TWO NEW PERSPECTIVES FOR ALGORITHMIC META-THEOREMS

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Algorithmic Meta-Theorems

Everything is a graph.
Algorithmic Meta-Theorems

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“All problems expressible in a certain logic can be solved efficiently on certain graphs”
Algorithmic Meta-Theorems

“All problems expressible in a certain logic can be solved efficiently on certain graphs”

MSO on treewidth
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- MSO on treewidth
- FO on sparse graphs
- FO(>{0}) for approximation on sparse graphs
- FO on unclustered complex networks
Many problems can be expressed in first-order (FO) logic.
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- independent set of size $k$:

$$
\exists x_1 \ldots \exists x_k \bigwedge_{i,j} x_i \not\sim x_j \land x_i \neq x_j
$$

- dominating set of size $k$:
Many problems can be expressed in first-order (FO) logic.

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- dominating set of size $k$:

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  \exists x_1 \ldots \exists x_k \forall y \bigvee_i y \sim x_i \lor y = x_i
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Many problems can be expressed in first-order (FO) logic.

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- some database queries
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Best algorithms on general graphs: $n^{O(k)}$
$\phi = \exists x_1 \ldots \exists x_k$ (length depends on $k$)

$k$-independent set

$k$-dominating set

...
Model-Checking

$\kappa$-independent set

$\kappa$-dominating set

logic

model-checking

...
Model-Checking

\[ \phi = \exists x_1 \ldots \exists x_k [... ] \]

length depends on \( k \)

\[ f(|\phi|)n \]

Algorithm

**MC(\( G, L \))**

*Input:* A graph \( G \in G \) and a sentence \( \phi \in L \)

*Parameter:* \(|\phi|\)

*Problem:* Is \( \phi \) true in \( G \)?

*Goal:* linear FPT run time \( f(|\phi|)n \)
Sparse Graph Classes

If $G$ has bounded treewidth then $\text{MC}(G, \text{MSO}) \in \text{FPT}$.

[Courcelle 1990]
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If $\mathcal{G}$ is nowhere dense then $MC(\mathcal{G}, \text{FO}) \in \text{FPT}$.

[Grohe, Kreutzer, Sieberz 2011]
If \( G \) has bounded treewidth then \( MC(G, MSO) \in FPT \).

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[Grohe, Kreutzer, Sieberz 2011]
Approximate Evaluation of First-Order Counting Queries
**Partial Dominating Set**

*Input:* A graph $G$ and $k, m \in \mathbb{N}$

*Parameter:* $k$

*Problem:* Are there $k$ vertices dominating $m$ vertices?

**Cannot be expressed in first-order logic (requires $\exists y_1 \ldots \exists y_m$).**

Can be solved on $H$-minor free graphs in time $(g(H)k)^{k n^{O(1)}}$.

[Amini, Fomin, Saurabh, 2008]

Can be solved on apex-minor-free graphs in time $2^{\sqrt{k}n^{O(1)}}$.

[Fomin, Lokshtanov, Raman, Saurabh, 2011]

Is W[1]-hard for 2-degenerate graphs.

[Golovach, Villanger 2008]
### Partial Dominating Set

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### Partial Dominating Set

**Input:** A graph $G$ and $k, m \in \mathbb{N}$

**Parameter:** $k$

**Problem:** Are there $k$ vertices dominating $m$ vertices?

\[
\exists x_1 \ldots \exists x_k \# y (\bigvee_i y \sim x_i \land y = x_i) \geq m
\]

Length of formula depends only on $k$ (and not on $m$)

\[\text{FO}\{>0\} = \text{FO} + \text{“there are at least/most } m \in \mathbb{N} \text{ elements”}\]
Partial Dominating Set

Input: A graph $G$ and $k, m \in \mathbb{N}$

Parameter: $k$

Problem: Are there $k$ vertices dominating $m$ vertices?

$\text{FO}(\{>0\}) = \text{FO} + \text{“there are at least/most } m \in \mathbb{N} \text{ elements”}$

$$\exists x_1 \ldots \exists x_k \#y \left( \bigvee_i y \sim x_i \land y = x_i \right) \geq m$$
Partial Dominating Set

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Problem: Are there $k$ vertices dominating $m$ vertices?

$FO(\{>0\}) = FO + \text{“there are at least/most } m \in \mathbb{N} \text{ elements”}$

$$\exists x_1 \ldots \exists x_k \#y (\bigvee_i y \sim x_i \land y = x_i) \geq m$$

Length of formula depends only on $k$ (and not on $m$)
Counting Logic

Definition of $\text{FO}(\{>0\})$:

Built recursively using:
- The rules of $\text{FO}$
- $\#y \varphi \geq m$ for every $m \in \mathbb{N}$ and $\text{FO}(\{>0\})$ formula $\varphi$

Example 1: PARTIAL DOMINATING SET

$$\exists x_1 \ldots \exists x_k \#y \left( \bigvee_{i} y \sim x_i \land y = x_i \right) \geq m$$

Example 2: $h$-Index

$$\#\text{mypaper} \left( \#\text{otherpaper cite(otherpaper, mypaper)} \geq h \right) \geq h$$
If $\mathcal{G}$ has bounded degree then $\text{MC}(\mathcal{G}, \text{FOC}) \in \text{FPT}$.

[Kuske, Schweikardt 2017]
If $\mathcal{G}$ has bounded degree then $\text{MC}(\mathcal{G}, \text{FOC}) \in \text{FPT}$.

[Kuske, Schweikardt 2017]

$\text{MC}(\mathcal{G}, \text{FO}(\{> 0\}))$ is $\text{AW}[\ast]$-hard on trees.

similar to [Grohe, Schweikardt 2018]
Bad News

contains k-clique

\(\iff\)

satisfies FO(\(>0\)) formula
Bad News

contains k-clique

\iff
satisfies FO(\{>0\}) formula
Bad News

contains k-clique

satisfies FO(\{>0\}) formula
Bad News

contains k-clique

satisfies FO(\{>0\}) formula
Are there $k$ vertices dominating at least $m = 5000$ vertices?
Are there \( k \) vertices dominating at least \( m = 4983 \) vertices?
Are there $k$ vertices dominating at least $m = 5017$ vertices?
Are there \( k \) vertices dominating at least \( m = 5017 \) vertices?

A formula \( \varphi \) is \( \varepsilon \)-stable on a graph \( G \) if scaling the counting literals by \( (1 \pm \varepsilon) \) does not change whether \( \varphi \) is true in \( G \).
**Theorem**

Let $\mathcal{G}$ be a graph class with bounded expansion and $\varepsilon > 0$. 
Theorem

Let $\mathcal{G}$ be a graph class with bounded expansion and $\varepsilon > 0$. There exists an algorithm which takes $G \in \mathcal{G}$, $\varphi \in \text{FO}(\{>0\})$, runs in time $f(|\varphi|)n$ and returns 😊, 😐, or 😞.
Approximate Model-Checking

**Theorem**

Let $\mathcal{G}$ be a graph class with bounded expansion and $\varepsilon > 0$. There exists an algorithm which takes $G \in \mathcal{G}$, $\varphi \in \text{FO}(\{>0\})$, runs in time $f(|\varphi|)n$ and returns ☻, ☹, or ☹.  

- If ☻ then $\varphi$ is true on $G$.  

- If ☹ then $\varphi$ is false on $G$.  

- If ☹ then $\varphi$ is $\varepsilon$-unstable on $G$.  


Theorem

Let $\mathcal{G}$ be a graph class with bounded expansion and $\varepsilon > 0$. There exists an algorithm which takes $G \in \mathcal{G}$, $\varphi \in \text{FO}(\{ > 0 \})$, runs in time $f(\|\varphi\|)n$ and returns $\bigcirc$, $\bigcirc$, or $\bigcirc$.

- If $\bigcirc$ then $\varphi$ is true on $G$.
- If $\bigcirc$ then $\varphi$ is false on $G$. 
Theorem

Let $\mathcal{G}$ be a graph class with bounded expansion and $\varepsilon > 0$. There exists an algorithm which takes $G \in \mathcal{G}$, $\varphi \in \text{FO}(\{>0\})$, runs in time $f(|\varphi|)n$ and returns $\begin{cases} \text{green}, & \text{if true} \\ \text{yellow}, & \text{if false} \\ \text{red}, & \text{if } \varepsilon \text{-unstable} \end{cases}$. 

- If $\begin{cases} \text{green}, & \text{true} \\ \text{red}, & \text{false} \end{cases}$ then $\varphi$ is true on $G$.
- If $\begin{cases} \text{yellow}, & \text{false} \end{cases}$ then $\varphi$ is false on $G$.
- If $\begin{cases} \text{red}, & \text{true} \end{cases}$ then $\varphi$ is $\varepsilon$-unstable on $G$. 
**PARTIAL DOMINATING SET:** \( \exists x_1 \ldots \exists x_k \#y \left( \bigvee_i y \sim x_i \land y = x_i \right) \geq m \)
Partial Dominating Set: \( \exists x_1 \ldots \exists x_k \#y (\bigvee_i y \sim x_i \land y = x_i) \geq m \)

There exists a set dominating \( \geq (1 + \varepsilon)m \) vertices.
Partial Dominating Set: \(\exists x_1 \ldots \exists x_k \#y \left( \bigvee_i y \sim x_i \land y = x_i \right) \geq m\)

There exists a set dominating \(\geq (1 + \varepsilon)m\) vertices.

All sets dominate \(< (1 - \varepsilon)m\) vertices.
**Partial Dominating Set:** \( \exists x_1 \ldots \exists x_k \# y \left( \bigvee_i y \sim x_i \land y = x_i \right) \geq m \)

There exists a set dominating \( \geq (1 + \varepsilon)m \) vertices.

All sets dominate \(< (1 + \varepsilon)m \) vertices and there exists a set dominating \( \geq (1 - \varepsilon)m \) vertices.

All sets dominate \(< (1 - \varepsilon)m \) vertices.
How about extensions of $\text{FO}(\{>0\})$?

$\text{FO}(\{>0\})$ allows comparing $\#y$ and $m \in \mathbb{N}$.

**Theorem**

Approximate model-checking becomes hard if also allow one of the following:

- Comparing $\#y$ and $\#z$ (e.g., $\#y \phi > \#z \psi$)
- Counting tuples $\#yz$ (e.g., $\#yz \phi > m$)
- Multiplying of counting terms (e.g., $\#y \phi \cdot \#z \psi > m$)
- Subtraction of counting terms (e.g., $\#y \phi - \#z \psi > m$)
How about extensions of $\text{FO} \{> 0\}$?

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Approximate model-checking becomes hard if also allow one of the following:

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How about extensions of FO($\{\geq 0\}$)?

FO($\{\geq 0\}$) allows comparing $\#y$ and $m \in \mathbb{N}$.

**Theorem**

Approximate model-checking becomes hard if also allow one of the following:

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How about extensions of FO(\{ > 0 \})?

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**Theorem**

Approximate model-checking becomes hard if also allow one of the following:

- comparing \#y and \#z  
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- counting tuples \#yz  
  (e.g., \#yz \varphi > m)
- multiplying of counting terms  
  (e.g., \#y \varphi \cdot \#z \psi > m)
- subtraction of counting terms  
  (e.g., \#y \varphi - \#z \psi > m)
Summary

$\text{FO}\{>0\}$ is

- hard to solve exactly on trees,
Summary

$\text{FO}(\{ > 0 \})$ is

- hard to solve exactly on trees,
- possible to approximate on bounded expansion.
Summary

$\text{FO}(\{>0\})$ is

- hard to solve exactly on trees,
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Slight extensions of $\text{FO}(\{>0\})$ are

- hard to approximate on trees.
FO($\{ > 0 \}$) is

- hard to solve exactly on trees,
- possible to approximate on bounded expansion.

Slight extensions of FO($\{ > 0 \}$) are

- hard to approximate on trees.

⇒ FO($\{ > 0 \}$) seems like “the right logic” for approximation on sparse graphs
Can we generalize our results to nowhere dense graph classes?
Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( \right) \]
Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

\[ m_1 \leq \# x_1 \left( m_2 \leq \# x_2 \left( \quad \right) \right) \]
We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( m_2 \leq \#x_2 \left( m_3 \leq \#x_3 \right) \right) \]
Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( m_2 \leq \#x_2 \left( m_3 \leq \#x_3 \varphi(x_1x_2x_3) \right) \right) \]
We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( m_2 \leq \#x_2 \left( m_3 \leq \#x_3 \varphi(x_1x_2x_3) \right) \right) \]

replace with quantifier-free FO

quantifier-free FO
Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( m_2 \leq \#x_2 \quad \phi'(x_1 x_2) \right) \]
We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( m_2 \leq \#x_2 \left[ \varphi'(x_1x_2) \right] \right) \]

replace with quantifier-free FO
We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \quad \phi''(x_1) \]
We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \quad \forall \varphi''(x_1) \quad \text{replace with quantifier-free FO} \]
We want to gradually simplify this formula.

quantifier-free FO

\varphi'\prime\prime\prime
Proof Sketch – Domination

\[ \#x_3 \ (x_3 \sim x_1 \lor x_3 \sim x_2) \geq m \]

replace with quantifier-free FO

\[ N(x_1) \cap N(x_2) \geq m \]
Proof Sketch – Small Intersection

\[ \approx m \]

\[ N(x_1) \cap N(x_2) \]

relatively small intersection
Proof Sketch – Small Intersection

\[ N(x_1) \cap N(x_2) \approx m \]

\[ N(x_1) + N(x_2) - \text{relatively small intersection} \approx m \]
Proof Sketch – Small Intersection

\[ N(x_1) \quad \text{relatively small intersection} \quad N(x_2) \]

\[ \geq m \]

\[ N(x_1) \quad + \quad N(x_2) \]

\[ \geq m \]
Proof Sketch – Small Intersection

\[ N(x_1) \approx m \]

\[ \approx m \]

\[ R_{\geq i}(x) \text{ true} \]

iff \[ |N(x)| \geq i \]
Proof Sketch – Small Intersection

\[ x_1 \quad \text{relatively small intersection} \quad x_2 \]

\[ N(x_1) \quad \nleftrightarrow \quad m \quad \nleftrightarrow \quad N(x_2) \]

\[ x_1 \quad \nleftrightarrow \quad m \quad \nleftrightarrow \quad x_2 \]

\[ N(x_1) \quad + \quad N(x_2) \quad \nleftrightarrow \quad m \]

\[ \bigvee_{i \in \{0, 1, 2, \ldots, m\}} R_{\geq i}(x_1) \land R_{\geq m-i}(x_2) \quad \text{iff} \quad |N(x)| \geq i \]

\[ R_{\geq i}(x) \text{ true} \]
Proof Sketch – Small Intersection

\[
\vee \quad R_{\geq i}(x_1) \land R_{\geq m-i}(x_2)
\]

\[i \in \{0 \varepsilon m, 1 \varepsilon m, 2 \varepsilon m, \ldots, m\}\]

\[\text{iff } |N(x)| \geq i\]
Proof Sketch – Small Intersection

\[ \frac{1}{\varepsilon} \bigvee_{i=0}^{1/\varepsilon} R_{\geq \varepsilon m_i}(x_1) \land R_{\geq m - \varepsilon m_i}(x_2) \]
Proof Sketch – Small Intersection

\[ \frac{1}{\varepsilon} \bigvee_{i=0}^{\frac{1}{\varepsilon}} R_{\geq \varepsilon m_i}(x_1) \land R_{\geq m - \varepsilon m_i}(x_2) \]
Proof Sketch – Small Intersection

\[
\frac{1}{\varepsilon} \bigvee_{i=0}^{R_{\geq \varepsilon mi}(x_1) \land R_{\geq m-\varepsilon mi}(x_2)}
\]

Diagram showing the intersection of regions with points \(x_1\) and \(x_2\).
Proof Sketch – Small Intersection

$$\frac{1}{\epsilon} \bigvee_{i=0}^{n} R_{\geq \epsilon m i}(x_1) \land R_{\geq m - \epsilon m i}(x_2)$$
Proof Sketch – Small Intersection

\[
\frac{1}{\varepsilon} \bigvee_{i=0}^{\varepsilon m} R_{\geq \varepsilon m i}(x_1) \land R_{\geq m - \varepsilon m i}(x_2)
\]
Proof Sketch – Small Intersection

\[ \frac{1}{\varepsilon} \bigvee_{i=0}^{\frac{1}{\varepsilon} \varepsilon m} R_{\geq \varepsilon m_i}(x_1) \land R_{\geq m - \varepsilon m_i}(x_2) \]
Proof Sketch – Small Intersection

\[ \frac{1}{\varepsilon} \bigvee_{i=0}^{1/\varepsilon} R_{\geq \varepsilon m}(x_1) \land R_{\geq m-\varepsilon m}(x_2) \]

Diagram:

- 
  - 
  - 
  - 
  - 
  - 
  - 

- \( x_1 \)
- \( x_2 \)

unstable
Proof Sketch – Large Intersection

\[ N(x_1) \cap N(x_2) \geq m \]

relatively large intersection
Proof Sketch – Large Intersection

\[ N(x_1) \cap N(x_2) \supseteq m \]

many \( x_2 \) with large intersection

\( x_1 \)
Proof Sketch – Large Intersection

\[ \forall \exists m \]

Many \( x_2 \) with large intersection \( \implies \) large 1-subdivided clique
Proof Sketch – Large Intersection

relatively large intersection

$N(x_1) \bigcap N(x_2) \cong m$

many $x_2$ with large intersection $\implies$ large 1-subdivided clique $\implies$ not bounded expansion

$\exists x_1$

$\exists x_2$
Proof Sketch – Large Intersection

\[ N(x_1) \cap N(x_2) \supseteq m \]

We assume (for simplicity) \( x_1 \) has only one \( x_2 \) with a large intersection.

Many \( x_2 \) with large intersection \( \implies \) large 1-subdivided clique \( \implies \) not bounded expansion

\[ x_1 \]

\[ x_2 \]

\[ x_2 \]

\[ x_2 \]

\[ x_2 \]
We assume (for simplicity) $x_1$ has only one $x_2$ with a large intersection. We call it $f(x_1)$. 

\[ \Rightarrow m \]

| many $x_2$ with large intersection | $\Rightarrow$ large 1-subdivided clique | $\Rightarrow$ not bounded expansion |
Proof Sketch – Large Intersection

\[ \forall x_1: N(x_1) \cap N(f(x_1)) \geq m \]

\[ Q_f(x) \text{ true if and only if } |N(x) \cup N(f(x))| \geq m \]
Proof Sketch – Large Intersection

\[ x_1 \text{ relatively large intersection } f(x_1) \]

\[ N(x_1) \cap N(f(x_1)) \ni m \]

\[ Q_f(x) \text{ true iff } |N(x) \cup N(f(x))| \geq m \]

**Final Formula:**

\[ (x_2 = f(x_1) \land Q_f(x_1)) \]
Proof Sketch – Large Intersection

\[ x_1 \text{ relatively large intersection } f(x_1) \]

\[ N(x_1) \cap N(f(x_1)) \supseteq m \]

\[ Q_f(x) \text{ true iff } |N(x) \cup N(f(x))| \geq m \]

Final Formula:

\[
\left( x_2 = f(x_1) \land Q_f(x_1) \right) \lor \\
\left( x_2 \neq f(x_1) \land \varphi_{\text{small}}(x_1, x_2) \right)
\]
Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( m_2 \leq \#x_2 \left( m_3 \leq \#x_3 \varphi(x_1x_2x_3) \right) \right) \]

replace with quantifier-free FO

quantifier-free FO
We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( m_2 \leq \#x_2 \right) \left( \varphi'(x_1 x_2) \right) \]

quantifier-free FO
We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \left( m_2 \leq \#x_2 \theta'(x_1x_2) \right) \]

replace with quantifier-free FO

quantifier-free FO
Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \quad \phi''(x_1) \]

quantifier-free FO
We want to gradually simplify this formula.

\[ m_1 \leq \#x_1 \phi''(x_1) \]

replace with quantifier-free FO
Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

quantifier-free FO

ϕ′′′
First-Order Model-Checking in Random Graphs and Complex Networks
Motivation

real world networks
real world networks

friendship networks
coauthor networks
internet backbone
world wide web
power-grid
biological processes
...
Motivation

real world networks

network science

random graph models

friendship networks
coauthor networks
internet backbone
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...

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- friendship networks
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profit?

engineering

practical algorithms?
The Real World

Some central properties:

- **Skewed degree distribution**: Fraction of vertices with degree \( k \) proportional to \( k^{-\alpha} \) with \( 2 \leq \alpha \leq 3 \).

- **Clustered**: If we have a common friend, we are likely friends as well.

- **Small-world property**: Everyone is close to everyone.
The Real World

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- Skewed degree distribution
  - Fraction of vertices with degree $k$ proportional to $k^{-\alpha}$ with $2 \leq \alpha \leq 3$

- Clustered
  - If we have a common friend we are likely friends as well

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  - Everyone is close to everyone
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Example: Preferential Attachment Model

Introduced by Barabási and Albert in 1999 to explain the structure of the world wide web.
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\[
\text{expected degree } \approx \sqrt{\frac{n}{i}}
\]
Previous Results

Some are here → Somewhere Dense

Nowhere Dense

and some here → Bounded Expansion

(Top.) Minor Free

Planar  Bounded Degree  Bounded Treewidth

[Grohe 2001], [Farrell et. al. 2015], [Demaine et. al. 2019], [Dreier et. al. 2020]
Meta-Theorem

A random graph model is 3-power-law-bounded if (roughly speaking):

\[ \frac{\text{fraction of vertices with degree } k}{k^3} \] is typically

- real networks: typically clustered
- optimal unclustered real networks:

Big Question: model-checking on clustered models?
Meta-Theorem

A random graph model is \textit{3-power-law-bounded} if (roughly speaking):

- fraction of vertices with degree $k$ is $O(k^{-3})$
- \textbf{real networks: typically $k^{-\alpha}$ with $2 \leq \alpha \leq 3$}
Meta-Theorem

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- fraction of vertices with degree $k$ is $O(k^{-3})$
  
  real networks: typically $k^{-\alpha}$ with $2 \leq \alpha \leq 3$

- unclustered
  
  real networks: typically clustered
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Theorem

Given a first-order sentence $\varphi$ and a graph $G$ sampled from a 3-power-law-bounded model, one can decide whether $\varphi$ is true on $G$ in expected time $f(|\varphi|)n^{1+\varepsilon}$ for every $\varepsilon > 0$. 
Meta-Theorem

A random graph model is \textit{3-power-law-bounded} if (roughly speaking):

- fraction of vertices with degree \( k \) is \( \mathcal{O}(k^{-3}) \)
  
  real networks: typically \( k^{-\alpha} \) with \( 2 \leq \alpha \leq 3 \)

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Meta-Theorem

A random graph model is 3-power-law-bounded if (roughly speaking):

- fraction of vertices with degree $k$ is $O(k^{-3})$
  - real networks: typically $k^{-\alpha}$ with $2 \leq \alpha \leq 3$ (optimal)

- unclustered
  - real networks: typically clustered (???)

Theorem

Given a first-order sentence $\varphi$ and a graph $G$ sampled from a 3-power-law-bounded model, one can decide whether $\varphi$ is true on $G$ in expected time $f(|\varphi|)n^{1+\varepsilon}$ for every $\varepsilon > 0$.

Big Question: model-checking on clustered models?
Example: Chung–Lu Model

A more direct way to get a desirable degree distribution.

\[
\frac{1}{\sqrt{i \cdot j}}
\]

\[
\text{expected degree} \approx \sqrt{\frac{n}{i}}
\]
A random graph model with vertices $1, \ldots, n$ is $3$-power-law-bounded if the probability that some subset of edges $E \subseteq \binom{1, \ldots, n}{2}$ is present is at most

$$\log(n)^O(|E|^2) \prod_{i,j \in E} \frac{1}{\sqrt{i \cdot j}}.$$
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Asymptotic Structure of 3-power-law-bounded models
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radius: $r$
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radius: $r$
Input: graph sampled from 3-power-law-bounded model
Algorithm
Gaifman’s theorem: consider only neighborhoods
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approximately find core
Algorithm

prune trees
prune protrusions
Algorithm

use brute force on core
repeat for every neighborhood
Runtime Analysis

Runtime: $n^{O(1)}$. 
Runtime Analysis

Runtime: \( n^{O(1)} \cdot \sum_{x=1}^{n} \Pr[x = x] \).
Runtime Analysis

Runtime: \( n^{O(1)} \cdot \sum_{x=1}^{n} \Pr\left[ \, \right] \cdot x |\varphi| \)
Runtime Analysis

Runtime: \( n^{O(1)} \cdot \sum_{x=1}^{n} \Pr[x \geq x] \cdot x \cdot |\varphi| \)

To get a run time of \( f(|\varphi|)n^{O(1)} \) we bound

\( \Pr[x \geq x] \) for every \( x \).
A random graph model is 3-power-law-bounded if (roughly speaking):

- fraction of vertices with degree $k$ is $O(k^{-3})$
  - real networks: typically $k^{-\alpha}$ with $2 \leq \alpha \leq 3$

- unclustered
  - real networks: typically clustered

**Theorem**

Given a first-order sentence $\varphi$ and a graph $G$ sampled from a 3-power-law-bounded model, one can decide whether $\varphi$ is true on $G$ in expected time $f(|\varphi|)n^{1+\varepsilon}$ for every $\varepsilon > 0$.

*Big Question*: model-checking on clustered models?
Let $\alpha > 2$. A random graph model $G$ is $\alpha$-power-law-bounded if for every $n \in \mathbb{N}$ there exists an ordering $v_1, \ldots, v_n$ of $V(G_n)$ such that for all $E \subseteq \binom{\{v_1, \ldots, v_n\}}{2}$

$$
\Pr[E \subseteq E(G_n)] \leq \prod_{v_i v_j \in E} \frac{(n/i)^{1/(\alpha-1)}(n/j)^{1/(\alpha-1)}}{n} \cdot \begin{cases}
2^{O(|E|^2)} & \text{if } \alpha > 3 \\
\log(n)^{O(|E|^2)} & \text{if } \alpha = 3 \\
O(n^\epsilon |E|^2) & \text{for every } \epsilon > 0 \text{ if } \alpha < 3.
\end{cases}
$$
A graph $H$ is an $r$-shallow topological minor of a graph $G$ if a graph obtained from $H$ by subdividing every edge up to $2r$ times is isomorphic to a subgraph of $G$. The set of all $r$-shallow topological minors of a graph $G$ is denoted by $G \triangledown r$.

A graph class $\mathcal{C}$ has bounded expansion if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$

$$\max_{H \in G \triangledown r} \frac{||H||}{|H|} \leq f(r).$$