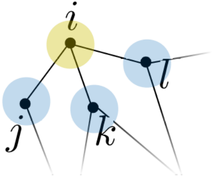


contains k-clique

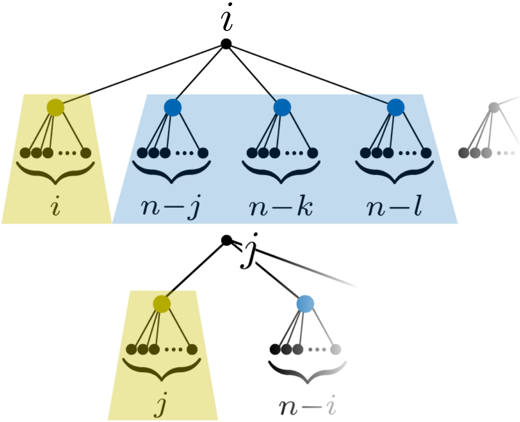
$\Leftrightarrow$



satisfies FO({>0}) formula

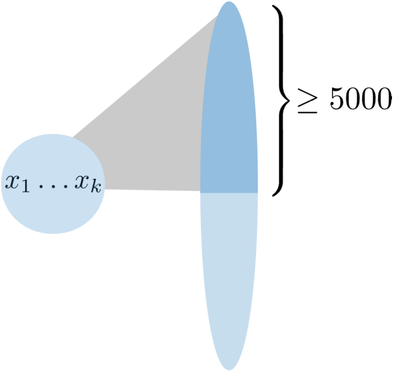


$\Leftrightarrow$



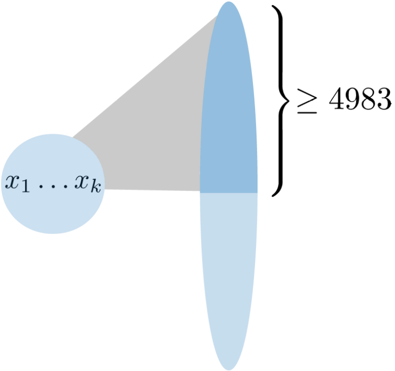
Stability

Are there k vertices dominating at least $m = 5000$ vertices?



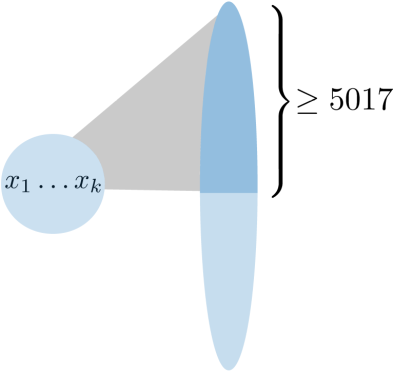
Stability

Are there k vertices dominating at least $m = 4983$ vertices?



Stability

Are there k vertices dominating at least $m = 5017$ vertices?



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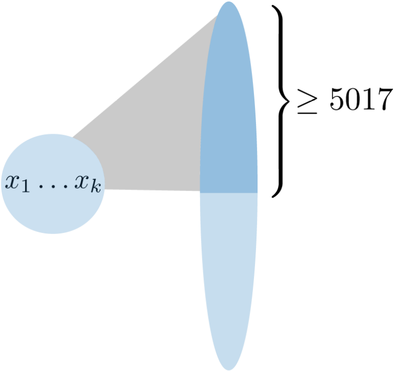

Let $\varepsilon > 0$. A formula φ is ε-*unstable* on a graph G if scaling the counting literals by $(1 \pm \varepsilon)$ changes whether φ is true in G.

Approximate Model-Checking

Theorem

Let $\mathcal{G}$ be a graph class with bounded expansion and $\varepsilon > 0$.

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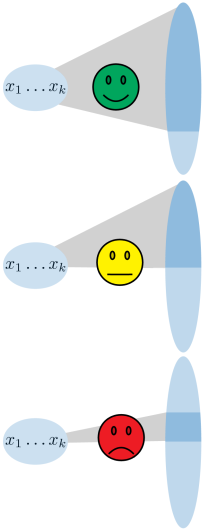
- *If*  *then φ is ε-unstable on G.*

Approximate Model-Checking

PARTIAL DOMINATING SET: $\exists x_1 \ldots \exists x_k \, \#y \left(\bigvee_i y \sim x_i \wedge y = x_i \right) \geq m$



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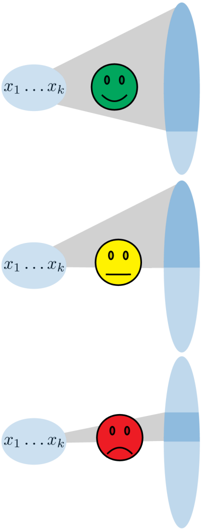

There exists a set dominating $\geq (1+\varepsilon)m$ vertices.

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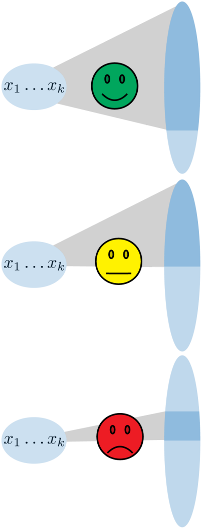

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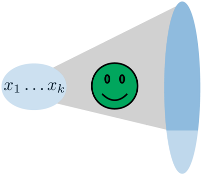

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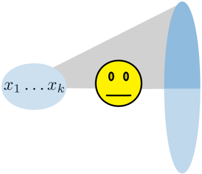
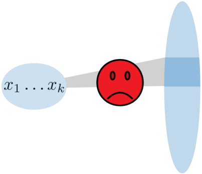


All sets dominate $< (1+\varepsilon)m$ vertices and there exists a set dominating $\geq (1-\varepsilon)m$ vertices.



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Exact Counting

Theorem

PartialDominatingSet can be solved in time $f(k)n$ on graph classes with bounded expansion.

Exact Counting

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This holds for all problems of the form

$$\exists x_1 \dots \exists x_k \# y \ \underbrace{\varphi(y x_1 \dots x_k)}_{\in \mathrm{FO}} \geq m.$$

How about extensions of FO($\{>0\}$)?

FO($\{>0\}$) allows comparing $\#y$ and $m \in \mathbf{N}$.

Theorem

Approximate model-checking becomes hard on trees if also allow one of the following:

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- multiplying of counting terms (e.g., $\#y\ \varphi \cdot \#z\ \psi > m$)

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- comparing $\#y$ and $\#z$ (e.g., $\#y\ \varphi > \#z\ \psi$)
- counting tuples $\#yz$ (e.g., $\#yz\ \varphi > m$)
- multiplying of counting terms (e.g., $\#y\ \varphi \cdot \#z\ \psi > m$)
- subtraction of counting terms (e.g., $\#y\ \varphi - \#z\ \psi > m$)

Summary

FO($\{>0\}$) is

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Slight extensions of FO($\{>0\}$) are

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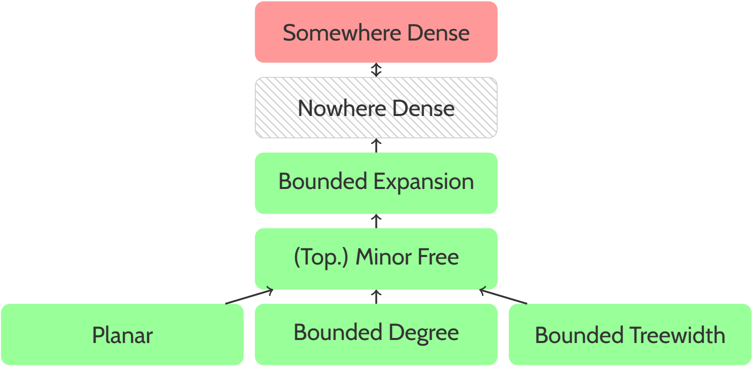
$\Rightarrow$ FO($\{>0\}$) seems like “the right logic” for approximation on sparse graphs

Big Question



Can we generalize our results to nowhere dense graph classes?

We want to gradually simplify this formula.

$$m_1 \leq \#x_1 \Big(\qquad\qquad\qquad\qquad\qquad\qquad \Big)$$

Proof — Quantifier Elimination

We want to gradually simplify this formula.

$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \left(\qquad\qquad\qquad\qquad \right)\right)$$

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$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \left(m_3 \leq \#x_3 \quad \overbrace{\varphi(x_1 x_2 x_3)}^{\text{quantifer-free FO}} \quad \right)\right)$$

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$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \quad \overbrace{\varphi'(x_1 x_2)}^{\text{quantifier-free FO}} \quad \right)$$

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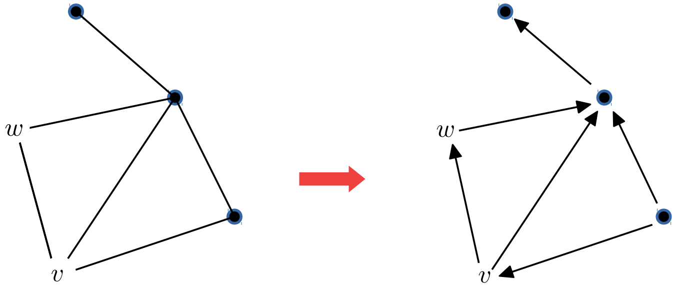
$$\overbrace{\varphi'''}^{\text{quantifier-free FO}}$$

Proof — Functional Representation



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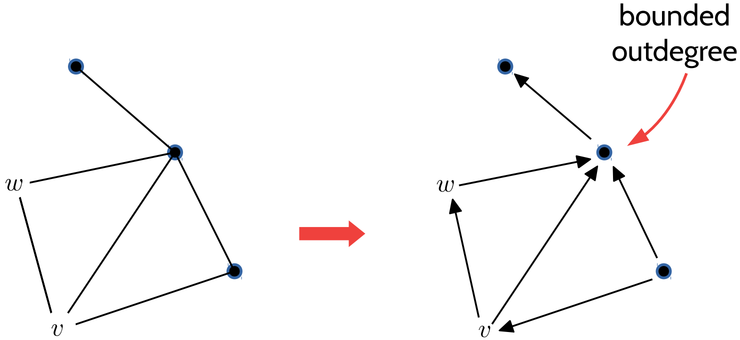

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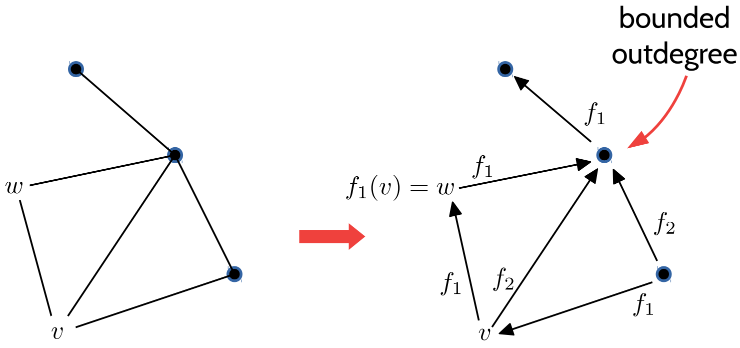

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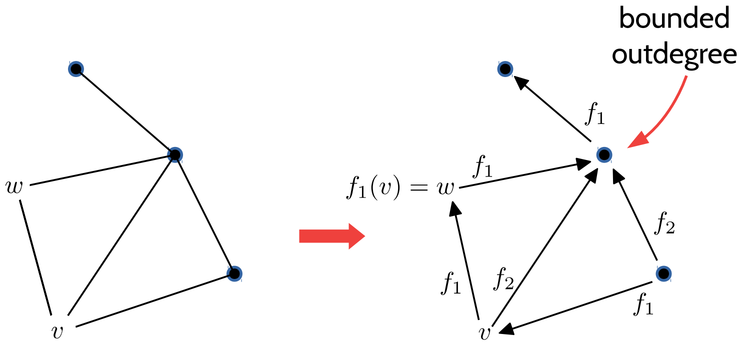
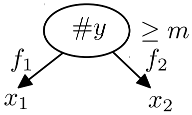


$x_1 \sim x_2$

$$\bigvee_i f_i(x_1) = x_2 \vee f_i(x_2) = x_1$$

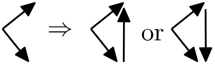
Proof — Equalities Only

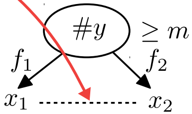
$$\#y\ \Big(\ f_1(y) = x_1 \wedge f_2(y) = x_2\ \Big) \geq m$$



Proof — Equalities Only

fraternal rule:





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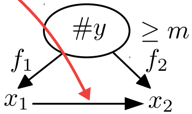

Proof — Equalities Only

$$\bigvee_h \Big(h(x_1) = x_2 \wedge \\ \#y \; \big(f_1(y) = x_1 \wedge f_2(y) = x_2 \big) \geq m \Big)$$

$$x_1 \xrightarrow[h]{} x_2 \quad \wedge \quad \begin{tikzcd} & \#y \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \geq m \\ x_1 & & x_2 \end{tikzcd}$$

Proof — Equalities Only

$$
\bigvee_h \Big(h(x_1) = x_2 \wedge \\
\quad \#y \; (\\
\qquad f_1(y) = x_1 \wedge \\
\qquad \cancel{f_2(y) = x_2} \\
\qquad f_2(y) = h(x_1) \\
\quad) \geq m \\
\Big)
$$

$$
x_1 \xrightarrow[h]{} x_2 \qquad \wedge \qquad
\begin{tikzcd}
& \#y \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \geq m \\
x_1 \arrow[rr, "h"'] & & {}
\end{tikzcd}
$$

Proof — Equalities Only

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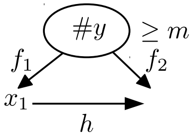

$\#y \; \varphi(yx_1) \geq m$

Proof — Equalities Only

$$\bigvee_h \Big(h(x_1) = x_2 \wedge \\ \#y \; \big(f_1(y) = x_1 \wedge \cancel{f_2(y) = x_2} \; f_2(y) = h(x_1) \big) \geq m \Big)$$

$$\begin{tikzcd} x_1 \arrow[r, "h"'] & x_2 \end{tikzcd} \quad \wedge \quad \begin{tikzcd} & \#y \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ x_1 \arrow[rr, "h"'] & & {} \end{tikzcd} \geq m$$

evaluate for
every vertex

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$$\iff \; R_h(x_1)$$

Proof — Equalities Only

$$\bigvee_h \Big(h(x_1) = x_2 \wedge R_h(x_1) \quad \cancel{\#y\ \big(f_1(y) = x_1 \wedge f_2(y) = x_2 \wedge f_2(y) = h(x_1) \big) \geq m} \quad \Big)$$

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Proof — Inequalities

$$
\begin{aligned}
\#y\ \ & f_1(y) = x_1 \wedge \\
& f_2(y) \neq x_2
\end{aligned}
$$

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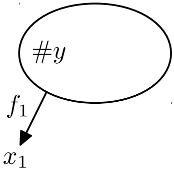

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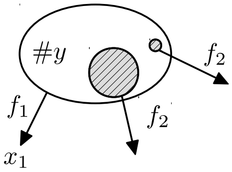

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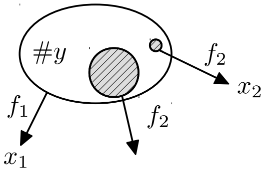

$$\#y\ f_1(y) = x_1 \wedge$$
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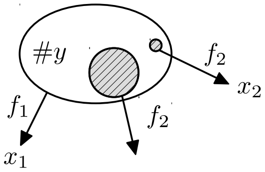

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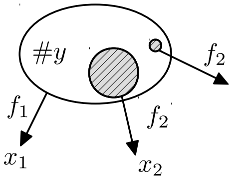

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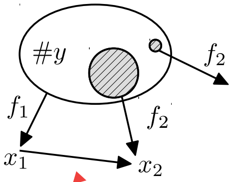

add extra edge and
proceed as before

$$\#y \; f_1(y) = x_1 \land$$
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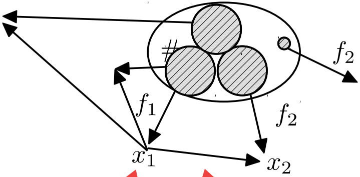

still bounded outdegree!

add extra edge and proceed as before

Proof — Quantifier Elimination

Gradually simplify formula.

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Proof — Quantifier Elimination

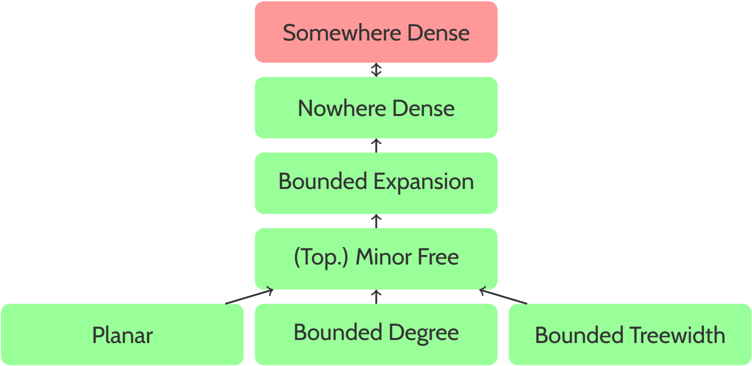
Gradually simplify formula.

$$\overbrace{\varphi'''}^{\text{quantifier-free FO}}$$

Gradually simplify formula.

Dense Graphs

Some Sparse Graph Classes

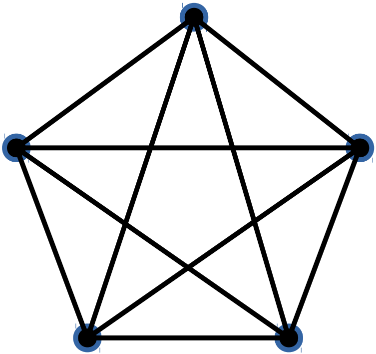


For graph classes $\mathcal{G}$ *closed under subgraphs*,
FO model-checking is tractable iff $\mathcal{G}$ is nowhere dense.

[Grohe, Kreutzer, Sieberz 2011]

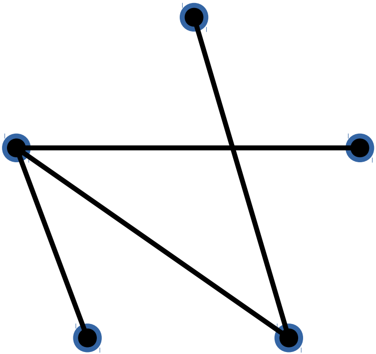
Dense Graphs

- What dense graph classes are tractable?



Dense Graphs

- What dense graph classes are tractable?
- Closure under subgraphs is not a good requirement.



Dense Graphs

- What dense graph classes are tractable?
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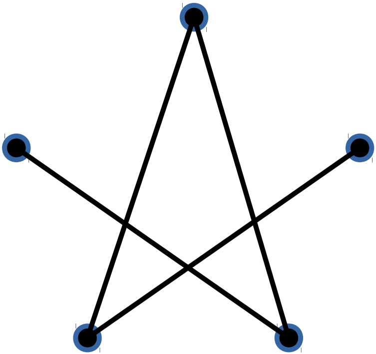

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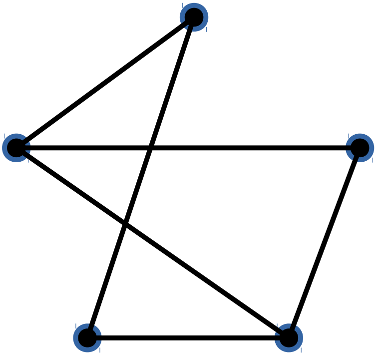

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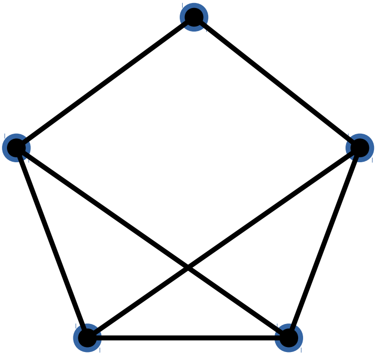

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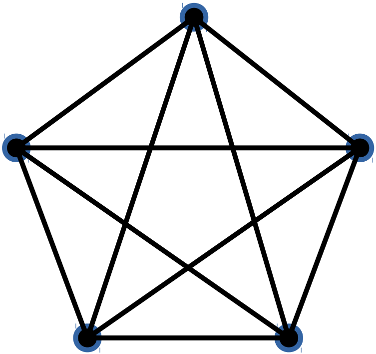

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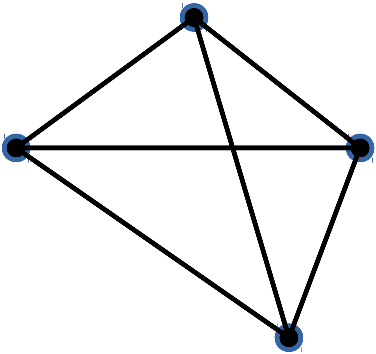

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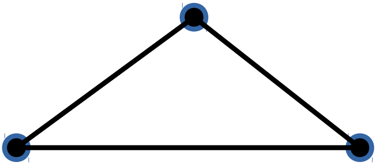

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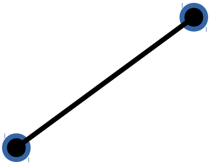

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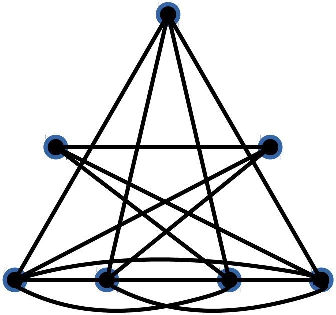
- **Goal:** For graph classes $\mathcal{G}$ *closed under induced subgraphs*, FO model-checking is tractable iff [...].

Example: Complements



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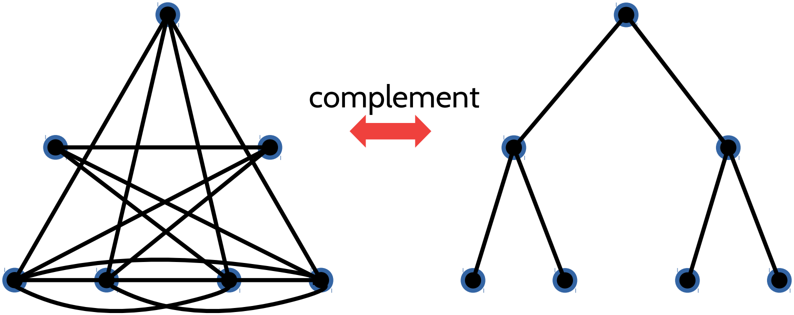
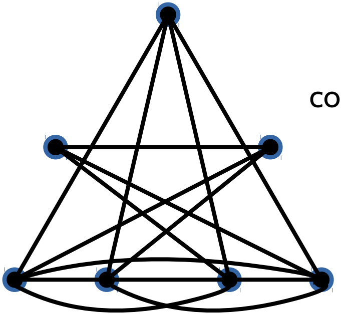
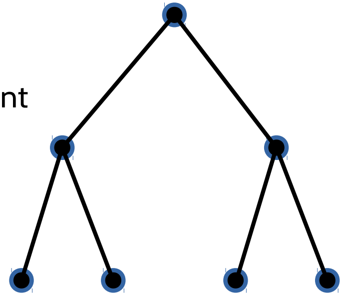


$x_1 \sim x_2$ $x_1 \not\sim x_2$

Example: Complements



complement



$$\exists x_1 \exists x_2 \exists x_3$$
$$x_1 \sim x_2 \wedge x_2 \sim x_3$$
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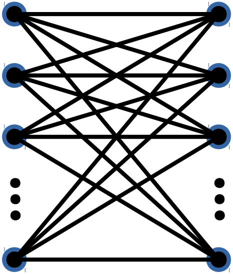
Example: Fully Bipartite



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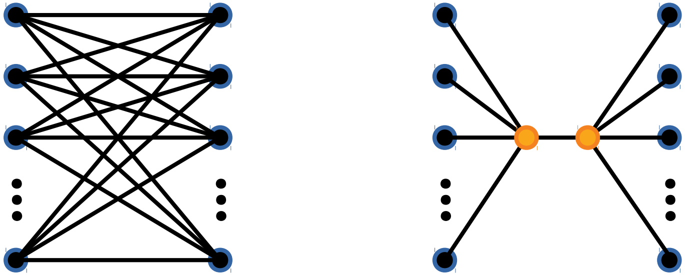

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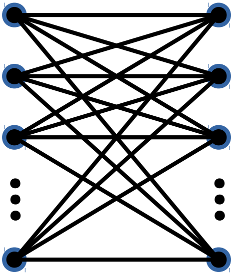
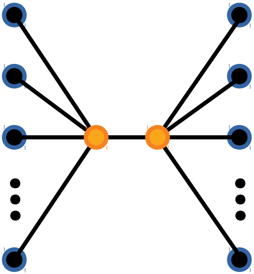


$$x \sim y$$



$$\mathrm{dist}(x, y) = 3$$

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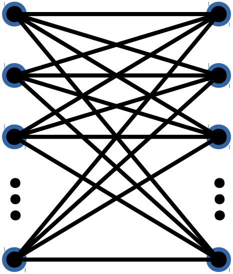
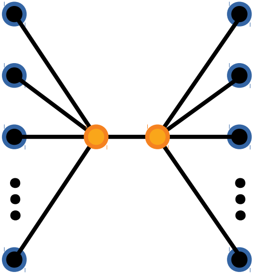


$$x \sim y$$

$$\exists x \ \varphi$$



$$\mathrm{dist}(x, y) = 3$$

$$\exists x \ \mathrm{blue}(x) \wedge \varphi$$

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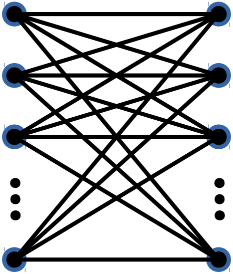

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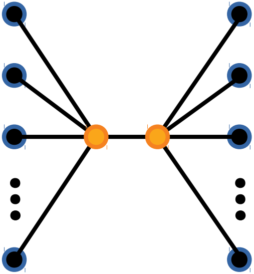

$$\exists x_1 \exists x_2 \exists x_3$$
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Interpretations

$$I = (\nu(x), \mu(x, y))$$

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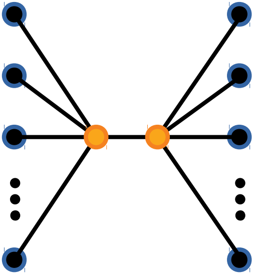


$I(G)$

vertices: $\{v \mid G \models \nu(v)\}$

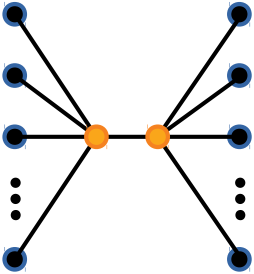
edges: $\{uv \mid G \models \mu(u, v)\}$

Interpretations

$$I = (\nu(x), \mu(x, y))$$

all blue vertices

G



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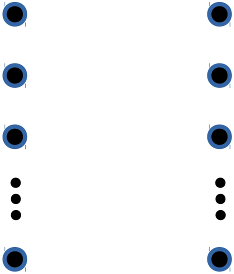

vertices: $\{v \mid G \models \nu(v)\}$

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Interpretations

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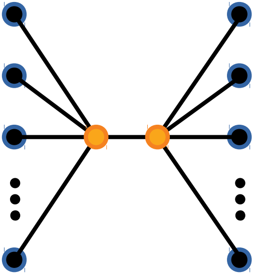
all pairs with distance 3

all blue vertices

G



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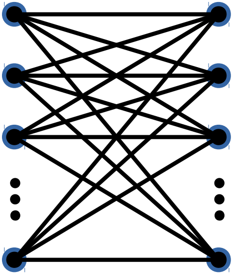

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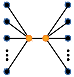
- The class of all  has treewidth 1

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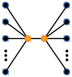
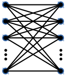
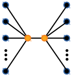
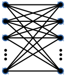
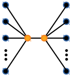
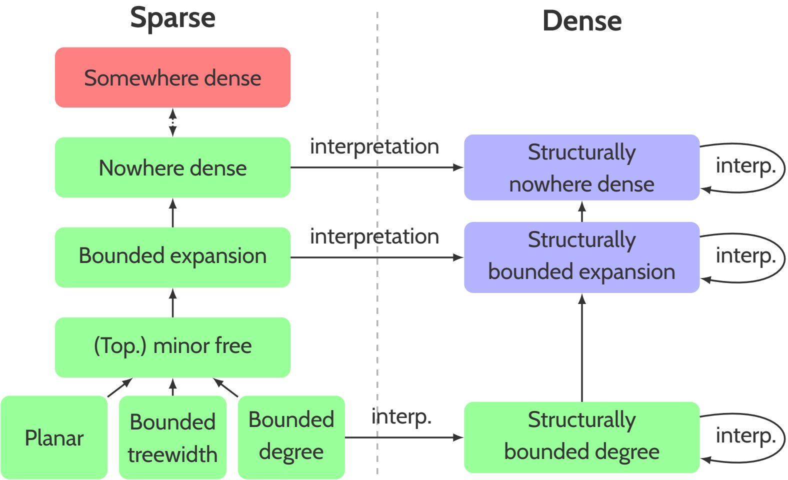
- The class of all  has treewidth 1
- For every  there is  with  $= I\Big($  $\Big)$.

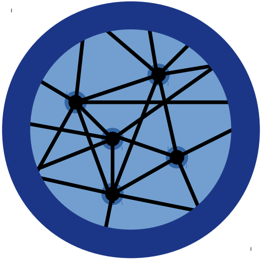
Model-Checking in Sparse and Dense Classes



Nowhere Dense: Grohe, Kreutzer, Sieberz 2011
Structurally Bounded Degree: Gajarský, Hlinenỳ, Obdržálek, Lokshtanov, Ramanujan 2016

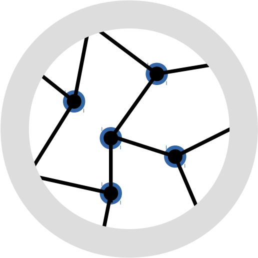
Structurally Bounded Expansion

Structurally
Bounded Expansion
$I(G)$



Structurally Bounded Expansion

Bounded Expansion
G



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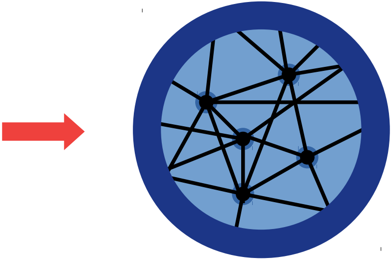

Bounded Expansion

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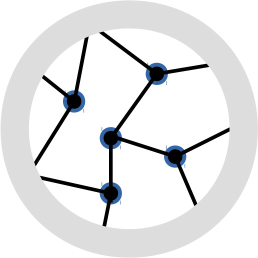

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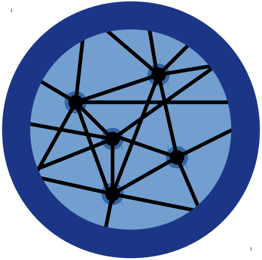

φ'

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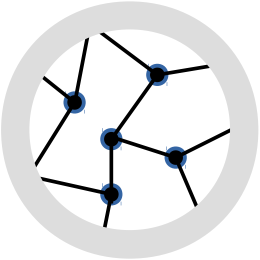
Structurally Bounded Expansion

Bounded Expansion

G



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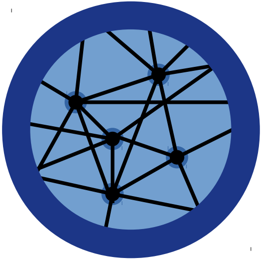

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MC-algorithm

Structurally Bounded Expansion

Bounded Expansion

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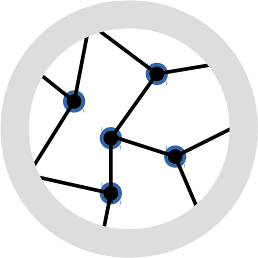

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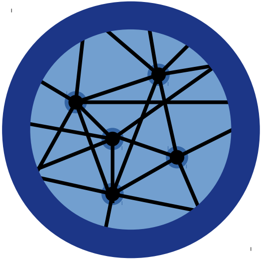

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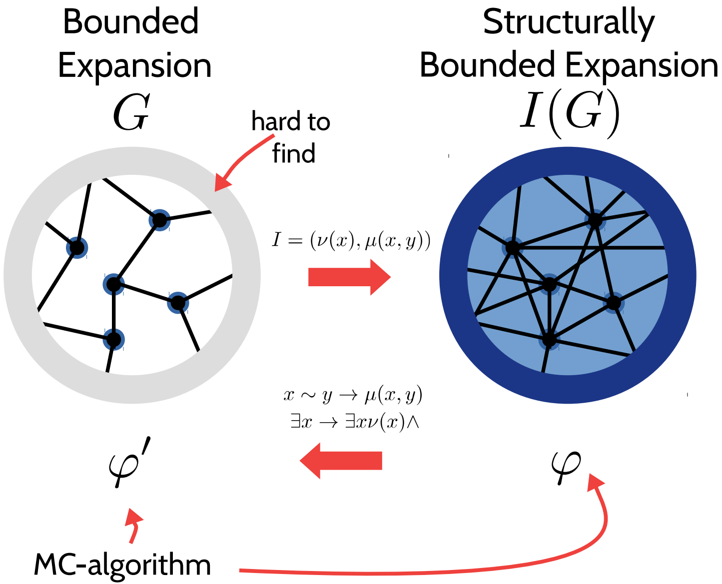
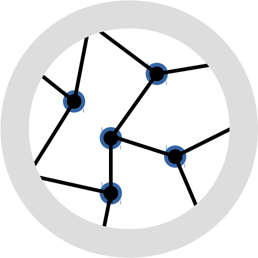


Structurally Bounded Degree

Degree 3

G



I

Structurally
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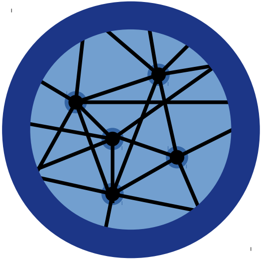
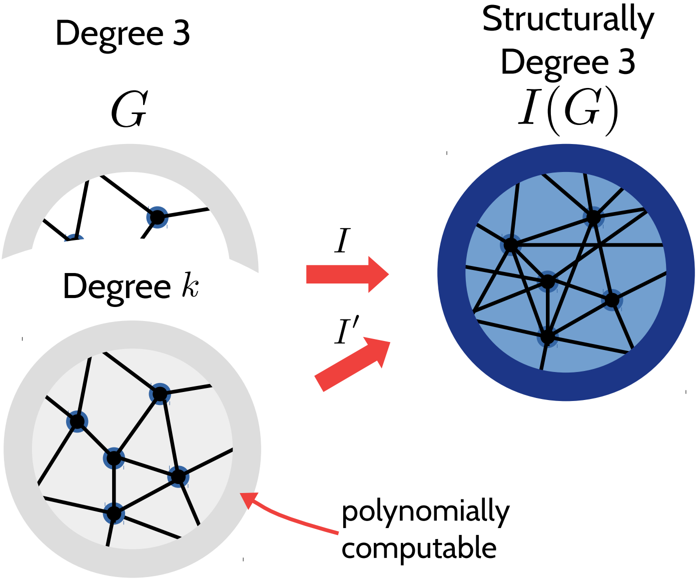
$I(G)$



NP-complete to find preimage

Structurally Bounded Degree



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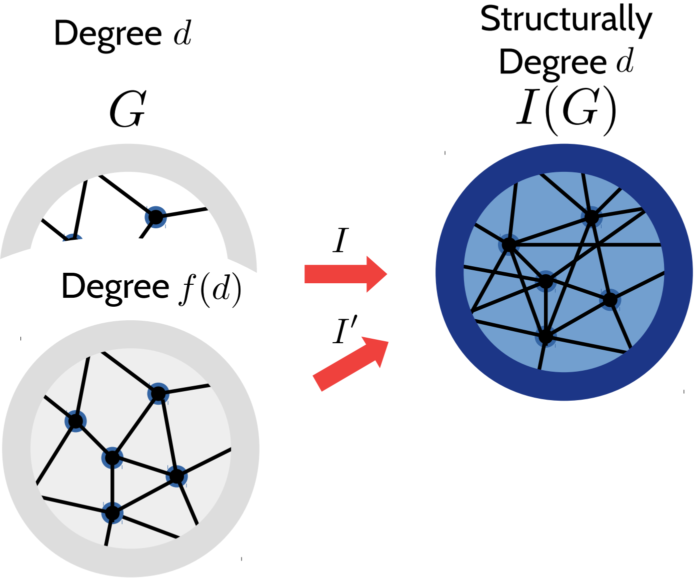

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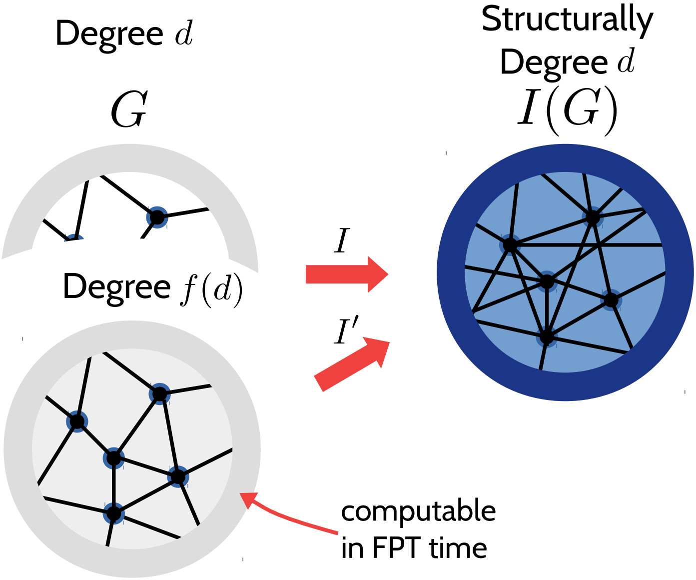

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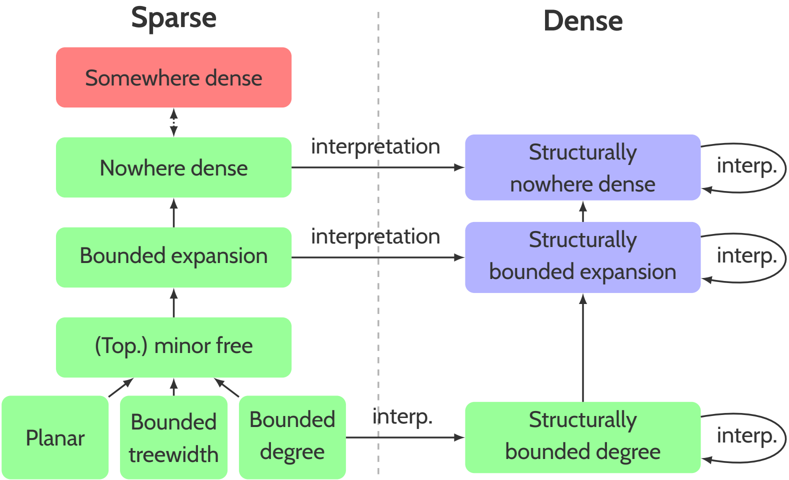

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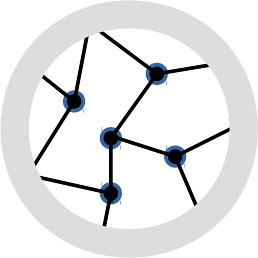
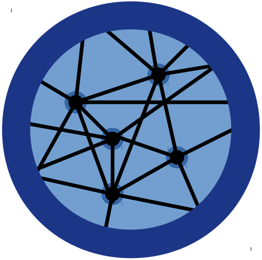
Big Question

Bounded Expansion

G



$\xrightarrow{I}$



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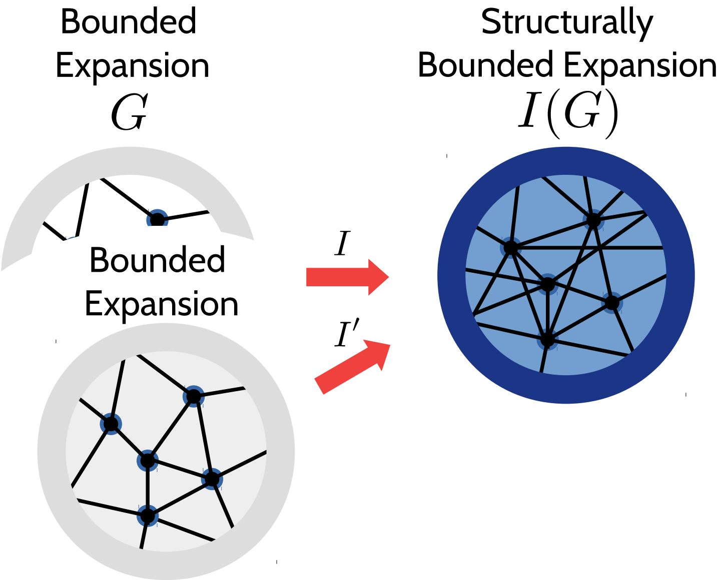

Big Question

Bounded
Expansion
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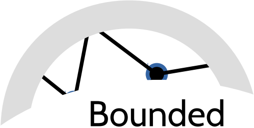
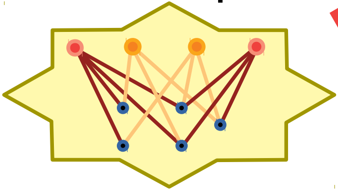
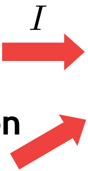
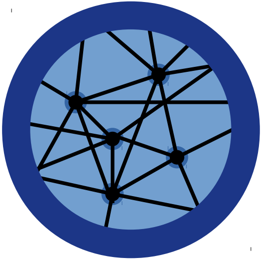


Bounded
Expansion
Lacon Decomposition





Structurally
Bounded Expansion
$I(G)$



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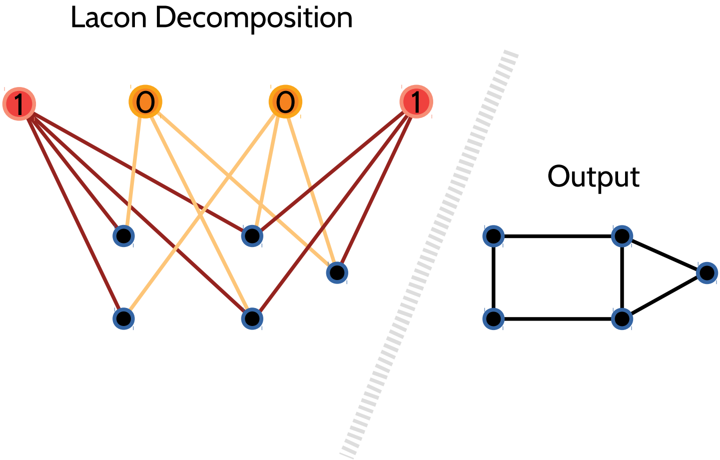

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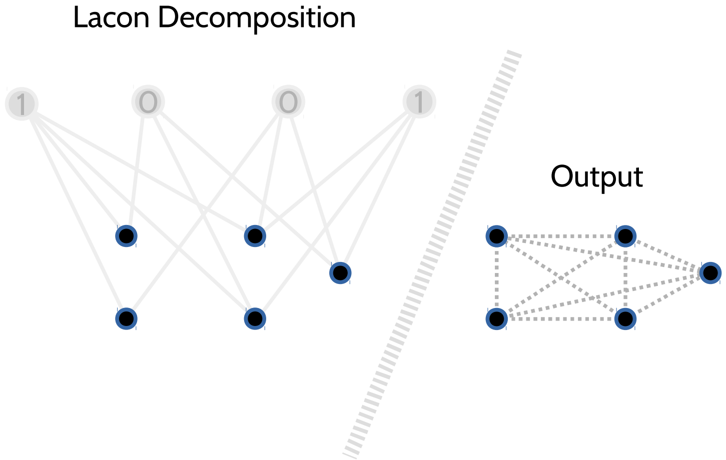
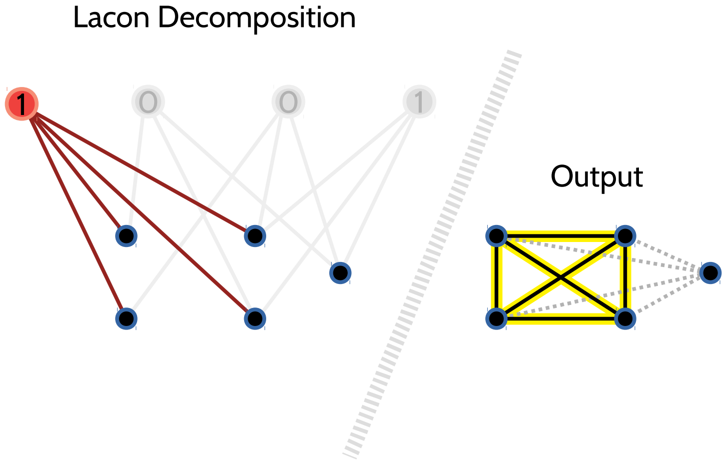
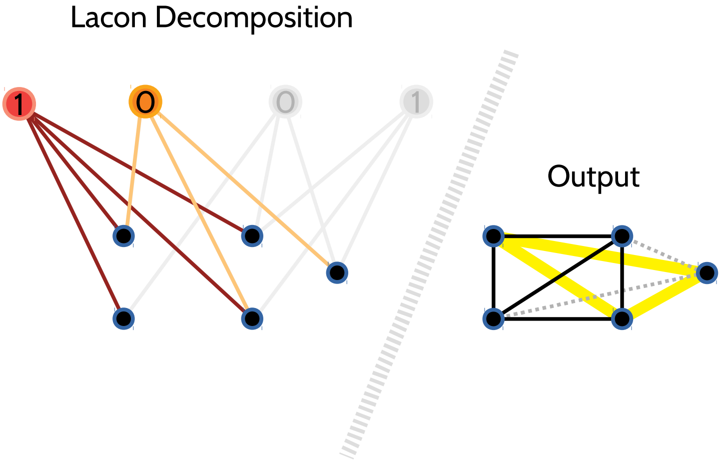
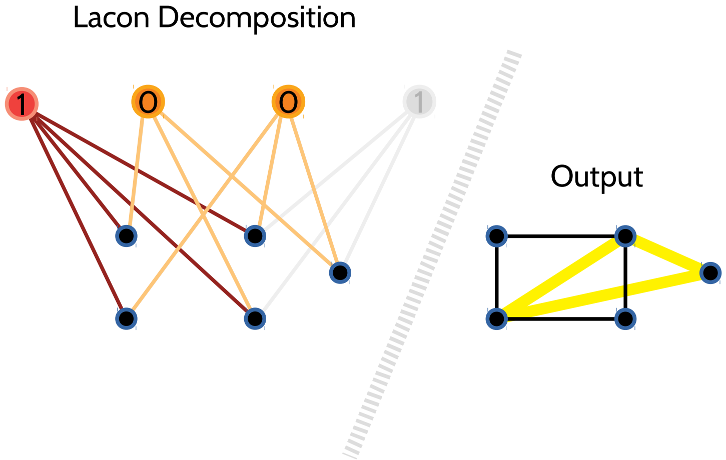


Lacon Decompositions

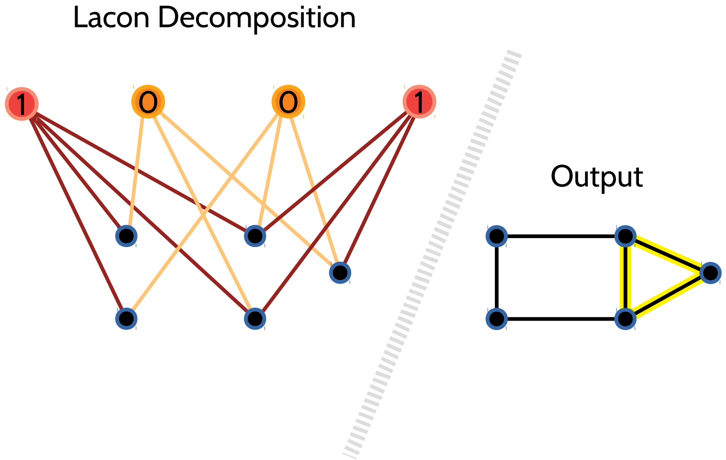


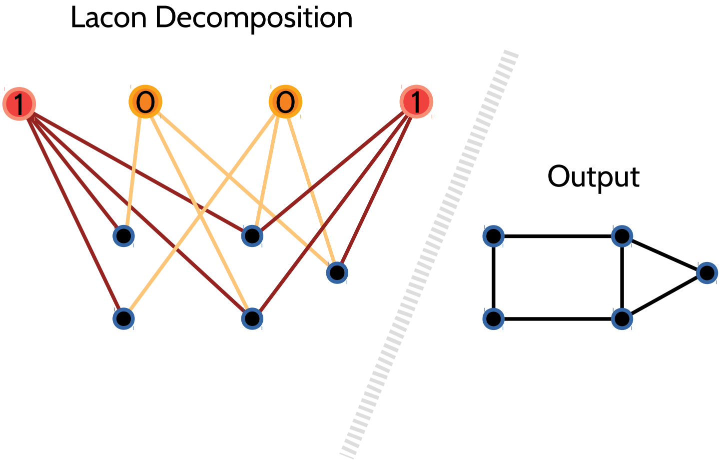

Lacon Decomposition
1
0
0
1
Output

Lacon Decompositions



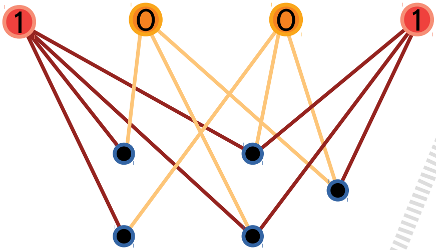
Lacon Decompositions



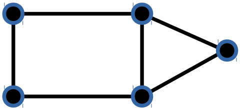

Lacon Decomposition
1
0
0
1
Output

Lacon Decompositions

Lacon Decomposition

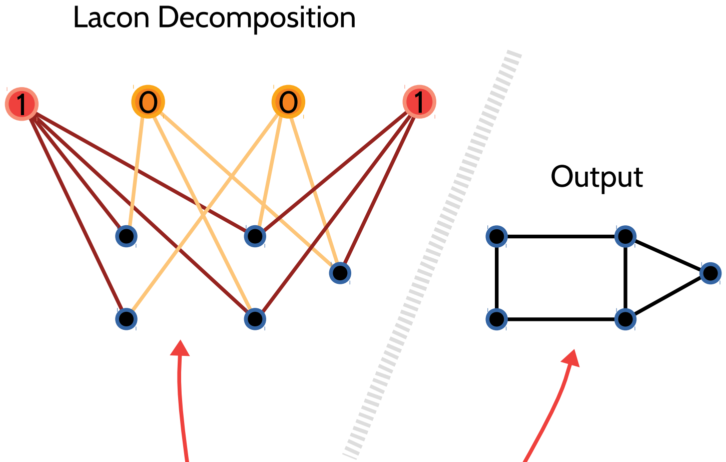


Output



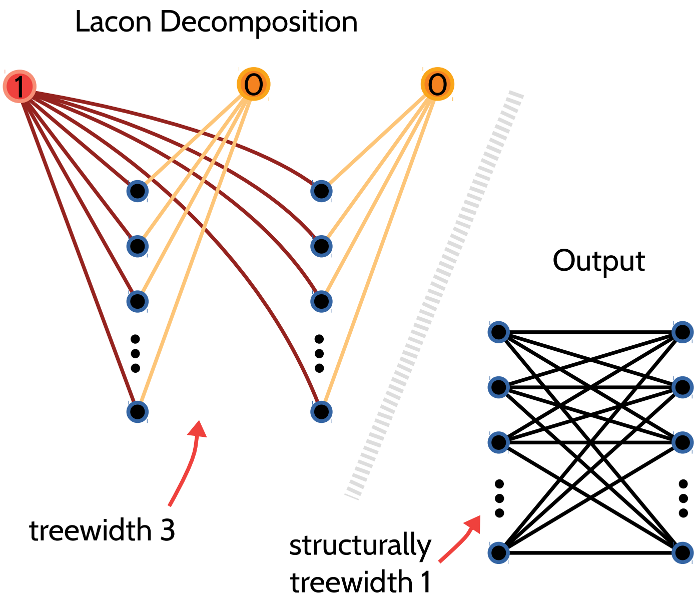
has lacon decomposition with property X $\implies$ structurally property X

Lacon Decompositions

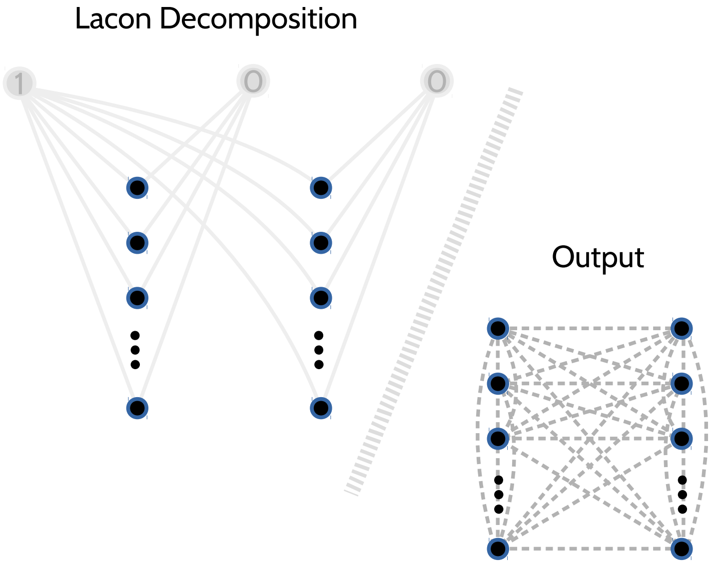


has lacon decomposition with property X $\iff$ structurally property X

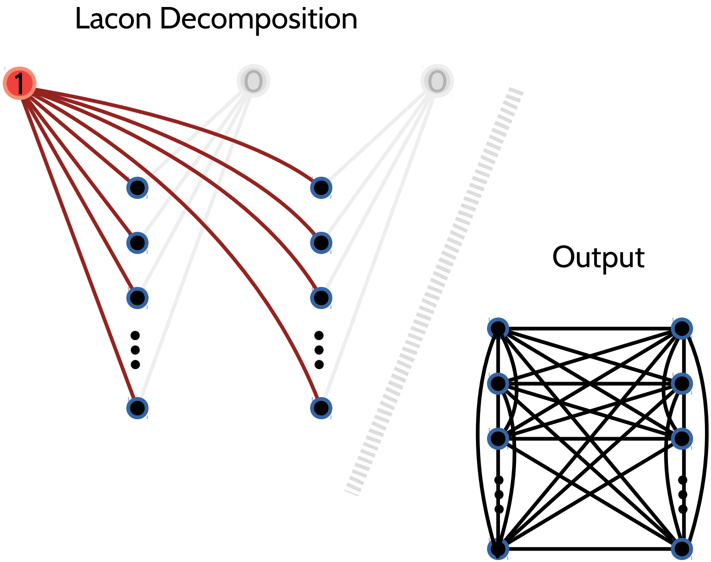
Lacon Decompositions

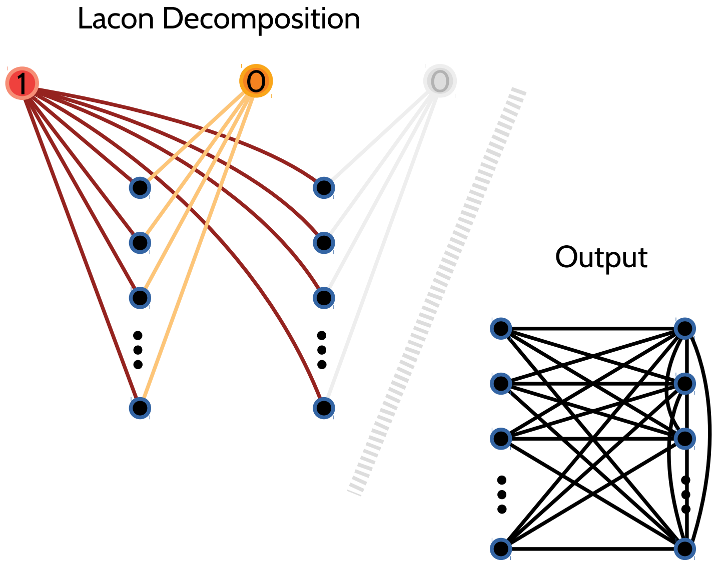


Lacon Decompositions

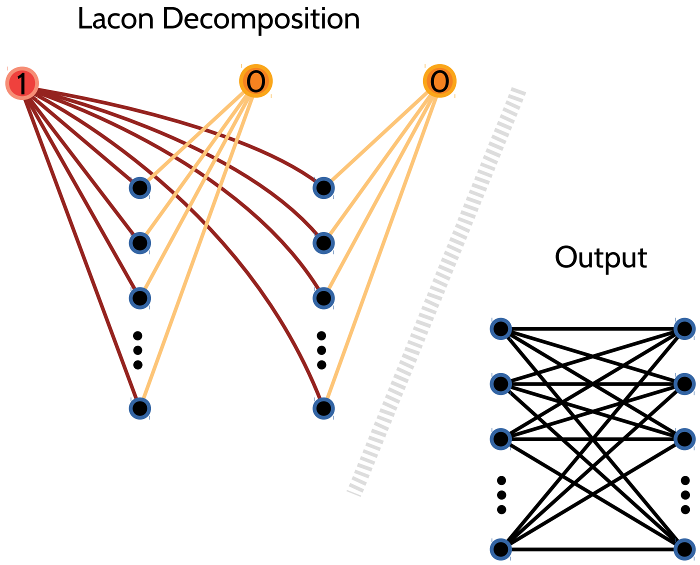


Lacon Decompositions



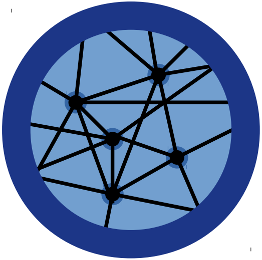

Lacon Decomposition
1
0
0
Output

Lacon Decompositions



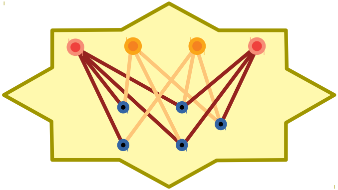
Big Question

Structurally
Bounded Expansion

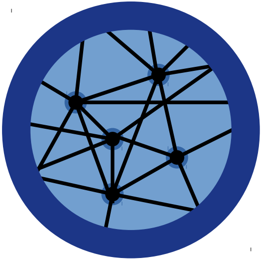


Big Question

has Lacon Decomposition
with Bounded Expansion



Structurally
Bounded Expansion



Big Question

has Lacon Decomposition
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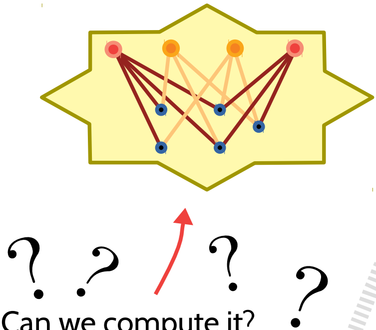
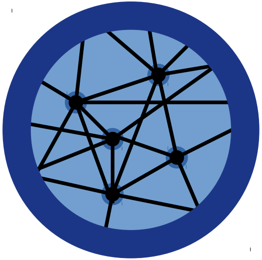
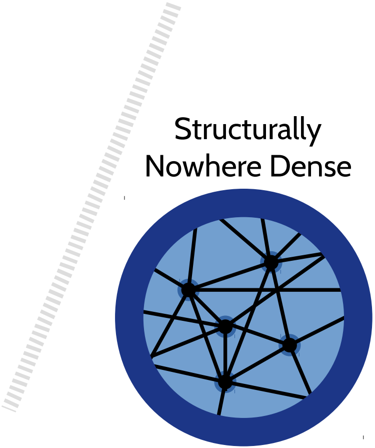


Can we compute it?
That would solve the
model-checking problem.

Structurally
Bounded Expansion

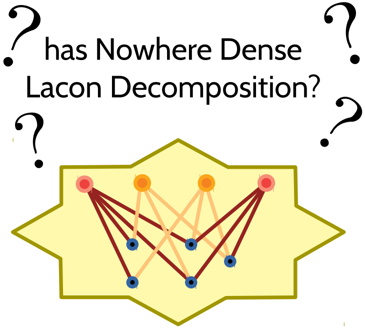


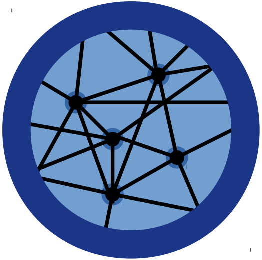
Big Question



Structurally Nowhere Dense

Big Question





End

Thanks!