Hardness of FO Model-Checking on Random Graphs

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Theoretical Computer Science
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FO-Logic on Graphs

FO-logic on graphs

- $\forall\ \exists$ quantification, adjacency ($\sim$) and equality ($=$) of vertices, $\neg$, $\vee$, $\wedge$
- Example: $\exists x\, \exists y\, \exists z\, \left(x \sim y \wedge y \sim z \wedge x \sim z\right)$

Algorithmic Metatheorems

Let $\mathcal{G}$ be a graph class.

FO-MC on $\mathcal{G}$	
Input:	A graph $G \in \mathcal{G}$ and a sentence $\varphi \in \mathrm{FO}$
Parameter:	$\|\varphi\|$
Problem:	$G \models \varphi$?

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- p-Clique
- p-IndependentSet
- p-DominatingSet
- …

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If **FO-MC** on $\mathcal{G}$ is FPT then many problems on $\mathcal{G}$ are FPT:

- p-CLIQUE $\quad \exists x_1 \dots x_k \left(\bigwedge_{i \neq j} x_i \sim x_j \right)$
- p-INDEPENDENTSET
- p-DOMINATINGSET
- …

Previous Results

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- *Dvořák, Král', Thomas 2010:* If $\mathcal{G}$ has bounded expansion, one can decide **FO-MC** on $\mathcal{G}$ in FPT time.
- *Kreutzer, Grohe, Siebertz 2011:* If $\mathcal{G}$ is nowhere dense, one can decide **FO-MC** on $\mathcal{G}$ in FPT time.

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Analyze runtime on *typical* instances.

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theoretical computer science:

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$$\Downarrow$$

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$$\mathrm{E}_{G\sim\mathcal{G}_n}\left[t(G,\varphi)\right] \leq f(|\varphi|)n^{O(1)}.$$

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Can these results be improved?

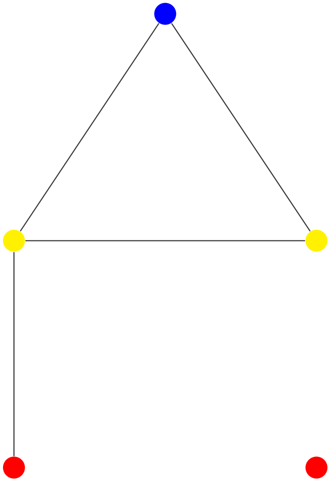
We don't know.

FO-Logic on Vertex-Colored Graphs

FO-logic on vertex-colored graphs

- $\forall\ \exists$ quantification, adjacency ($\sim$) and equality ($=$) of vertices, $\neg$, $\vee$, $\wedge$
- unary predicates for each color
- Example: $\exists x\, \exists y\, \exists z\, \big(x \sim y \wedge y \sim z \wedge x \sim z \wedge \mathrm{blue}(x)\big)$

Triangle with One Blue Vertex



$$\exists x \, \exists y \, \exists z \, ($$
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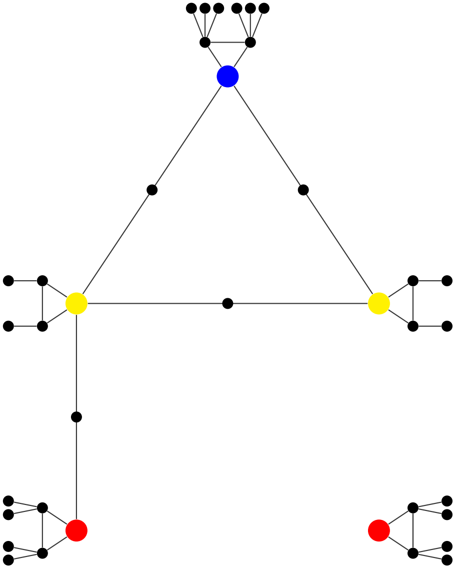

$$\exists x\, \exists y\, \exists z\, \big(\\ x \sim y \land y \sim z \land x \sim z \\ \land\, \mathrm{blue}(x) \\ \big)$$

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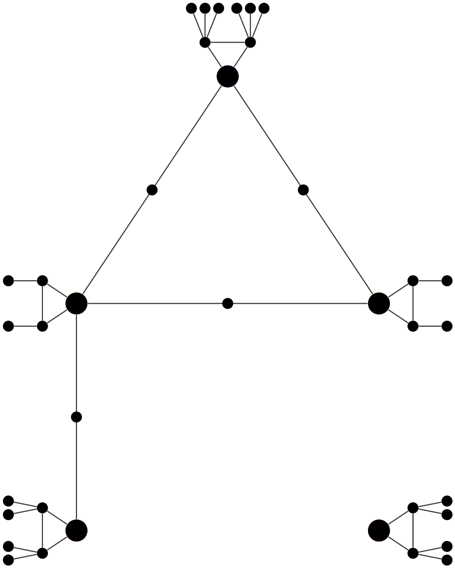

$$\exists x\, \exists y\, \exists z\, (\\ x \sim y \wedge y \sim z \wedge x \sim z \\ \wedge\, \text{blue}(x) \\ )$$

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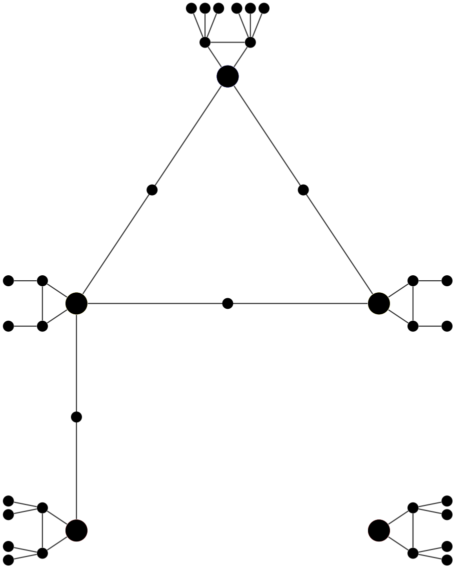

$$\exists x\, \exists y\, \exists z\, \big($$

$$x\lhd \wedge y\lhd \wedge z\lhd$$

$$\wedge\, x \mathbin{-\bullet-} y \wedge y \mathbin{-\bullet-} z \wedge x \mathbin{-\bullet-} z$$

$$\wedge\, x\lhd\!\equiv$$

$$\big)$$

Takeaway

If we can solve the model-checking problem on a graph, we can usually also solve it on on vertex-colorings of the graph.

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Definition

$\mathcal{A}$ solves **colour-FO-MC** on $\mathcal{G}$ in expected FPT time if for all coloring functions C

$$\mathrm{E}_{G\sim\mathcal{G}_n}\left[t(C(G), \varphi)\right] \leq f(|\varphi|)n^{O(1)}.$$

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Results

Theorem (Hardness Example)

Let $0 < \varepsilon < 1$. **colour-FO-MC** cannot be solved in expected FPT time on Erdős–Rényi graph $G(n, \frac{n^\varepsilon}{n})$ or $G(n, \frac{1}{2})$ unless $\mathrm{AW}[*] \subseteq \mathrm{FPT/poly}$.

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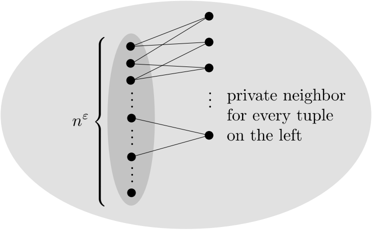
(hardness also holds for parameterized variant of Levin's average polynomial time)

Idea

Assume we can find with probability at least 1/2 the following structure in a random graph model.



Then **colour-FO-MC** cannot be solved in expected FPT time on this model.

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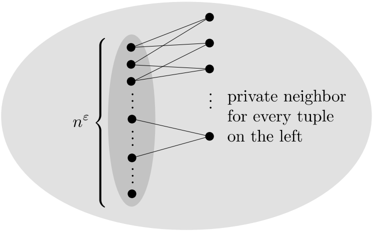

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Proof: We assume one can solve **colour-FO-MC** efficiently and show that one can solve **FO-MC** efficiently on *all graphs*.

Idea

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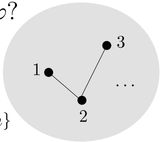
input: graph H, FO-sentence φ

question: $H \models \varphi$?



$V(H) = \{1, \ldots, n\}$

4

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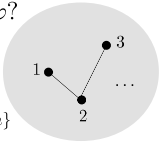

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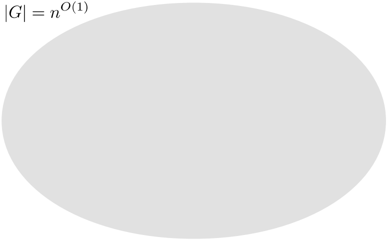
sample random graph G

$|G| = n^{O(1)}$



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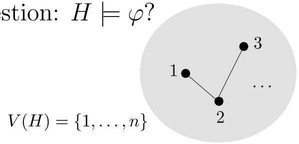

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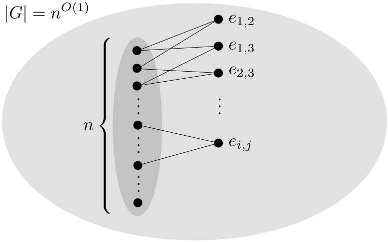

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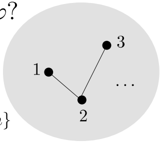

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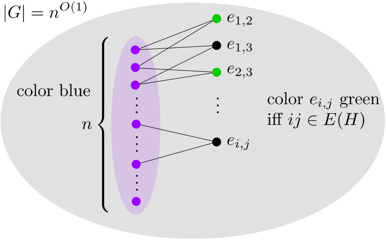

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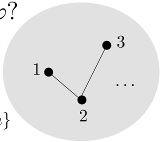

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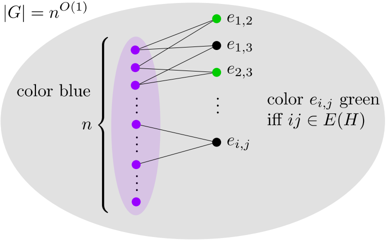

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construct φ'

from φ by replacing

$\exists\, x \longrightarrow \exists\, x$

$\forall\, x \longrightarrow \forall\, x$

$x \sim y \longrightarrow x$ — • — y

4

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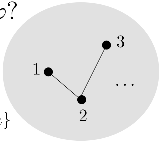

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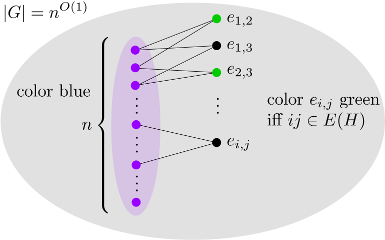

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$H \models \varphi \Leftrightarrow G \models \varphi'$

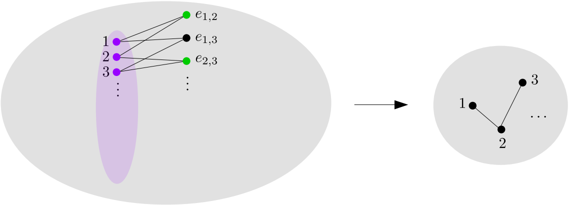
solve $G \models \varphi'$ in

expected FPT time

Reductions

interpretation $(\nu(x), \mu(x,y))$ maps graph G to graph G' with

- $V(G') = \{v \mid G \models \nu(v)\}$ (e.g. $\nu(v) = \text{blue}(v)$)
- $E(G') = \{uv \mid G \models \mu(u,v)\}$ (e.g. $\mu(x,y) = x$ —●— y)



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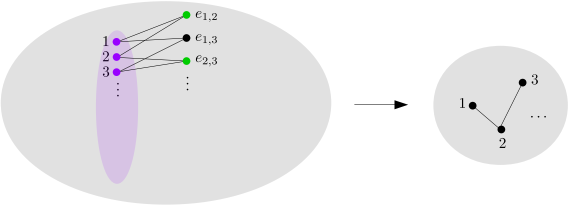

Theorem (Reductions)

If $\mathcal{X}$ is hard and $\mathcal{Y}$ can be colored and interpreted to behave like $\mathcal{X}$ then $\mathcal{Y}$ is hard.

Conclusion

- Hardness of a natural problems on a natural distributions under $\mathrm{AW}[*] \not\subseteq \mathrm{FPT/poly}$.
- Our average case algorithms for **colour-FO-MC** are optimal.

Open Questions

Can we show that the problem
is hard *without* colors?

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Can we show that the problem
is hard *without* colors?

tractable?

Requires technique that does
not work with colors.

hard?

Try proving
distW[1]-hardness.
Maybe random colors?