Lacon- and Shrub-Decompositions

A New Characterization of First-Order Transductions of Bounded Expansion Classes

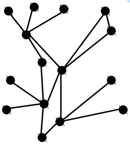
Jan Dreier
TU Wien, Austria

LICS 2021

Algorithmic Meta-Theorems

Algorithmic Meta-Theorems

MSO on treewidth



Algorithmic Meta-Theorems

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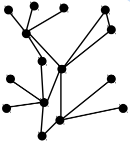

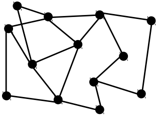

Algorithmic Meta-Theorems

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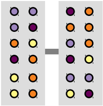

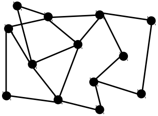

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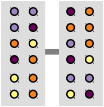

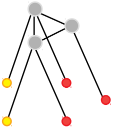

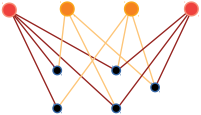

Some Sparse Graph Classes

Planar

Bounded Degree

Bounded Treewidth

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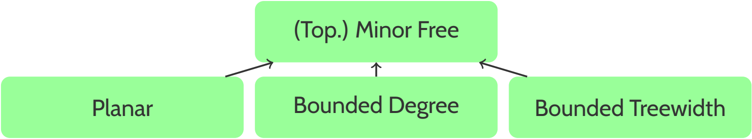

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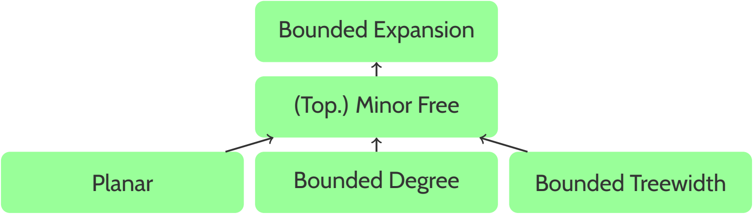

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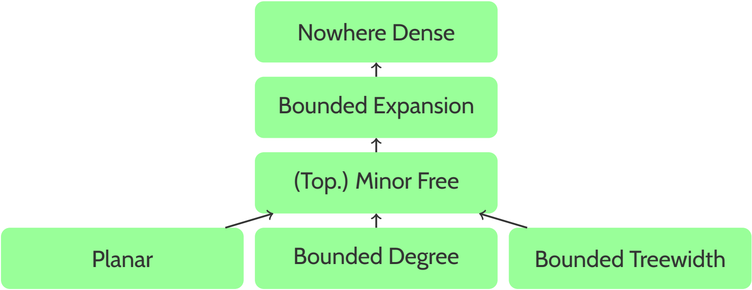

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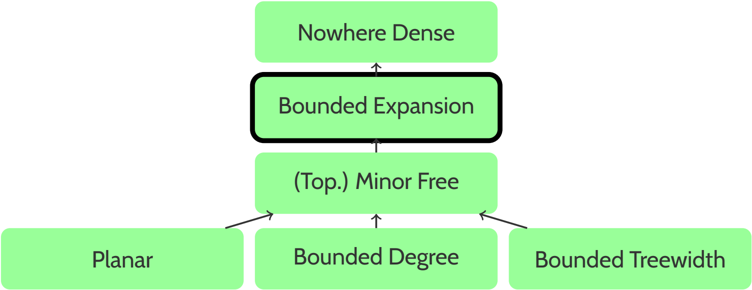
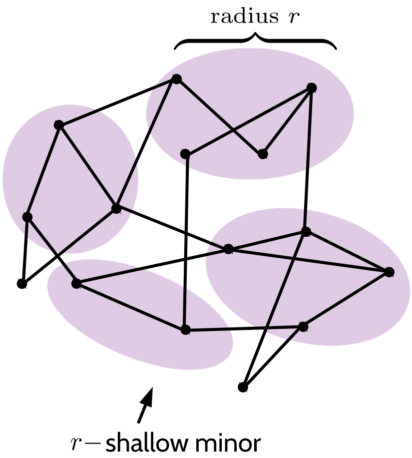


Definition of Bounded Expansion



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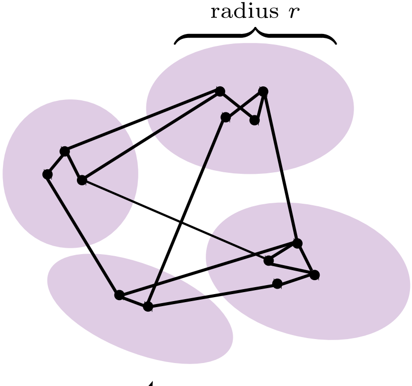

$r-$shallow minor

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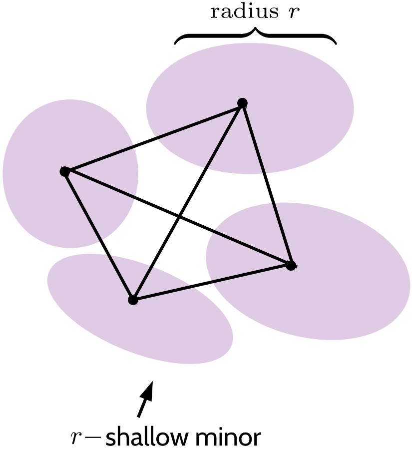

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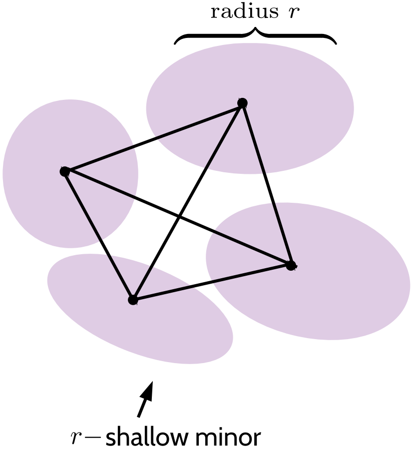

$$\frac{|E|}{|V|} \leq f(r)$$

for every r-shallow minor of every graph in the graph class.

FO Model-Checking [Dvořák, Král, Thomas 2010]

First-order formulas φ can be evaluated on bounded expansion classes in time $f(|\varphi|)n$.

Algorithmic Meta-Theorems

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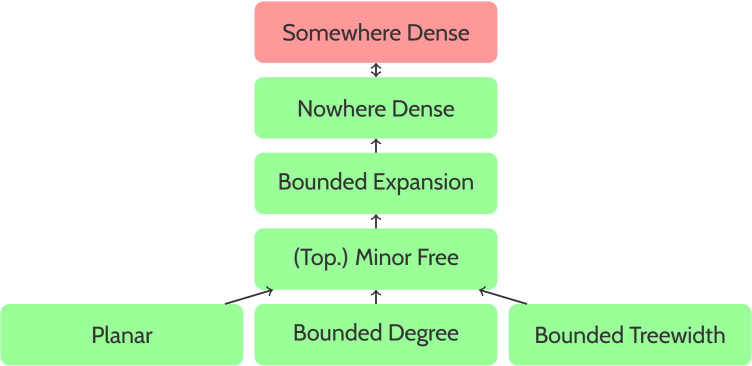
On bounded expansion: $f(k)n$

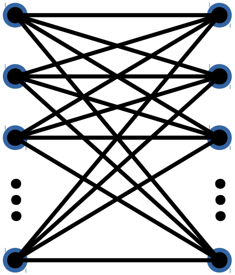
Exact Characterization



For graph classes $\mathcal{G}$ *closed under subgraphs*,
FO model-checking is tractable iff $\mathcal{G}$ is nowhere dense.

[Grohe, Kreutzer, Sieberz 2011]

Example: Fully Bipartite



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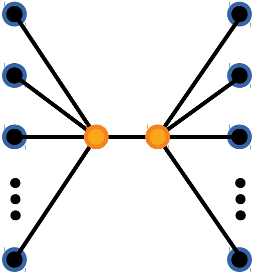

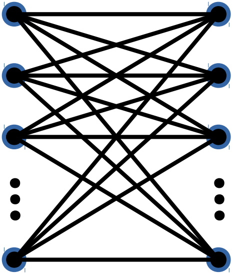

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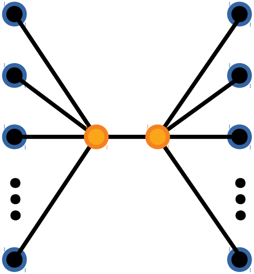
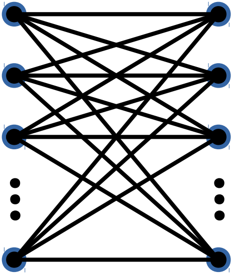


$$\mathrm{dist}(x, y) = 3$$



$$x \sim y$$

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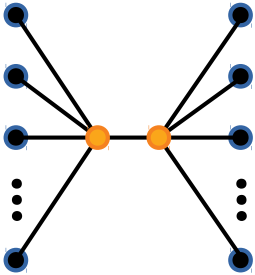
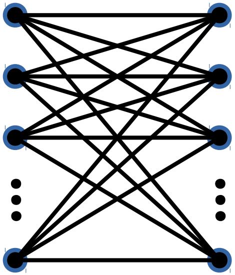


$\mathrm{dist}(x, y) = 3$

$\exists x \ \mathrm{blue}(x) \wedge \varphi$



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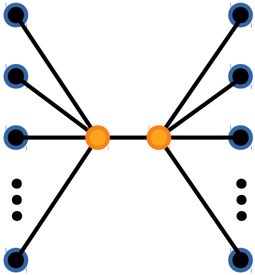

$$\exists x_1 \exists x_2 \exists x_3$$
$$\text{blue}(x_1) \wedge \text{blue}(x_2) \wedge \text{blue}(x_3)$$
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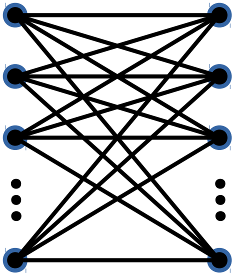

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Interpretations

$$I = (\nu(x), \mu(x, y))$$

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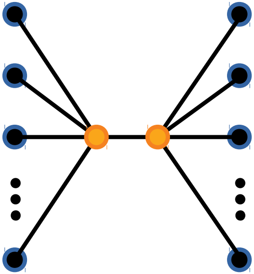

$I(G)$

vertices: $\{v \mid G \models \nu(v)\}$

edges: $\{uv \mid G \models \mu(u, v)\}$

Interpretations

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is blue

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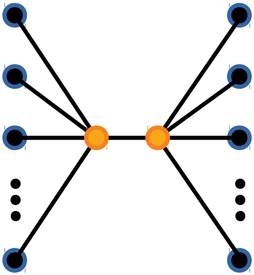

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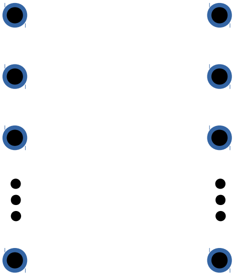

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have distance three

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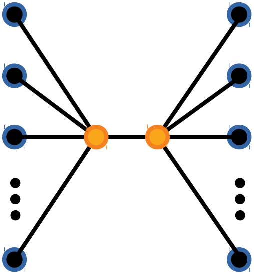

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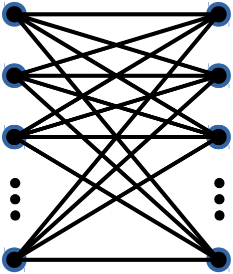

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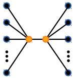
- The class of all  has treewidth 1

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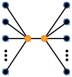
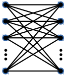
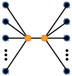
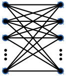
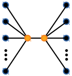
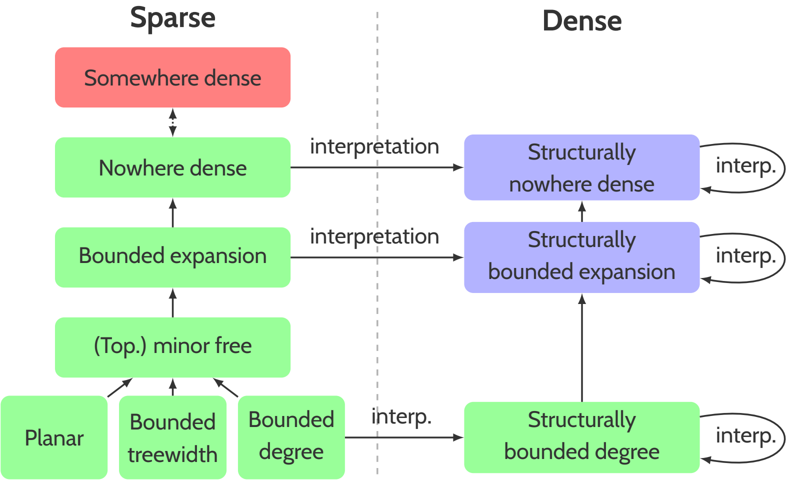
- The class of all  has treewidth 1
- For every  there is  with  $= I\Big($  $\Big)$.

Model-Checking in Sparse and Dense Classes

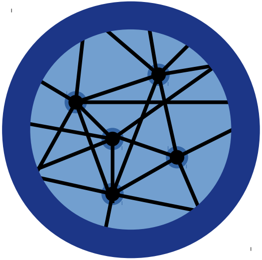


Nowhere Dense: Grohe, Kreutzer, Sieberz 2011
Structurally Bounded Degree: Gajarský, Hlineňỳ, Obdržálek, Lokshtanov, Ramanujan 2016

Big Question

Structurally
Bounded Expansion

$$I(G)$$



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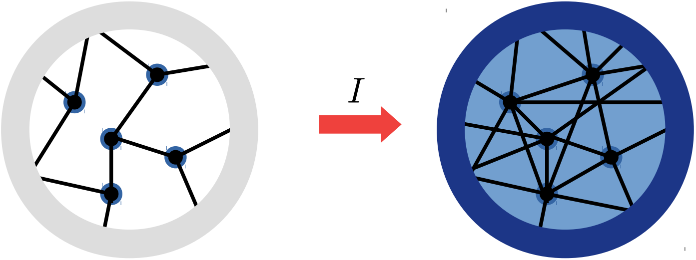

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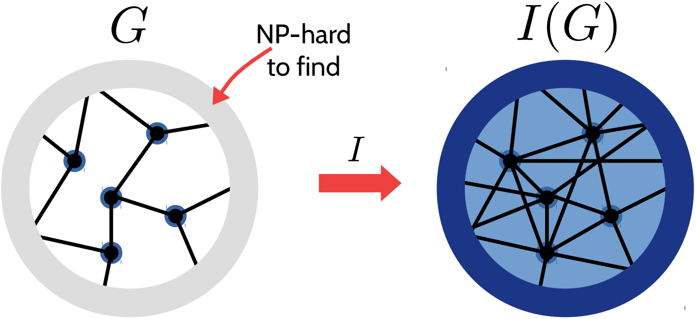

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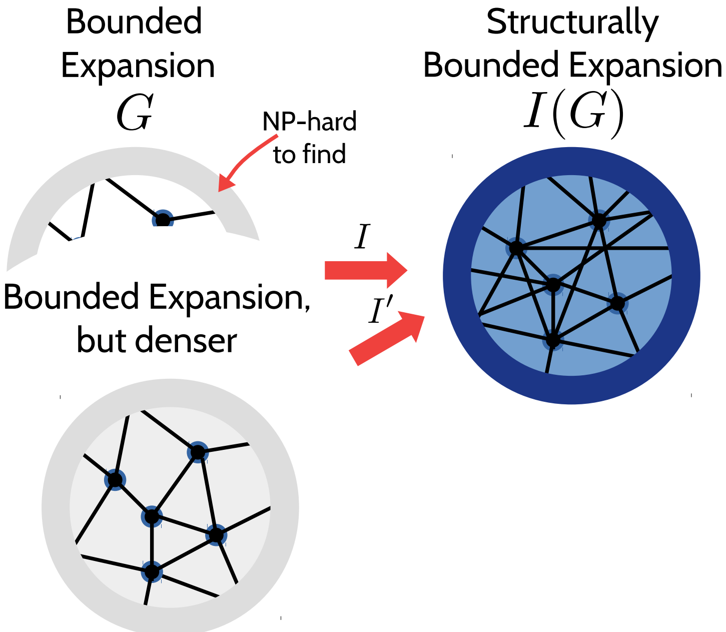

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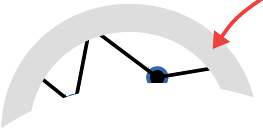

NP-hard to find

Structurally Bounded Expansion

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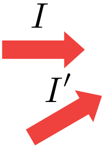

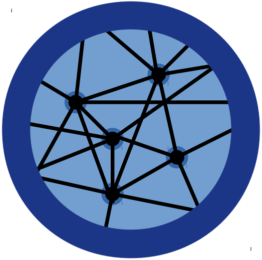

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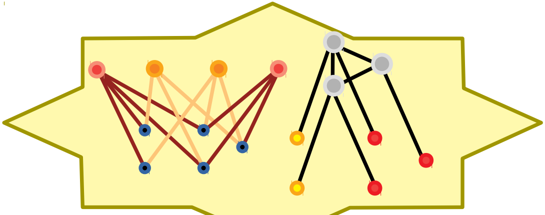

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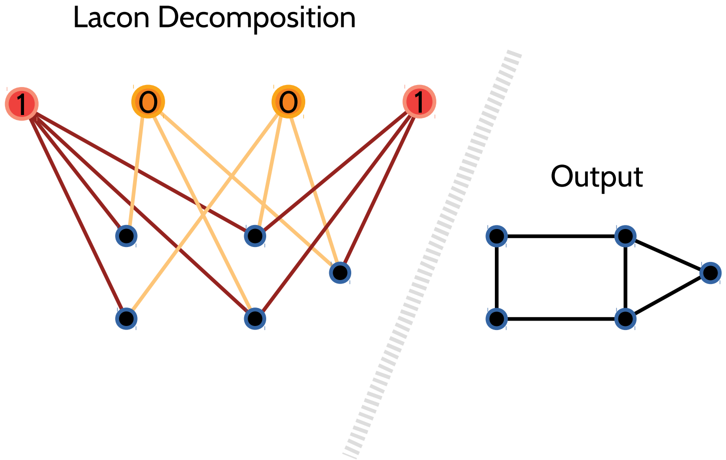

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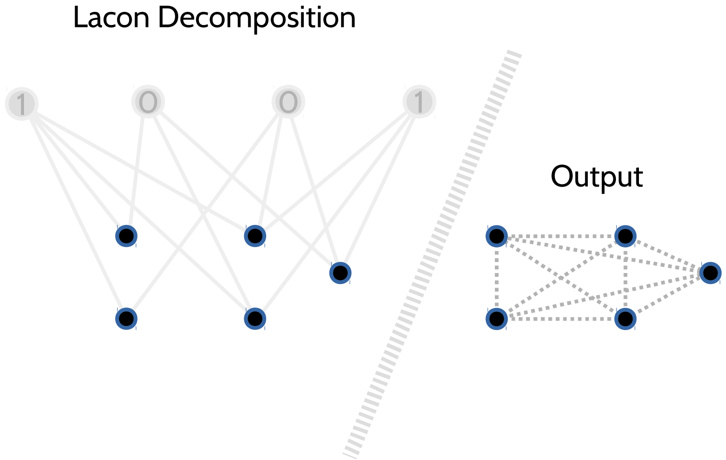

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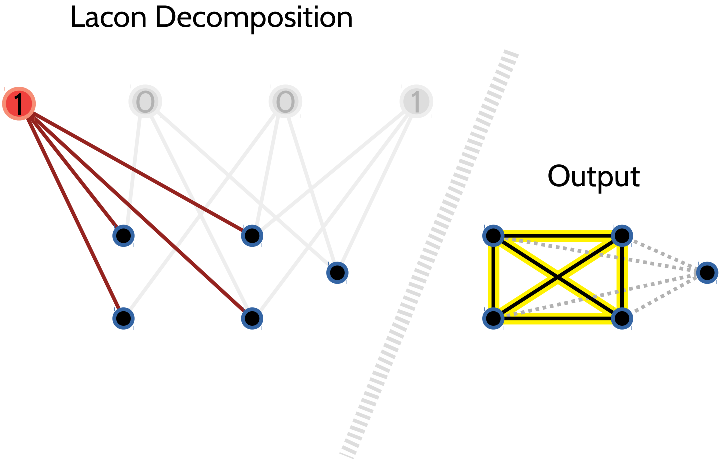

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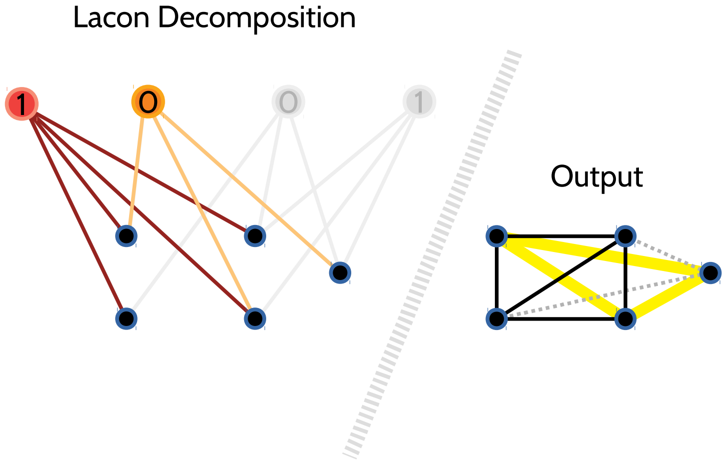

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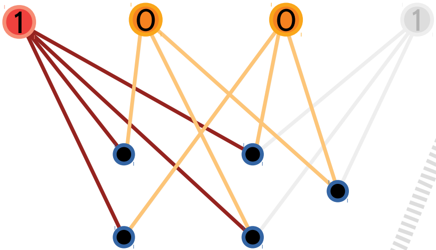
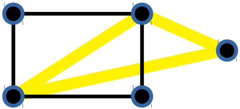


Output



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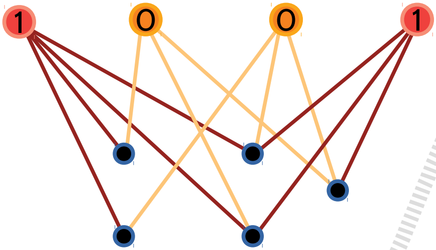

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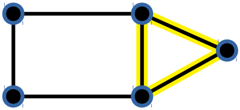

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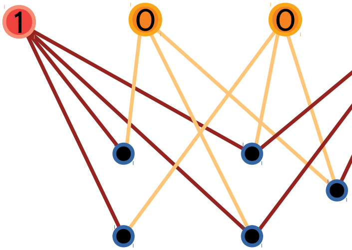

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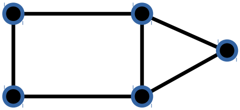
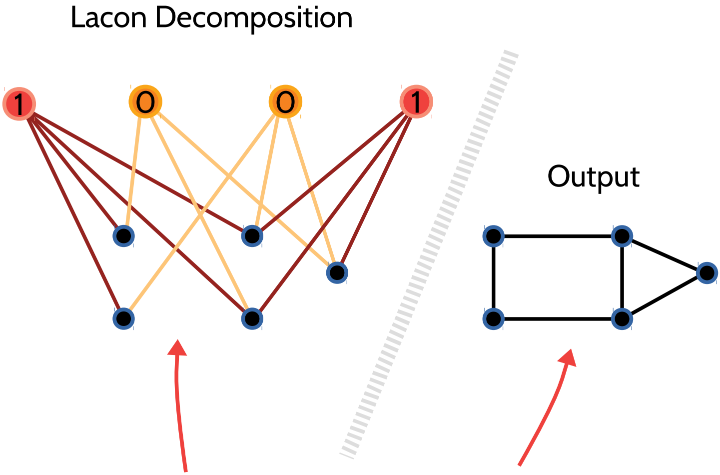
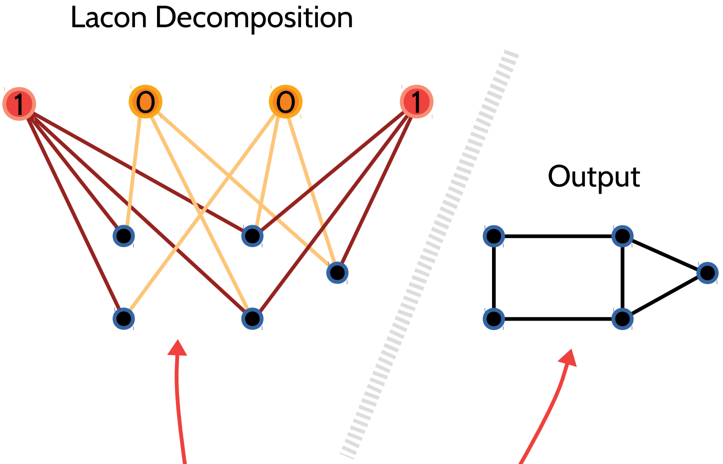


Lacon Decompositions



has lacon decomposition with property X $\implies$ structurally property X

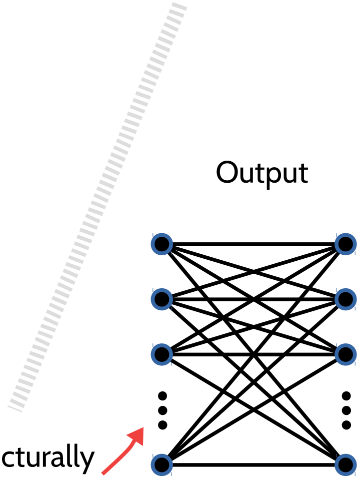
Lacon Decompositions



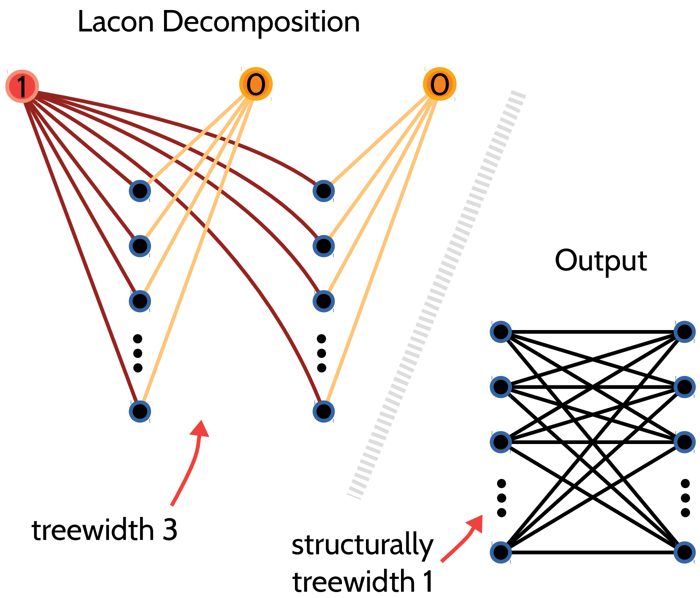
has lacon decomposition with property X $\iff$ structurally property X

Lacon Decompositions

Output

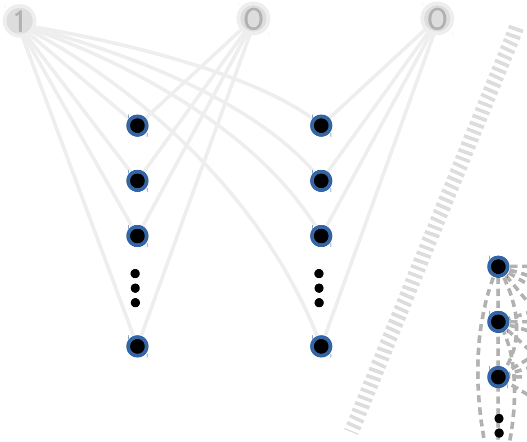


structurally
treewidth 1

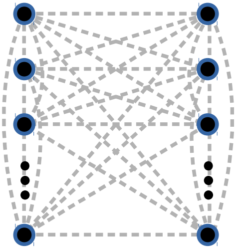

Lacon Decomposition
1
0
0
Output
treewidth 3
structurally
treewidth 1

Lacon Decompositions

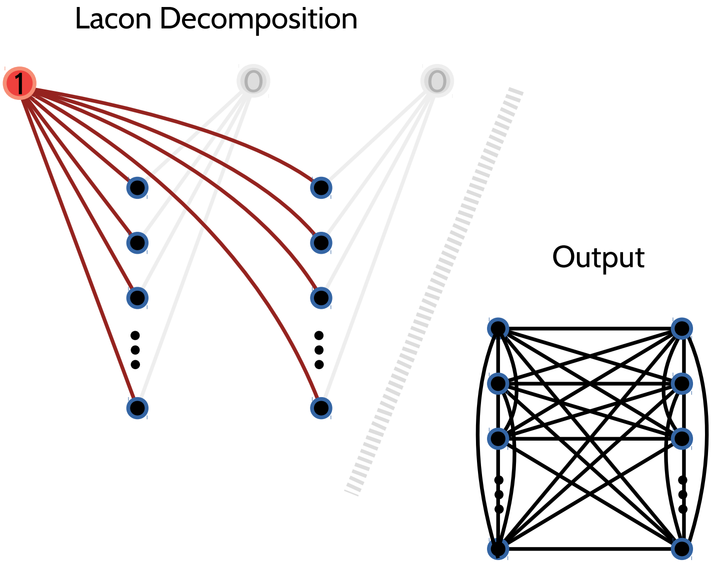
Lacon Decomposition



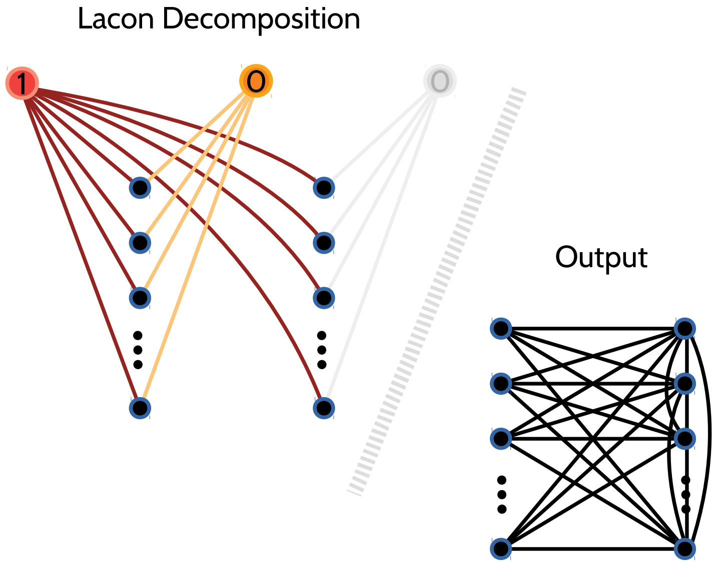
Output



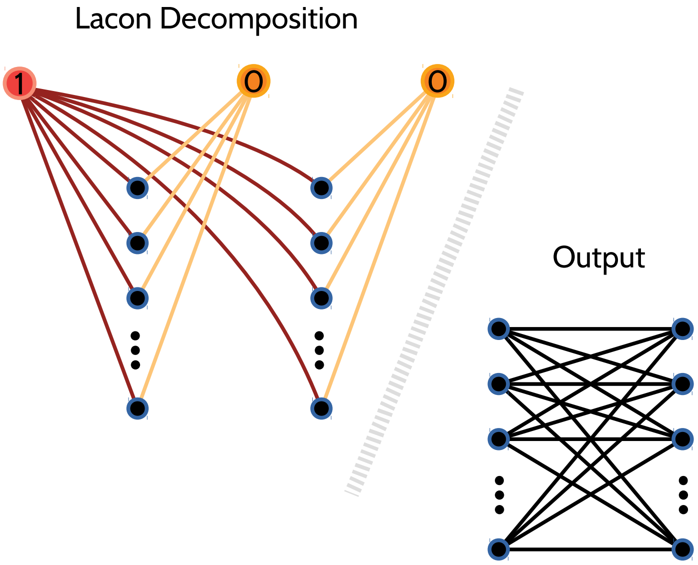
Lacon Decompositions



Lacon Decompositions



Lacon Decompositions

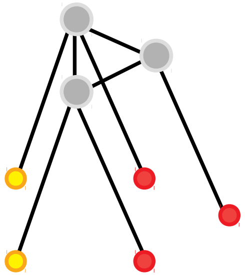


Shrub Decompositions

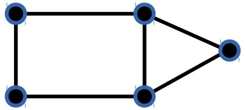
connect vertices with...

- distance 2
- distance 3 and same color

Shrub Decomposition



Output

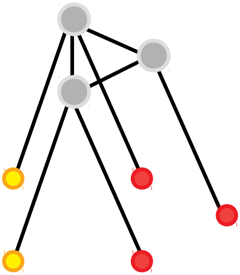


Shrub Decompositions

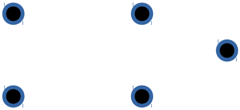
connect vertices with...

-
-

Shrub Decomposition



Output



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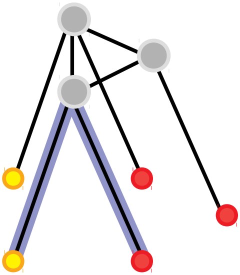
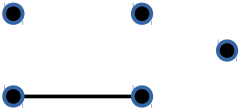
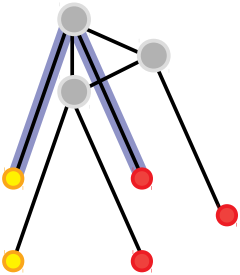


Output



Shrub Decompositions

Shrub Decomposition



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-

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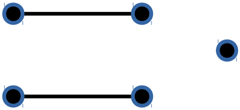

Shrub Decompositions

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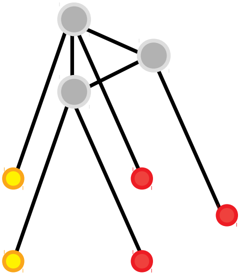
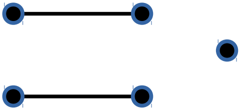
Shrub Decomposition



Output



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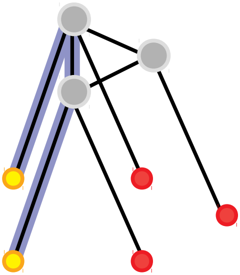
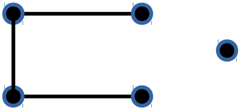


Output



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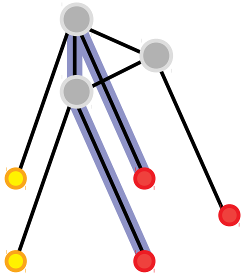
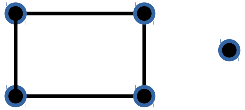
Shrub Decomposition

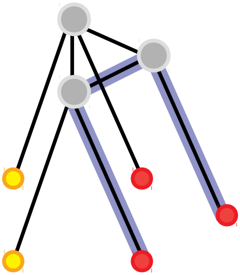


Output



Shrub Decompositions

Shrub Decomposition



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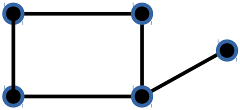
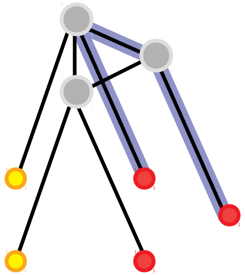


Shrub Decompositions

Shrub Decomposition



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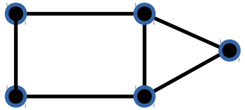
Output



Shrub Decompositions

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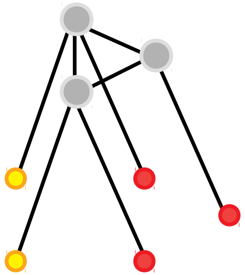
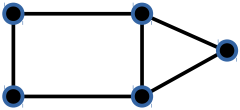
Shrub Decomposition



Output



has shrub decomposition with property X $\Longleftrightarrow$ structurally property X

Result

Theorem

Let $\mathcal{G}$ be a graph class. The following statements are equivalent.

Result

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Result

Theorem

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  - **bounded number of colors,**
  - **bounded diameter.**

Result

Theorem

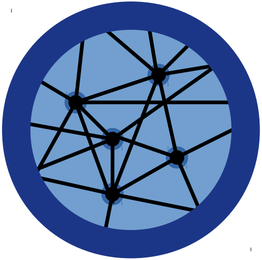
Let $\mathcal{G}$ be a graph class. The following statements are equivalent.

- $\mathcal{G}$ has **structurally bounded expansion.**
- $\mathcal{G}$ has **lacon decompositions** with
  - **bounded expansion,**
  - **bounded target vertex degree.**
- $\mathcal{G}$ has **shrub decompositions** with
  - **bounded expansion,**
  - **bounded number of colors,**
  - **bounded diameter.**
- $\mathcal{G}$ has **low shrubdepth covers** [1].

[1] Gajarský, Kreuzer, Nešetřil, Ossona de Mendez, Siebertz, Toruńczyk 2018

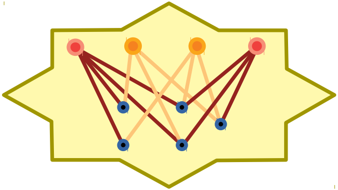
Questions

Structurally
Bounded Expansion

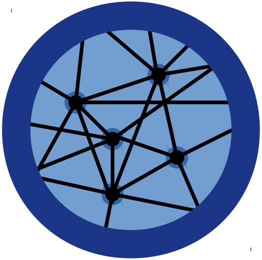


Questions

has Decomposition
with Bounded Expansion



Structurally
Bounded Expansion



Questions

has Decomposition
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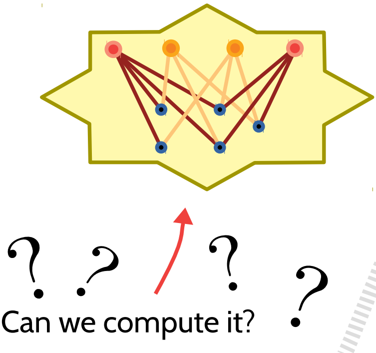
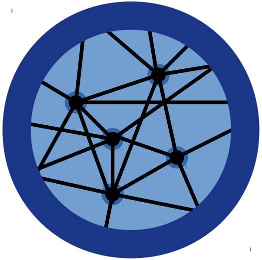


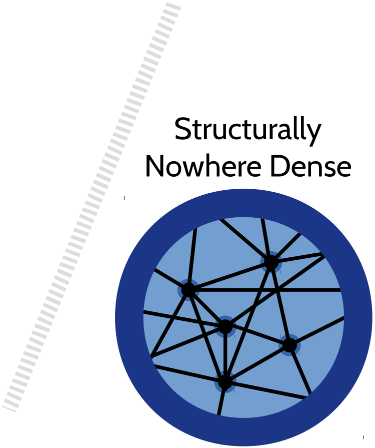
Can we compute it?

Structurally
Bounded Expansion



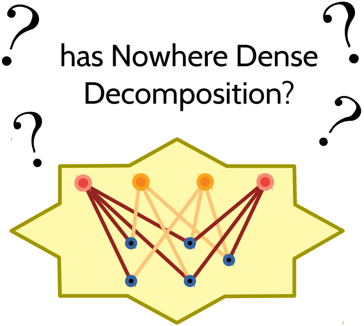
Questions

Structurally
Nowhere Dense

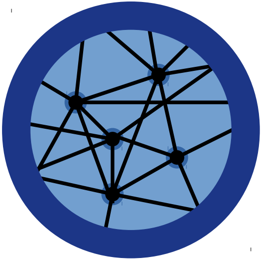


Questions

has Nowhere Dense Decomposition?



Structurally Nowhere Dense



End

Thanks!

dreier@ac.tuwien.ac.at