Motif Counting in Preferential Attachment Graphs

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December 11, 2019

Theoretical Computer Science
RWTH Aachen University, Germany

FSTTCS 2019

Subgraph Counting

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- *Curticapean, Dell, Marx:* in time $k^{O(k)}n^{0.174k+o(k)}$

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Analyze runtime on *typical* instances.

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$\Downarrow$

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Real-World Complex Networks

Some properties of complex networks:

- **Degree distribution**
  Most people have little but a few have a lot.
- **Small world**
  Everyone is close to everyone.
- **Clustering**
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- **Clustering** ✗
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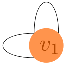
Preferential Attachment

- G_m^n graph with vertices $v_1, \ldots, v_n$ and m edges per vertex

- G_2^1:



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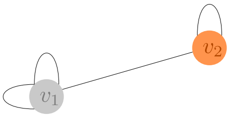

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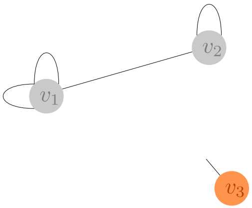

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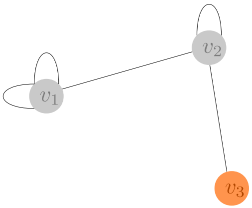

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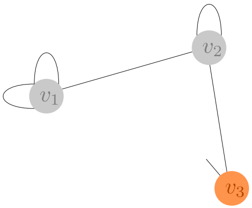

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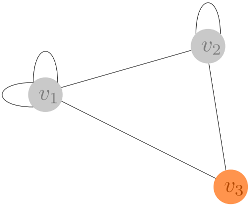

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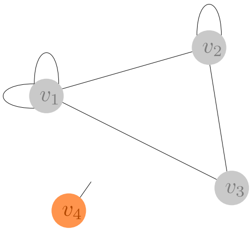

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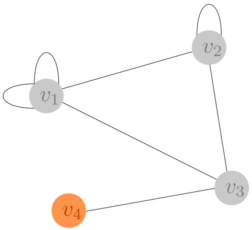

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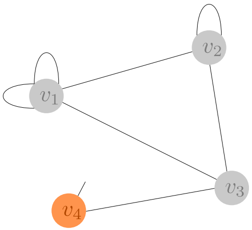

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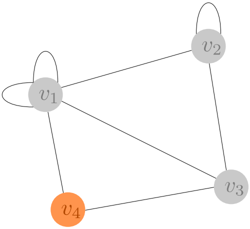

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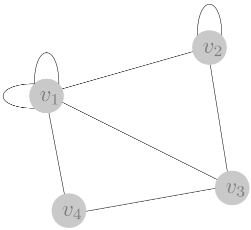


Our Result

Theorem

One can compute how often H occurs in G_m^n in expected FPT time $f(|H|)m^{O(|H|^6)}\log(n)^{O(|H|^{12})}n$.

In particular, for fixed $m \in \mathbf{N}$ and H the run time is $\log(n)^{O(1)}n$.

Outline

Simple Algorithm

$\downarrow$

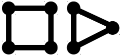
Bound on number of certain subgraphs

$\downarrow$

Bound on degrees of individual vertices

Counting Disconnected Subgraphs

It is sufficient to count connected subgraphs.

\# 

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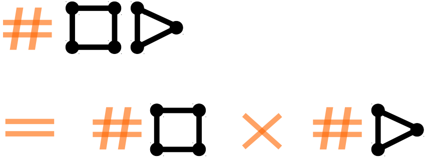

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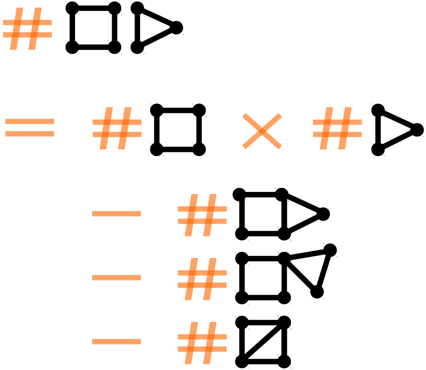
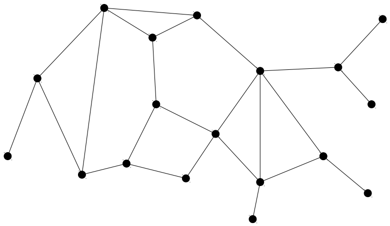


Example: Counting the number of squares



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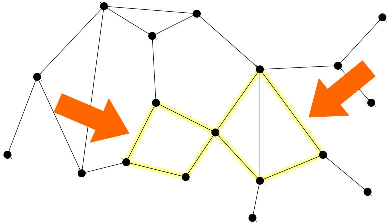

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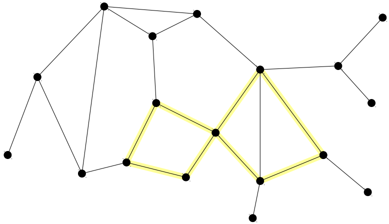

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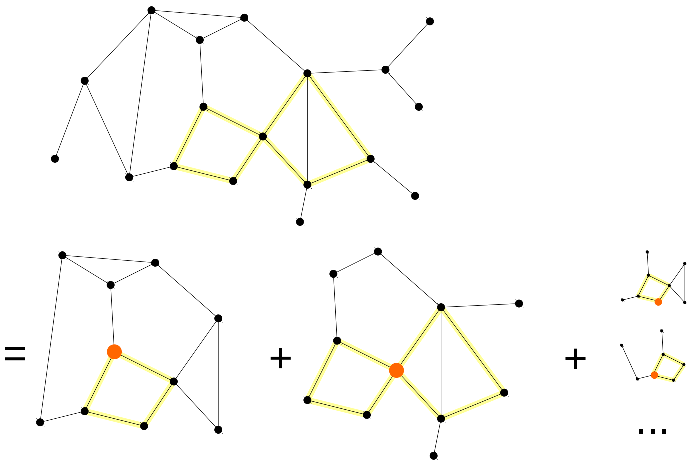

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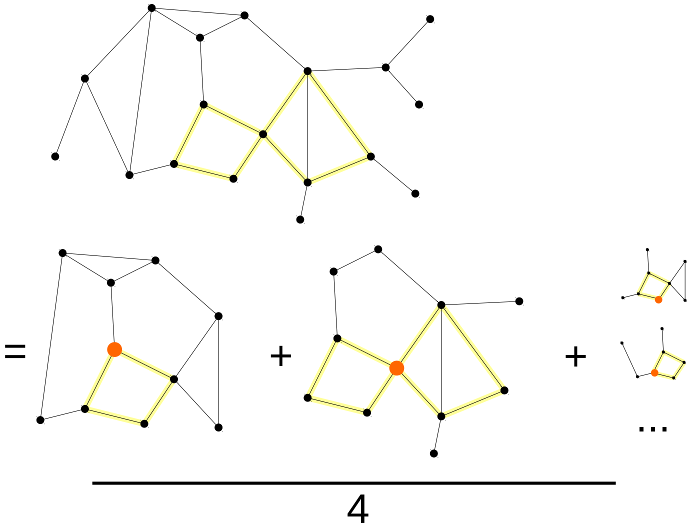

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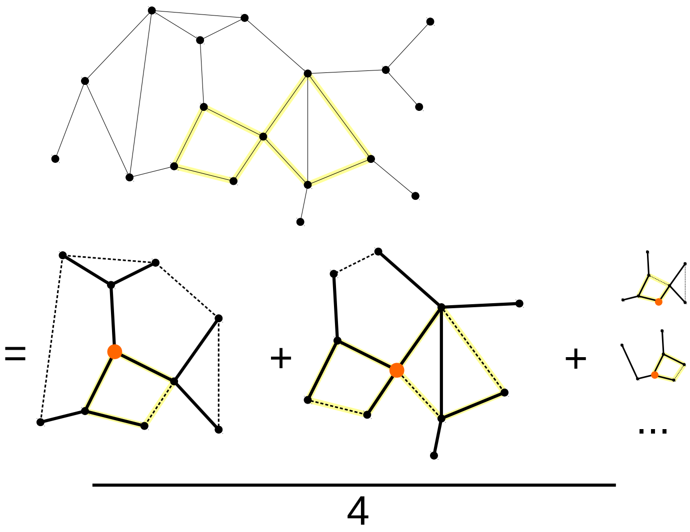

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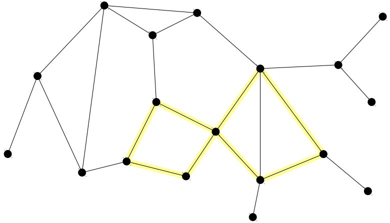

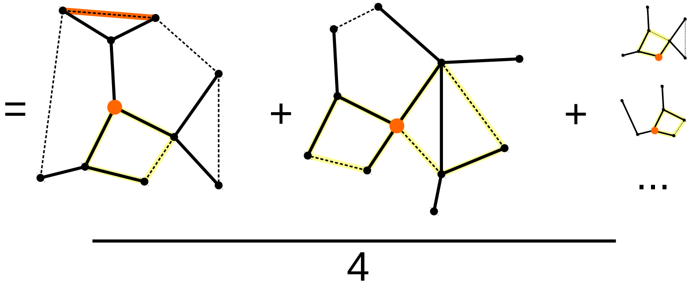

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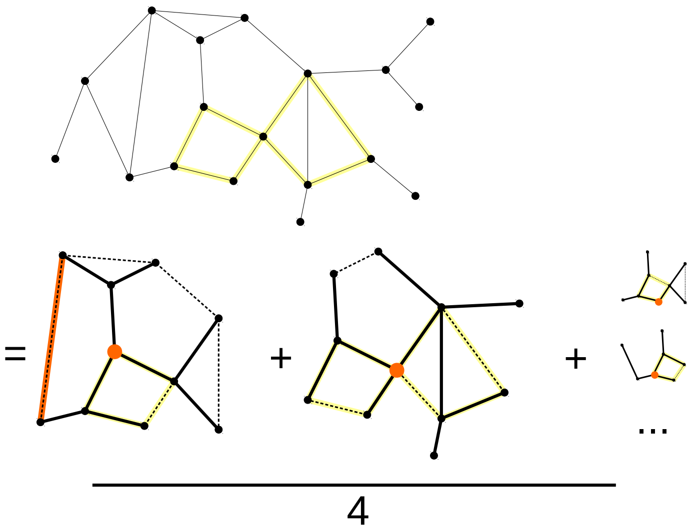

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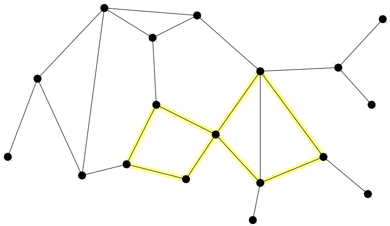

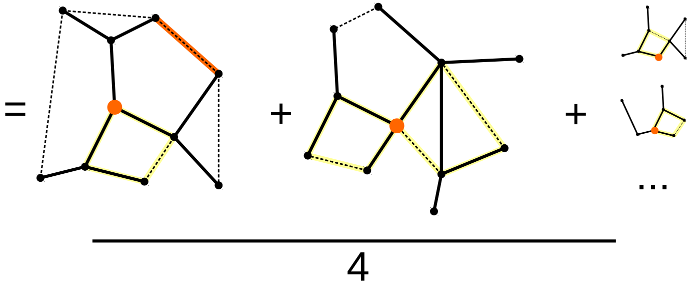

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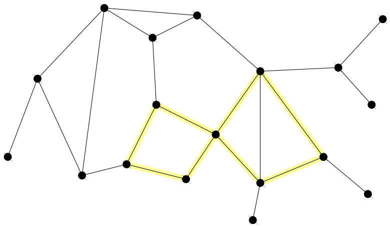

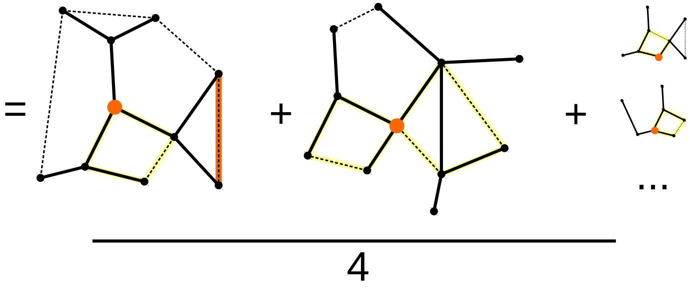

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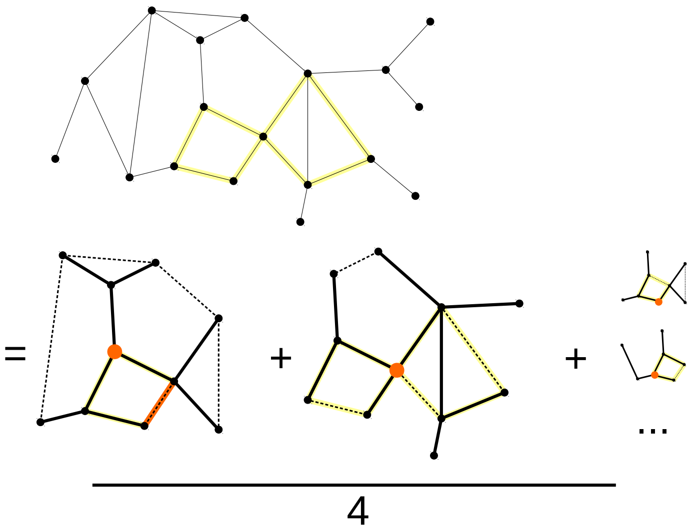

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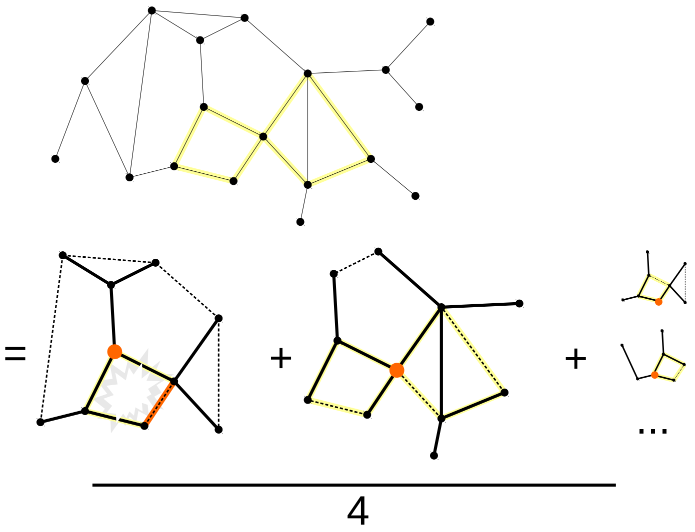

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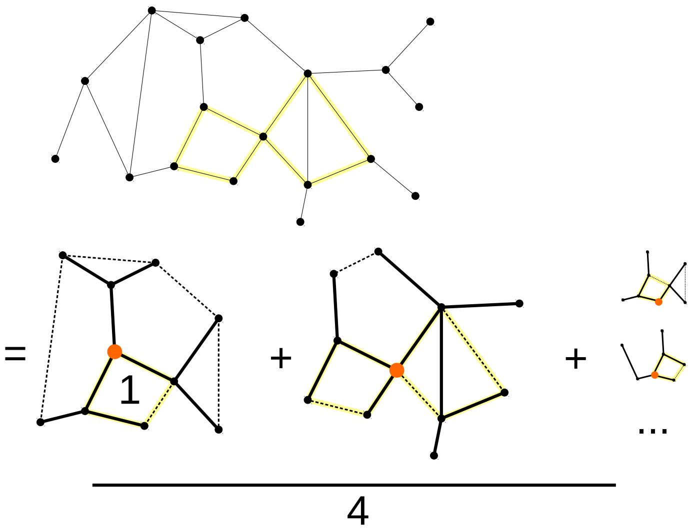

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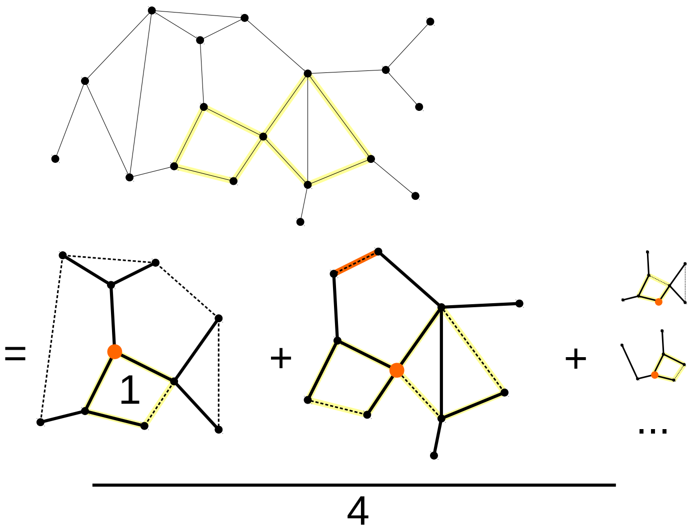

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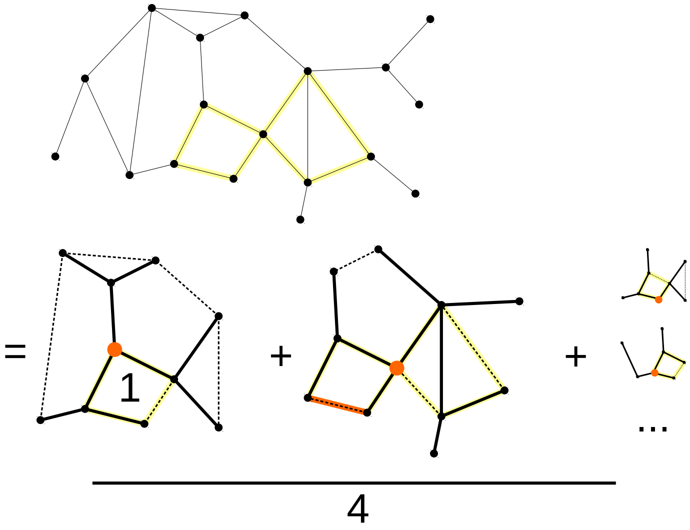

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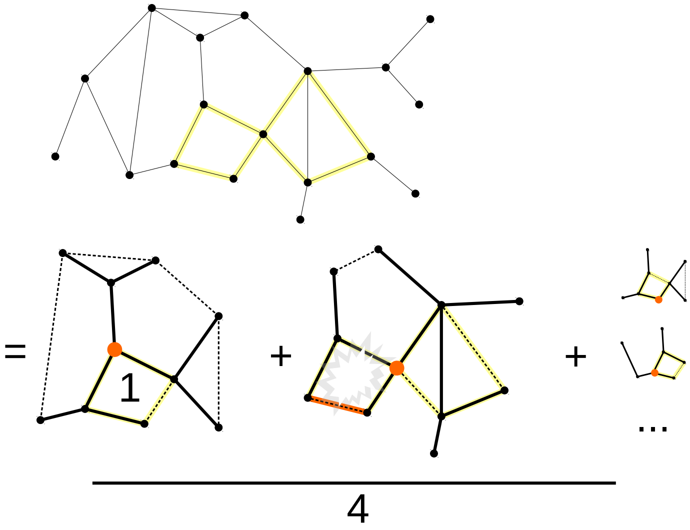

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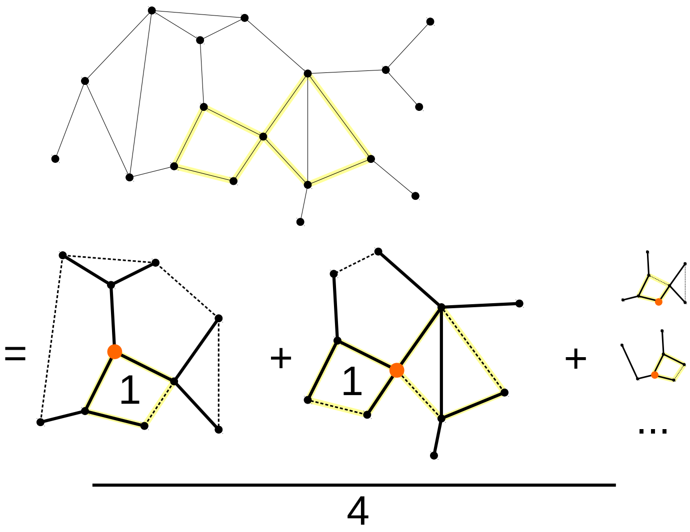

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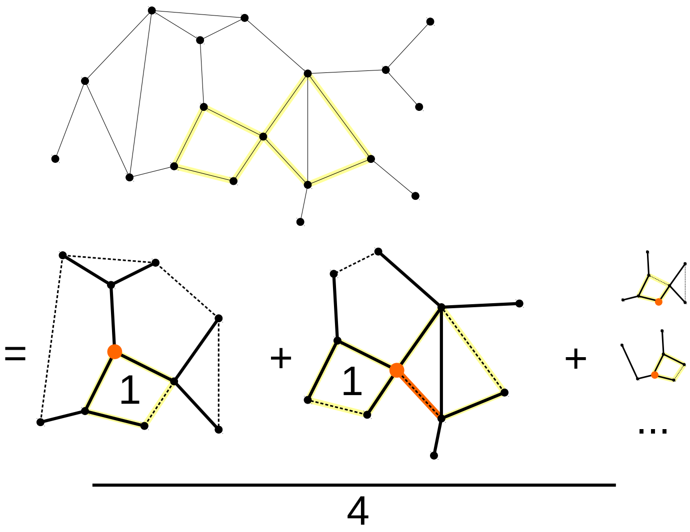

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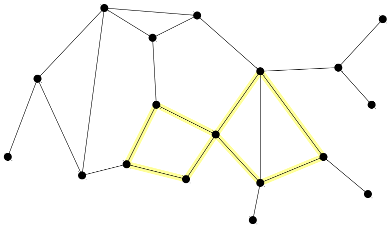

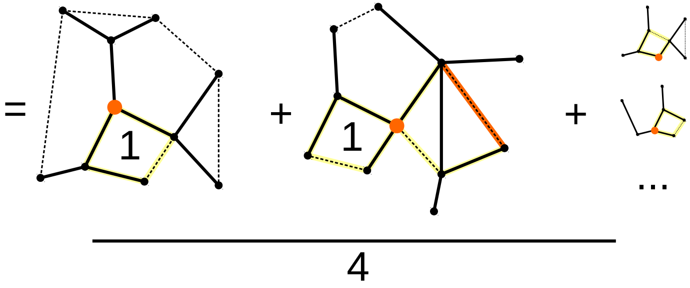

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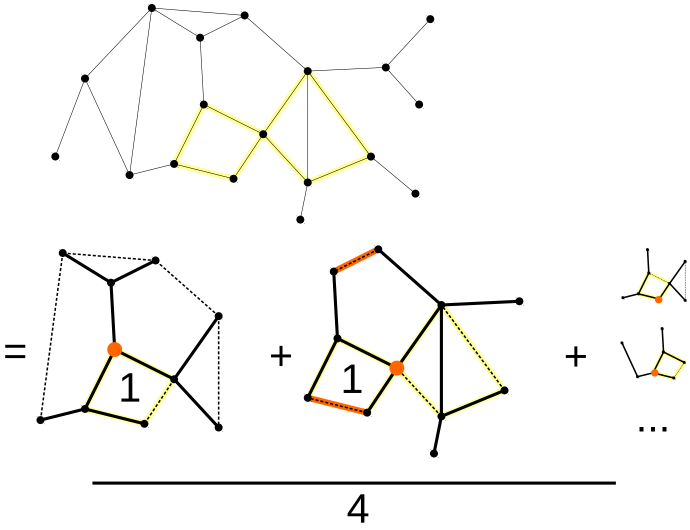

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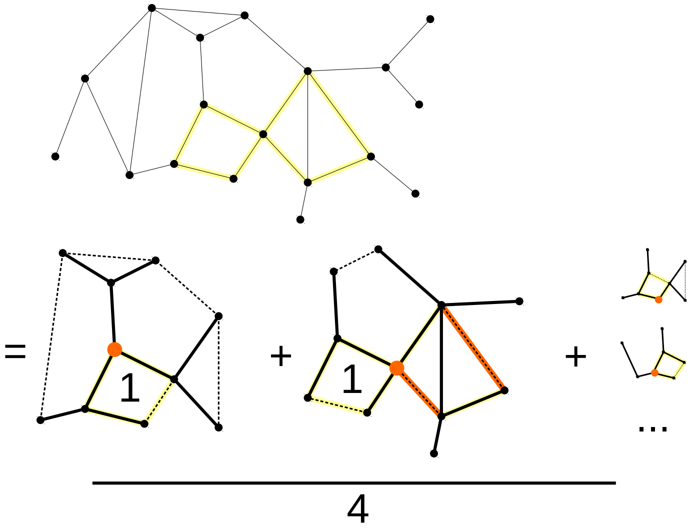

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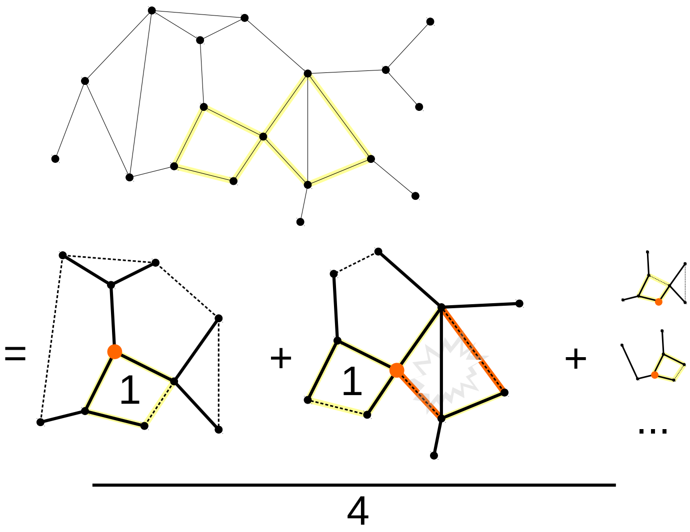

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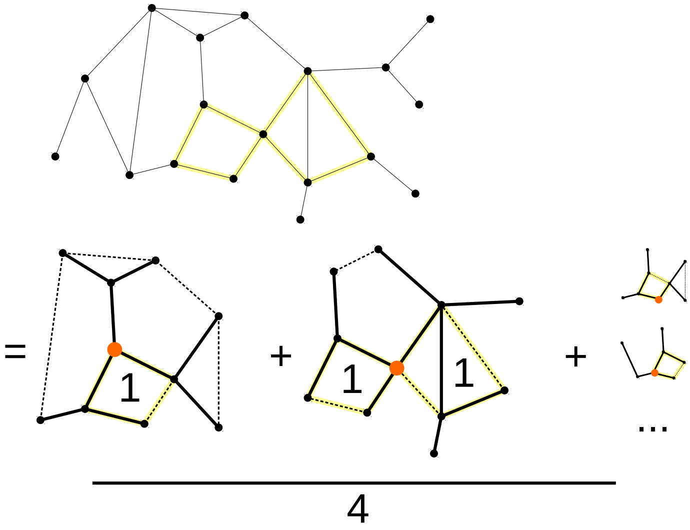

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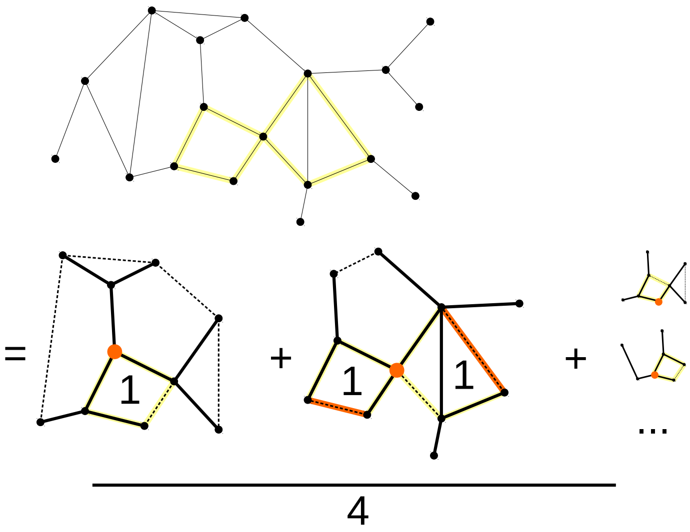

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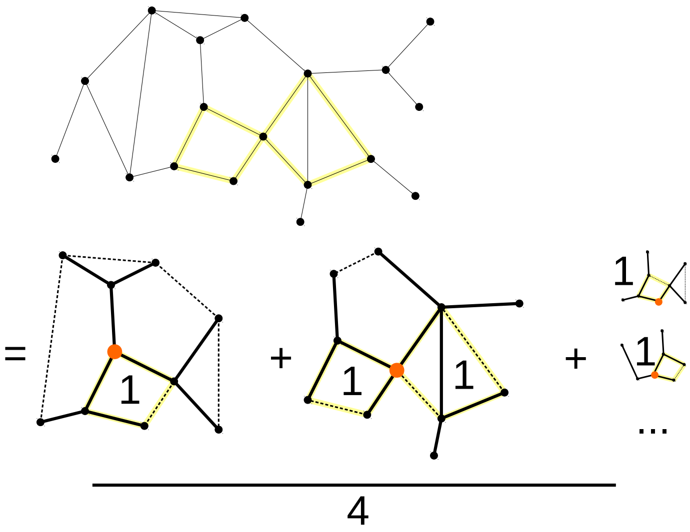

Bounding the Run Time

We need to bound the number
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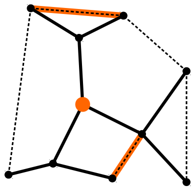

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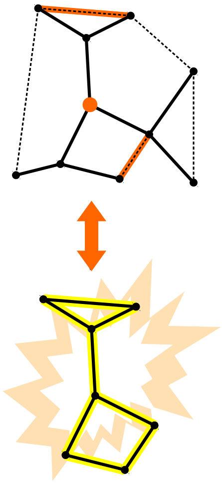

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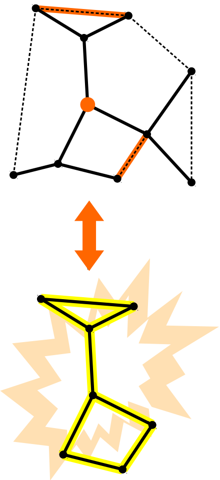

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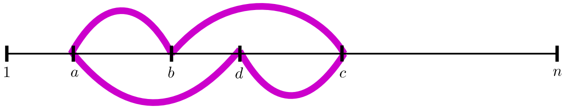
$= \Pr[$  $]$

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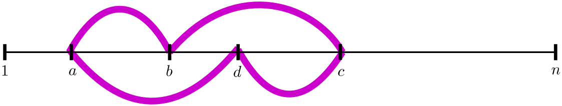

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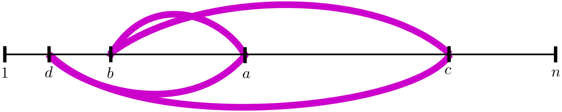

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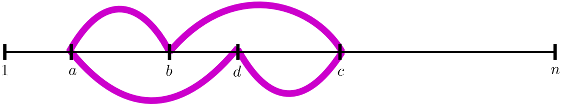
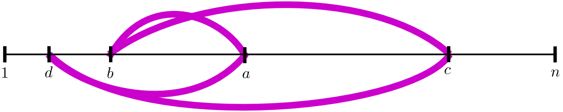
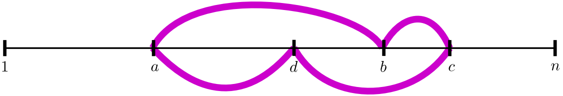
$= \Pr[$  $]$

$+ \Pr[$  $]$

$+ \Pr[$  $]$

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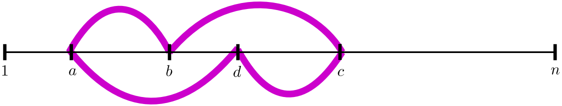
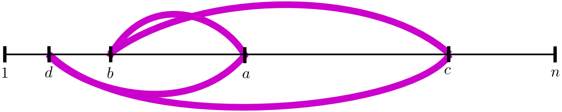
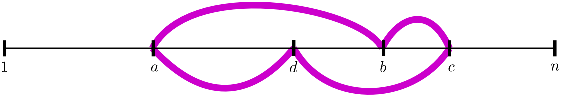
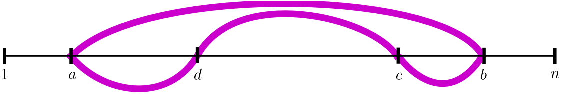
$= \Pr[$  $]$

$+ \Pr[$  $]$

$+ \Pr[$  $]$

$+ \Pr[$  $]$

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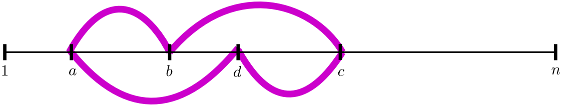
$= \Pr[$ 

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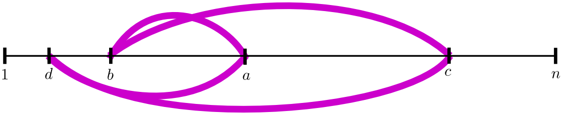
$+ \Pr[$ 

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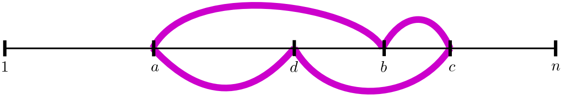
$+ \Pr[$ 

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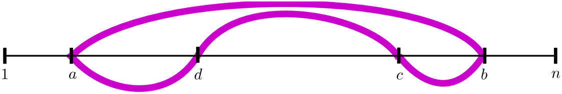
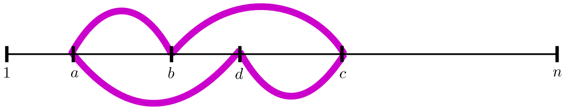
$+ \Pr[$ 

$]$

$+ \cdots$

Bounding Number of Leaf-Free Subgraphs

$\Pr[$ 

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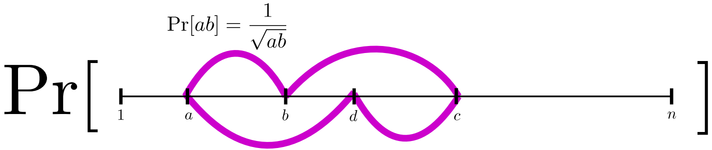
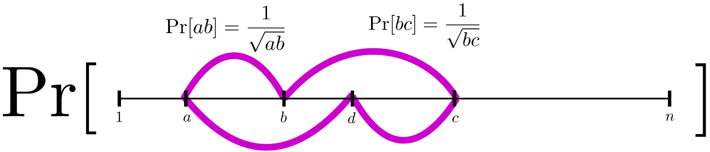
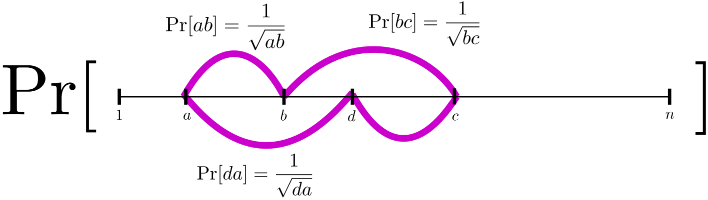
Bounding Number of Leaf-Free Subgraphs



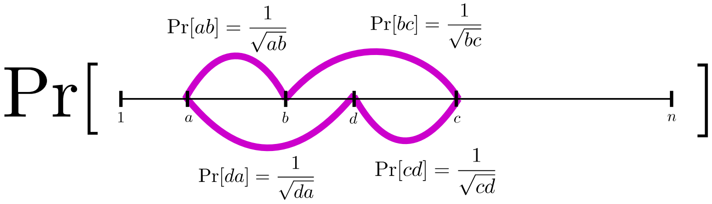
Bounding Number of Leaf-Free Subgraphs



Bounding Number of Leaf-Free Subgraphs

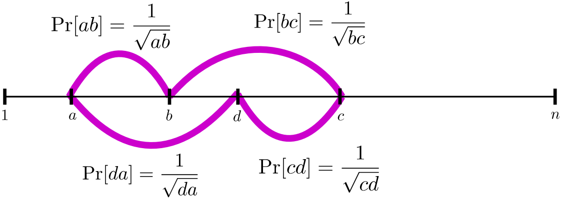


Bounding Number of Leaf-Free Subgraphs



Bounding Number of Leaf-Free Subgraphs

Pr[



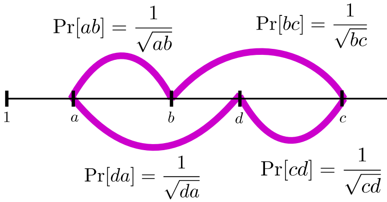
]

$$= \Pr[ab] \cdot \Pr[bc] \cdot \Pr[cd] \cdot \Pr[da]$$

$$= \frac{1}{abcd}$$

Bounding Number of Leaf-Free Subgraphs

Pr[



]

$$= \log(n)^{O(1)} \cdot \Pr[ab] \cdot \Pr[bc] \cdot \Pr[cd] \cdot \Pr[da]$$

$$= \frac{\log(n)^{O(1)}}{abcd}$$

Bounding Number of Leaf-Free Subgraphs

$$\text{expectation of } \# \square \text{ in } G_m^n$$

Bounding Number of Leaf-Free Subgraphs

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$$= \sum_{a=1}^{n}$$

Bounding Number of Leaf-Free Subgraphs

$$\text{expectation of } \# \square \text{ in } G_m^n$$

$$= \sum_{a=1}^{n} \sum_{b=1}^{n}$$

Bounding Number of Leaf-Free Subgraphs

$$\text{expectation of } \# \square \text{ in } G_m^n$$

$$= \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n}$$

Bounding Number of Leaf-Free Subgraphs

expectation of $\# \square$ in G_m^n

$$= \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n}$$

Bounding Number of Leaf-Free Subgraphs

expectation of $\# \square$ in G_m^n

$$= \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n} \frac{\log(n)^{O(1)}}{abcd}$$

Bounding Number of Leaf-Free Subgraphs

$$\text{expectation of } \#\square \text{ in } G_m^n$$

$$= \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n} \frac{\log(n)^{O(1)}}{abcd}$$

$$= \log(n)^{O(1)} \sum_{a=1}^{n} \frac{1}{a} \sum_{b=1}^{n} \frac{1}{b} \sum_{c=1}^{n} \frac{1}{c} \sum_{d=1}^{n} \frac{1}{d}$$

Bounding Number of Leaf-Free Subgraphs

expectation of $\# \square$ in G_m^n

$$= \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n} \frac{\log(n)^{O(1)}}{abcd}$$

$$= \log(n)^{O(1)} \sum_{a=1}^{n} \frac{1}{a} \sum_{b=1}^{n} \frac{1}{b} \sum_{c=1}^{n} \frac{1}{c} \sum_{d=1}^{n} \frac{1}{d}$$

$$= \log(n)^{O(1)}$$

(similarly for every leaf-free graph)

Total Runtime

Analysis of run time:

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- Number of vertices: $O(n)$

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- Subroutine calls per vertex: $\log(n)^{O(1)}$

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With a more careful analysis: $\log(n)^{O(1)} n$

Related: FPT First-Order Model Checking

First-Order Model Checking

- $\exists$ and $\forall$ quantification over vertices, equality, adjacency
- cannot be solved in time $f(|\varphi|)n^{o(|\varphi|)}$ under ETH
- *Grohe, Kreutzer, Siebertz:* for fixed formula decidable in time $O(n^{1+\varepsilon})$ on nowhere-dense graph classes

Theorem

Let $\varepsilon > 0$, $m \in \mathbf{N}$ be fixed.
One can decide every property expressible in FO logic on G^n_m in expected time $O(n^{1+\varepsilon})$.

Thank you!