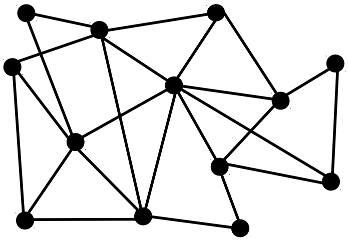
Two New Perspectives for Algorithmic Meta-Theorems

Jan Dreier

November 16 2020

Everything is a graph.



Algorithmic Meta-Theorems

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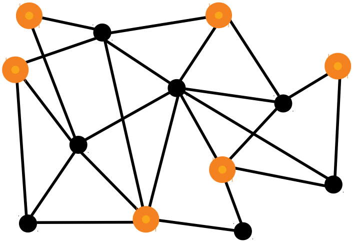

Algorithmic Meta-Theorems

*"All problems expressible in a certain logic
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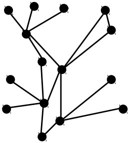
MSO on treewidth



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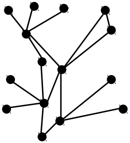

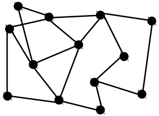

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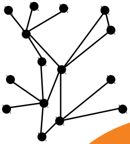

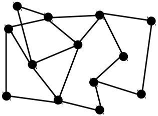

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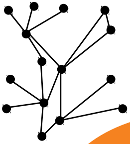

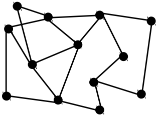

FO({>0}) for ***approximation*** on sparse graphs

FO on unclustered ***complex networks***

Logic

Many problems can be expressed in first-order (FO) logic.

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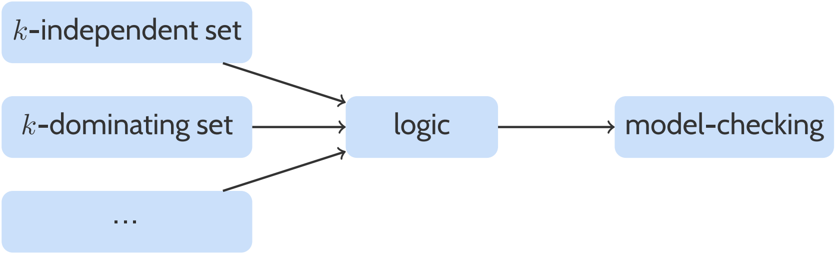
- some database queries

Best algorithms on general graphs: $n^{O(k)}$

Model-Checking



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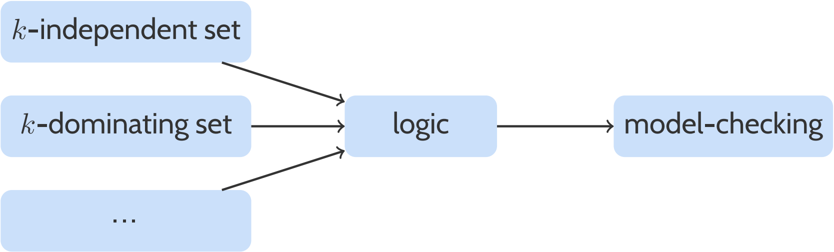

MC($\mathcal{G}$, L)

Input:	A graph $G \in \mathcal{G}$ and a sentence $\varphi \in \mathrm{L}$
Parameter:	$\lvert\varphi\rvert$
Problem:	Is φ true in G?

Goal: linear FPT run time $f(\lvert\varphi\rvert)n$

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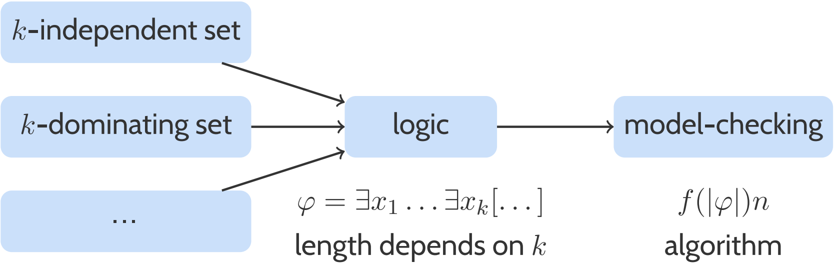

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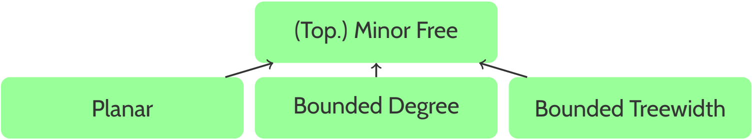
Sparse Graph Classes

Planar | Bounded Degree | Bounded Treewidth

If $\mathcal{G}$ has bounded treewidth then MC($\mathcal{G}$, MSO) $\in$ FPT.

[Courcelle 1990]

Sparse Graph Classes



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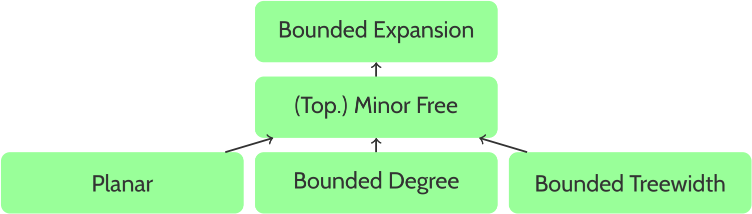

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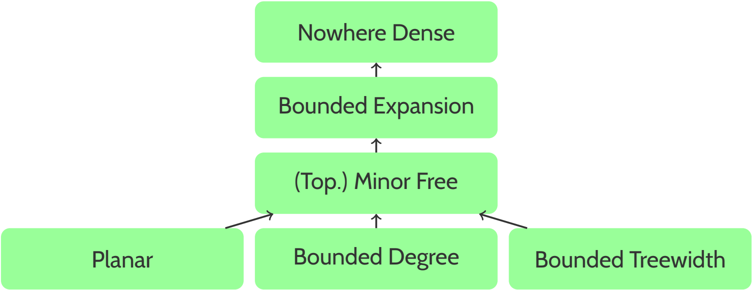

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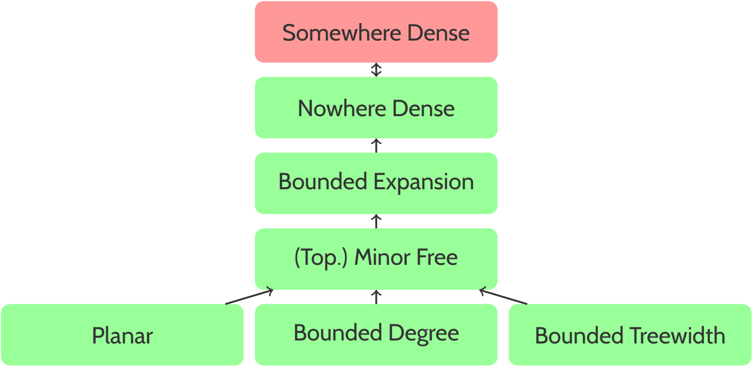

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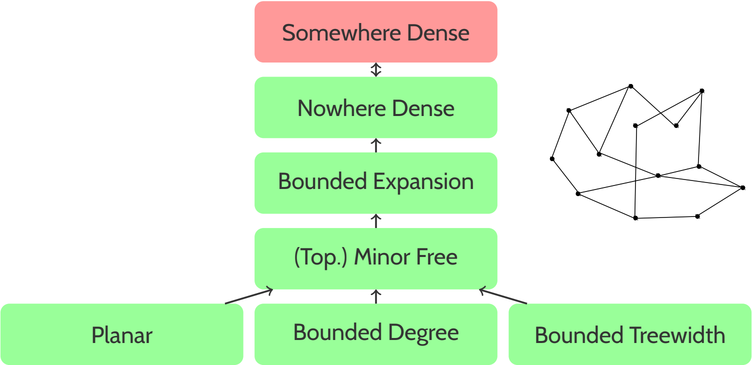

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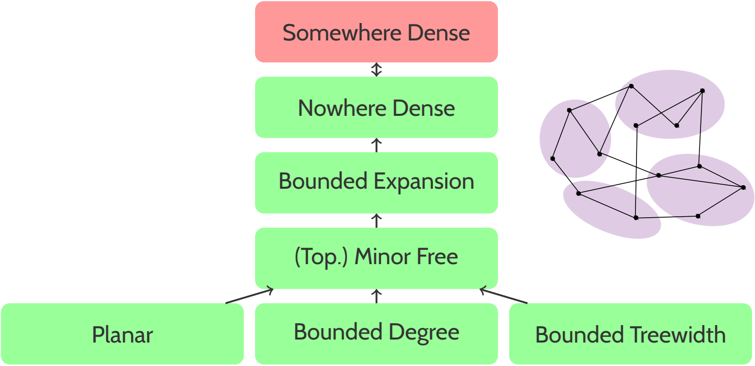

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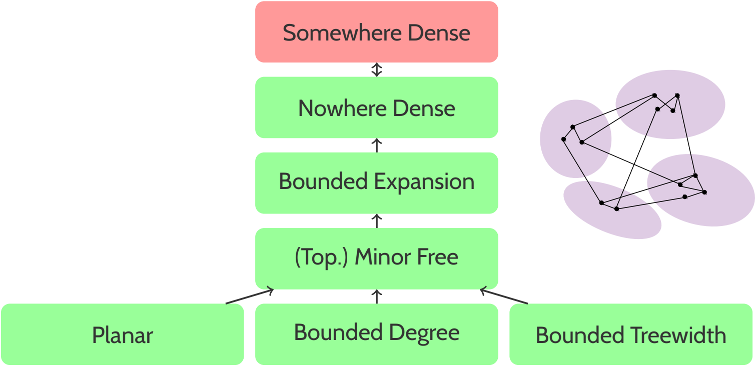

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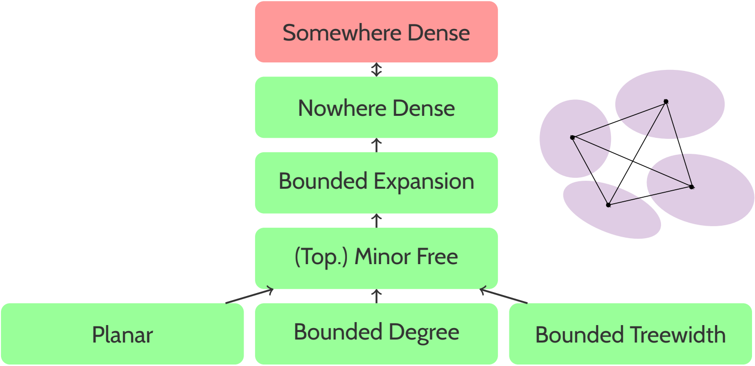

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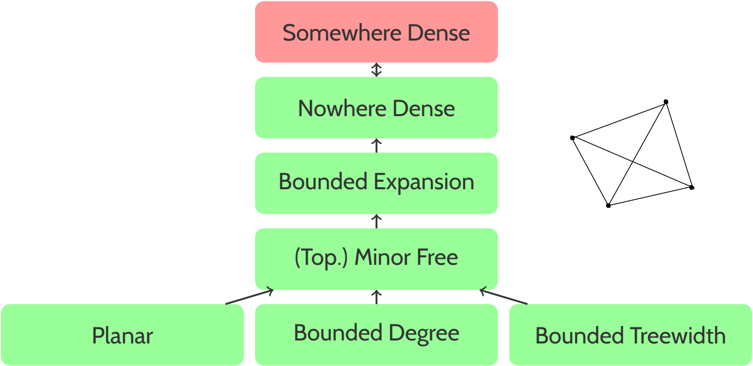

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Approximate Evaluation of First-Order Counting Queries

Counting Problems

Partial Dominating Set

Input:	A graph G and $k, m \in \mathbf{N}$
Parameter:	k
Problem:	Are there k vertices dominating m vertices?

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Can be solved on H-minor free graphs in time $(g(H)k)^k n^{O(1)}$.

[Amini, Fomin, Saurabh, 2008]

Can be solved on apex-minor-free graphs in time $2^{\sqrt{k}} n^{O(1)}$.

[Fomin, Lokshtanov, Raman, Saurabh, 2011]

Is W[1]-hard for 2-degenerate graphs.

[Golovach, Villanger 2008]

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Length of formula depends only on k (and not on m)

Counting Logic

Definition of FO($\{>0\}$)

built recursively using

- the rules of FO
- $\#y\ \varphi \geq m$ for every $m \in \mathbb{N}$ and FO($\{>0\}$) formula φ

Example 1: Partial Dominating Set

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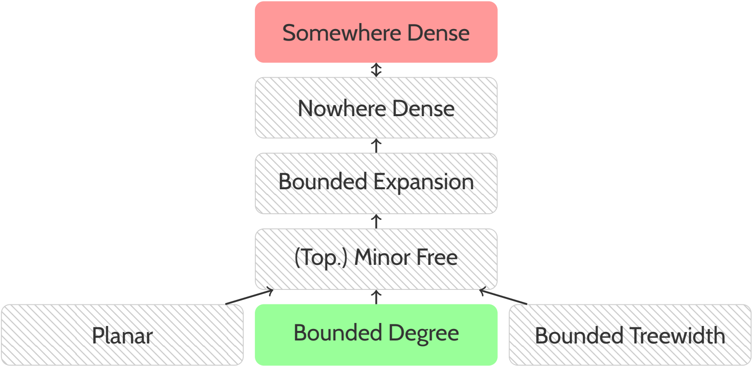
Example 2: h-Index

$$\#\text{mypaper} \left(\#\text{otherpaper cite}(\text{otherpaper}, \text{mypaper}) \geq h \right) \geq h$$

Good News



If $\mathcal{G}$ has bounded degree then MC($\mathcal{G}$, FOC) $\in$ FPT.

[Kuske, Schweikardt 2017]

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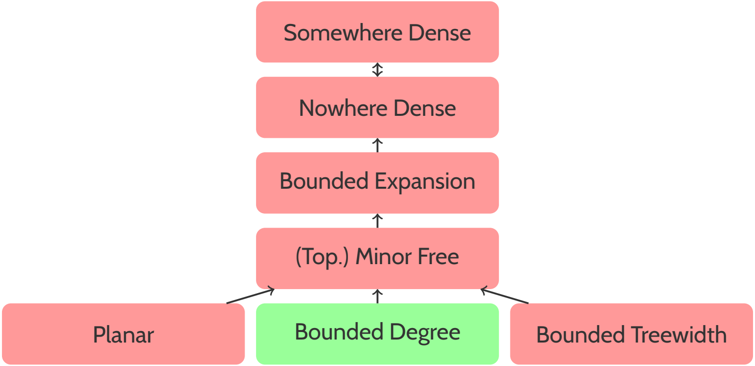

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MC($\mathcal{G}$, FO($\{>0\}$)) is AW[$*$]-hard on trees.

similar to [Grohe, Schweikardt 2018]

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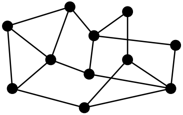
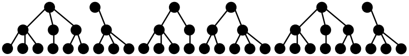


contains k-clique

$\Leftrightarrow$



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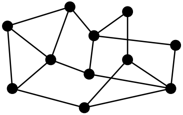

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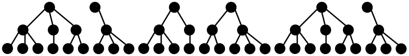

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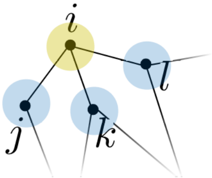

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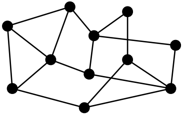

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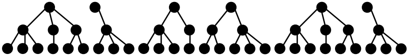

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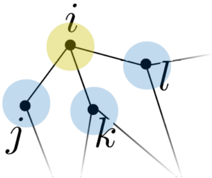

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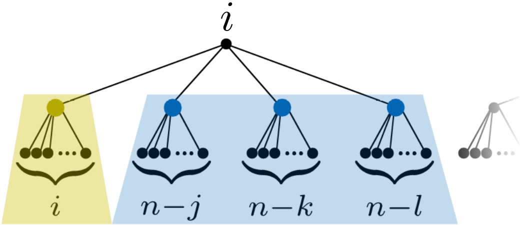

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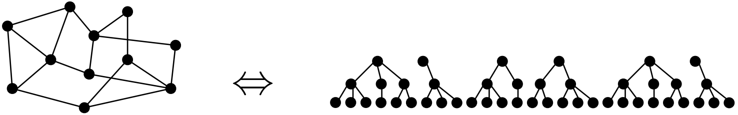
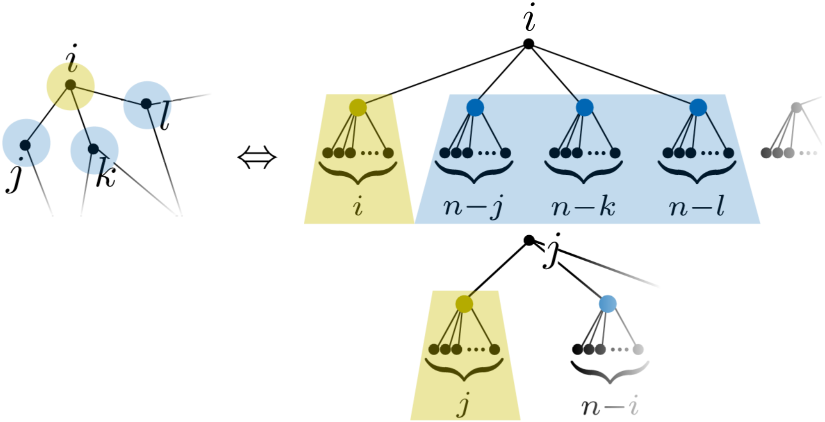
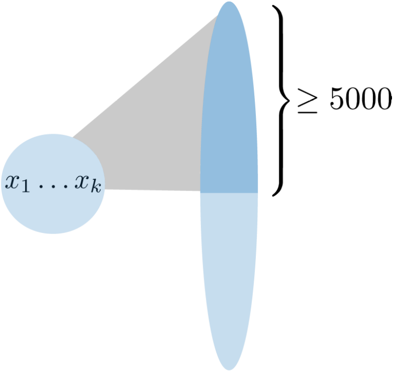


contains k-clique ⇔ satisfies FO({>0}) formula



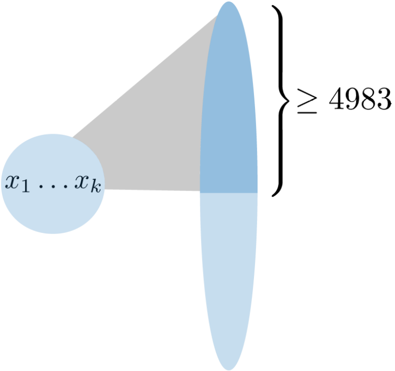
Stability

Are there k vertices dominating at least $m = 5000$ vertices?



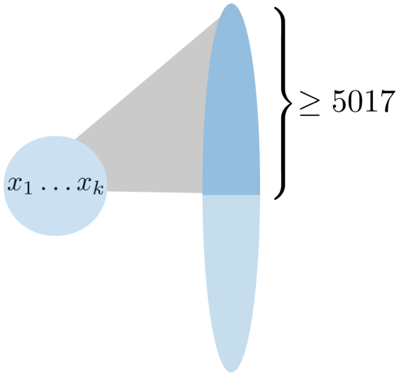
Stability

Are there k vertices dominating at least $m = 4983$ vertices?



Stability

Are there k vertices dominating at least $m = 5017$ vertices?



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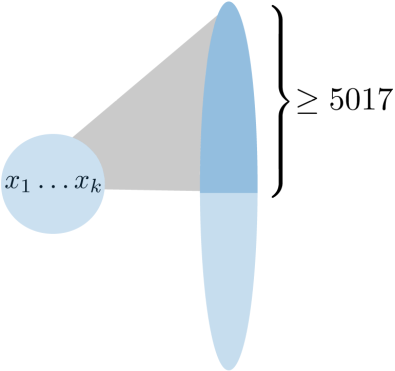

A formula φ is ε-*stable* on a graph G if scaling the counting literals by $(1 \pm \varepsilon)$ does not change whether φ is true in G.

Theorem

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- *If*  *then φ is ε-unstable on G.*

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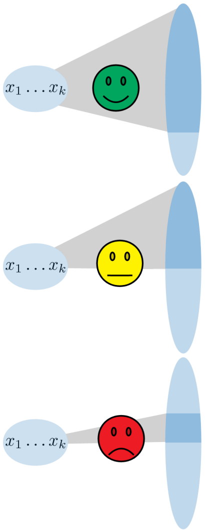
PARTIAL DOMINATING SET: $\exists x_1 \ldots \exists x_k \, \#y \left(\bigvee_i y \sim x_i \wedge y = x_i \right) \geq m$



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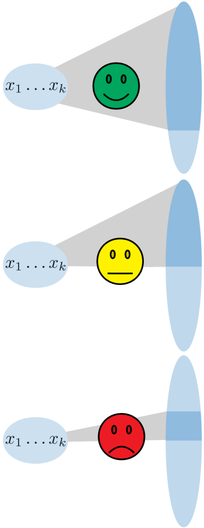

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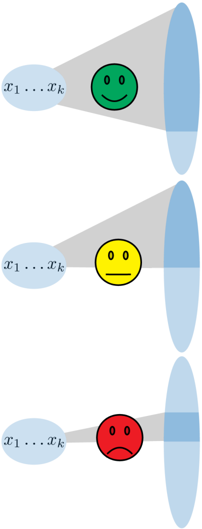

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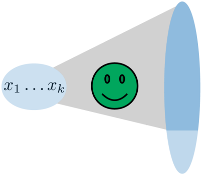

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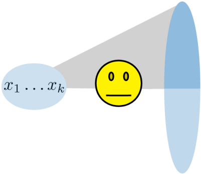
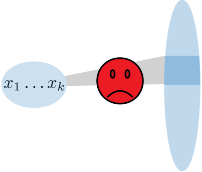


All sets dominate $< (1+\varepsilon)m$ vertices and there exists a set dominating $\geq (1-\varepsilon)m$ vertices.



All sets dominate $< (1-\varepsilon)m$ vertices.

How about extensions of FO($\{>0\}$)?

FO($\{>0\}$) allows comparing $\#y$ and $m \in \mathbf{N}$.

Theorem

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- multiplying of counting terms (e.g., $\#y\ \varphi \cdot \#z\ \psi > m$)
- subtraction of counting terms (e.g., $\#y\ \varphi - \#z\ \psi > m$)

Summary

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Slight extensions of FO($\{>0\}$) are

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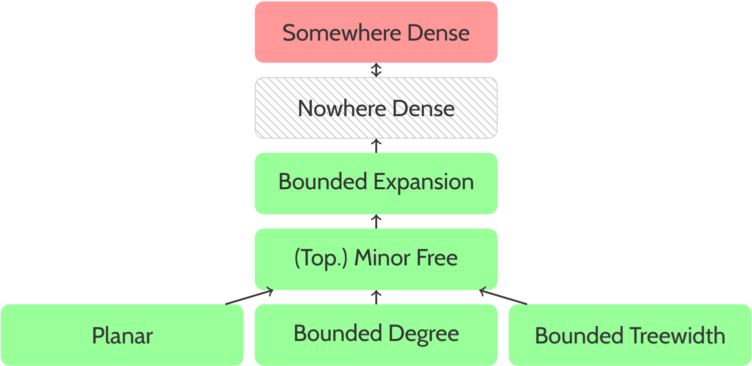
$\Rightarrow$ FO($\{>0\}$) seems like "the right logic" for approximation on sparse graphs

Big Question



Can we generalize our results to nowhere dense graph classes?

Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

$$m_1 \leq \#x_1 \left(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right)$$

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Proof Sketch – Quantifier Elimination

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quantifier-free FO

$$\underbrace{m_1 \leq \#x_1 \quad \overbrace{\varphi''(x_1)}}$$

replace with quantifier-free FO

Proof Sketch – Quantifier Elimination

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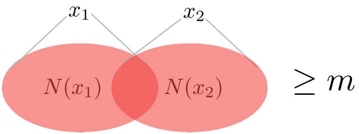
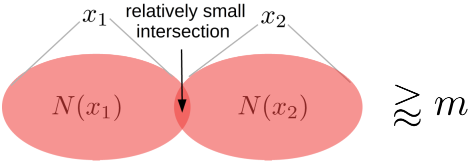
$$\overbrace{\varphi'''}^{\text{quantifier-free FO}}$$

Proof Sketch – Domination

$$\underbrace{\#x_3\ (x_3 \sim x_1 \vee x_3 \sim x_2) \geq m}_{\text{replace with quantifier-free FO}}$$



Proof Sketch – Small Intersection



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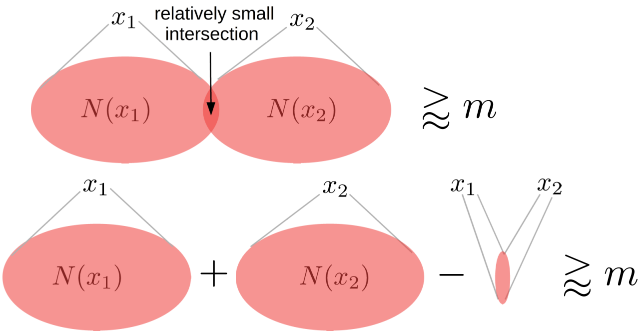

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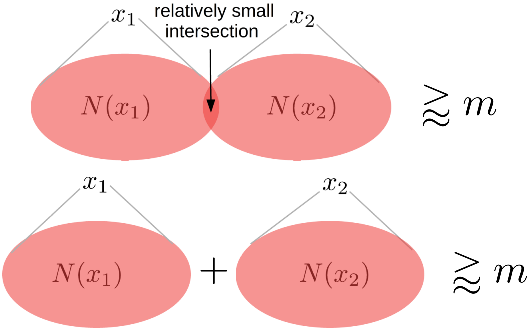

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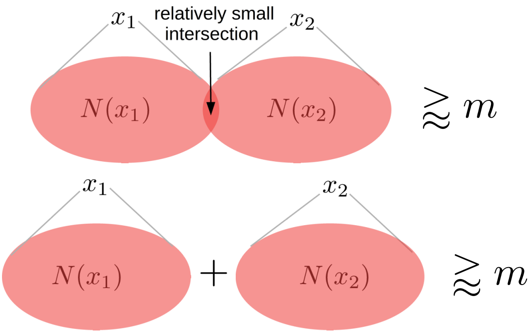
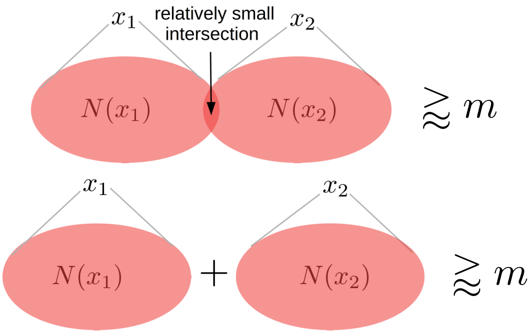


$R_{\geq i}(x)$ true
iff $|N(x)| \geq i$

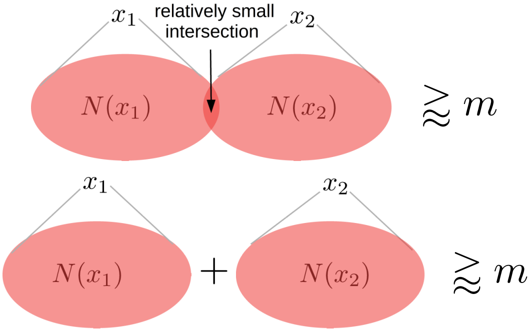
Proof Sketch – Small Intersection



$$\bigvee_{i \in \{0,1,2,\ldots,m\}} R_{\geq i}(x_1) \wedge R_{\geq m-i}(x_2)$$

$R_{\geq i}(x)$ true iff $|N(x)| \geq i$

Proof Sketch – Small Intersection



$$\bigvee_{i \in \{0\varepsilon m, 1\varepsilon m, 2\varepsilon m, \ldots, m\}} R_{\geq i}(x_1) \wedge R_{\geq m-i}(x_2)$$

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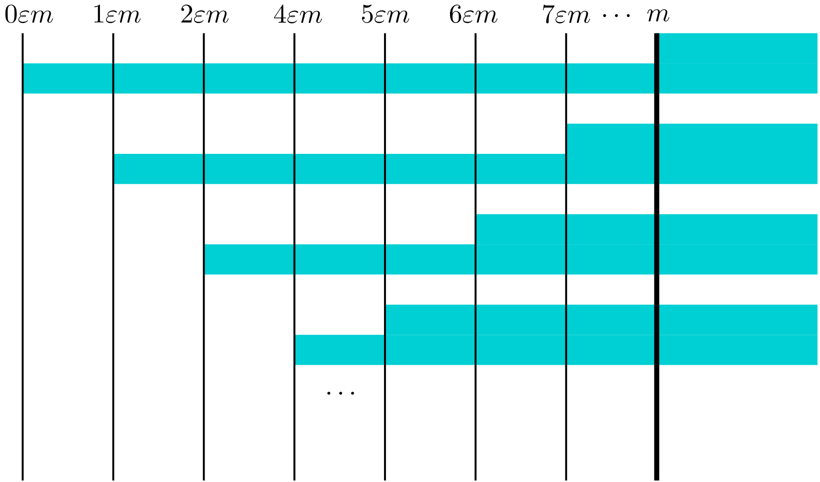
$$\bigvee_{i=0}^{1/\varepsilon} R_{\geq \varepsilon m i}(x_1) \wedge R_{\geq m - \varepsilon m i}(x_2)$$



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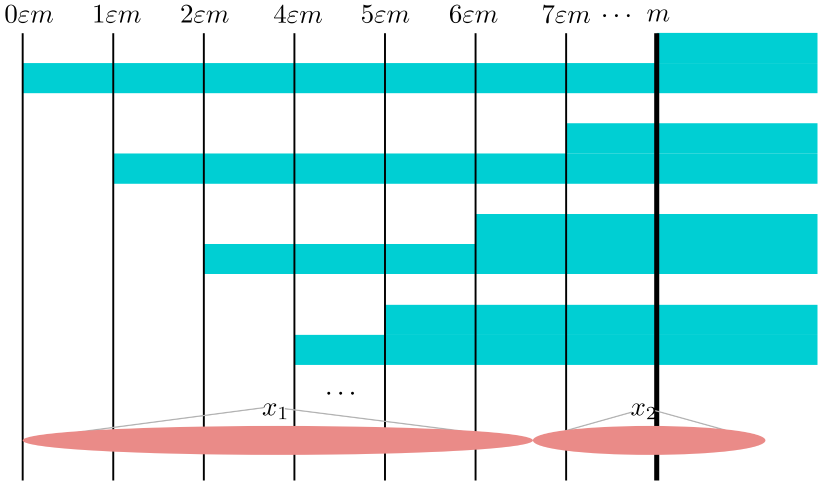

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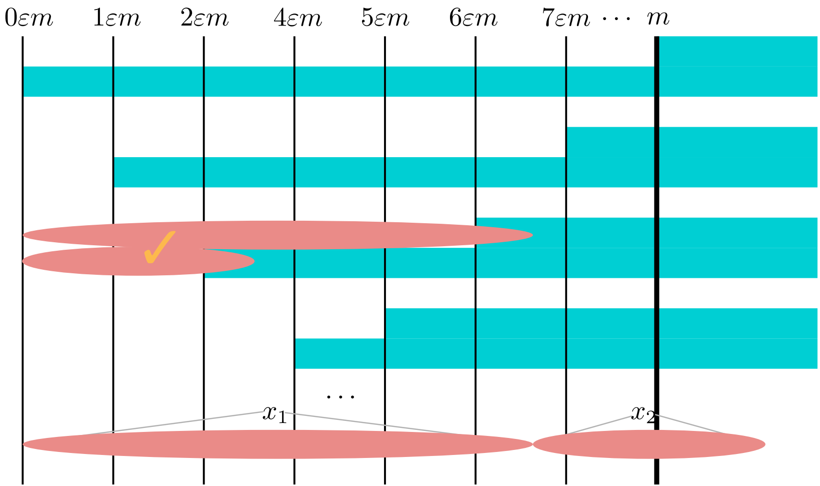

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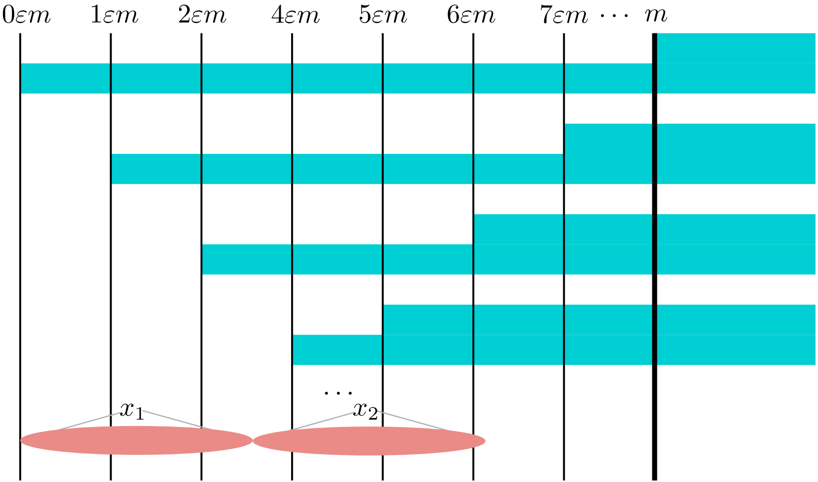

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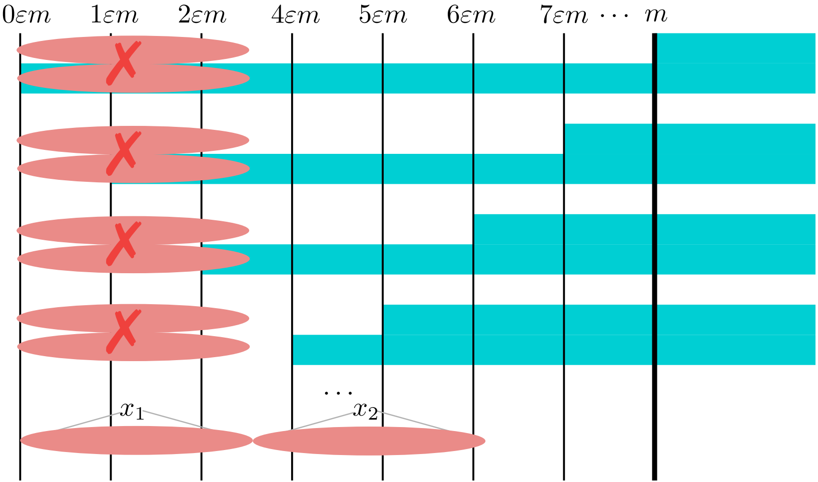

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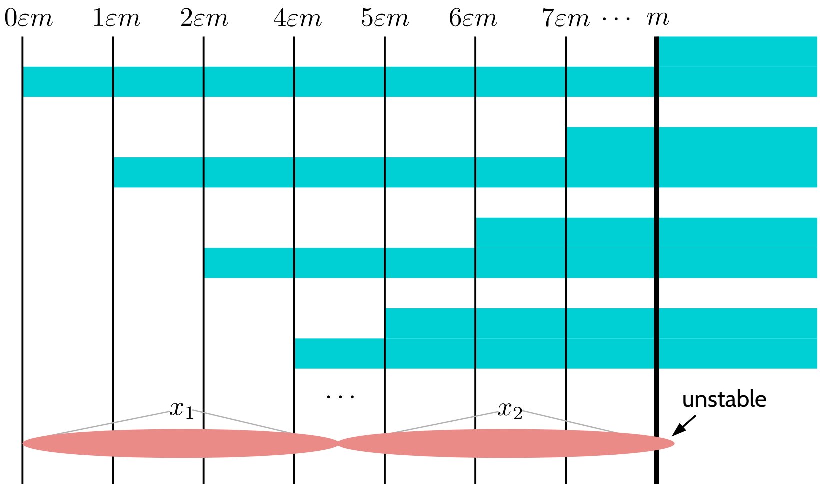

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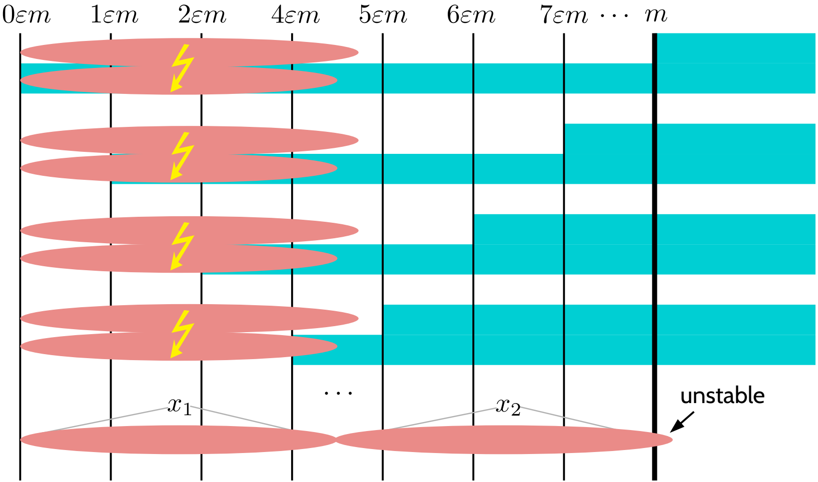
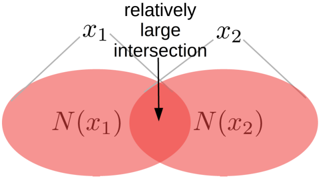
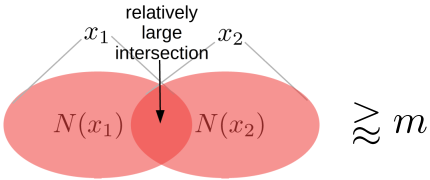


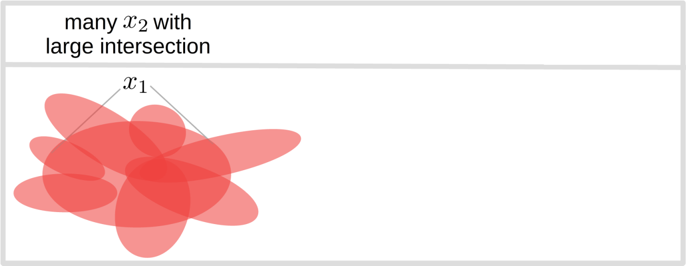
Proof Sketch – Large Intersection



$\gtrapprox m$

Proof Sketch – Large Intersection





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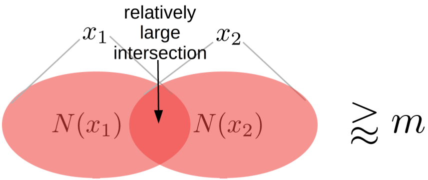

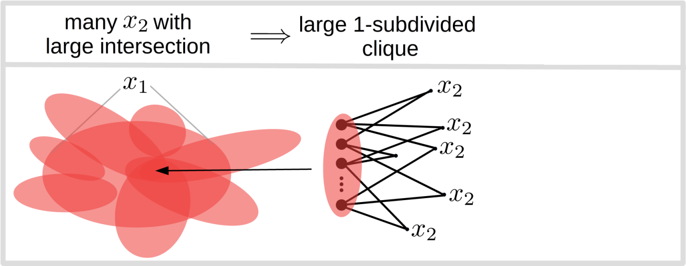

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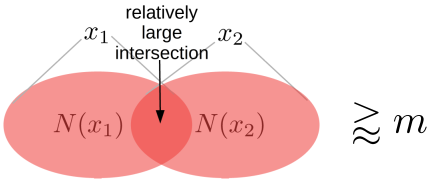
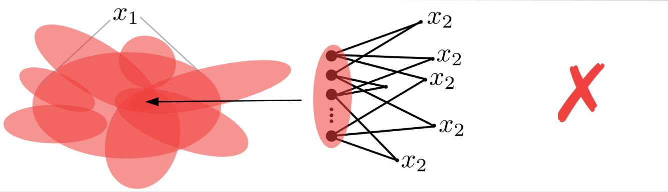


many x_2 with large intersection $\implies$ large 1-subdivided clique $\implies$ not bounded expansion



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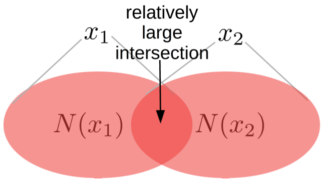
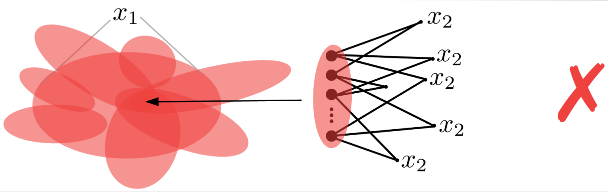


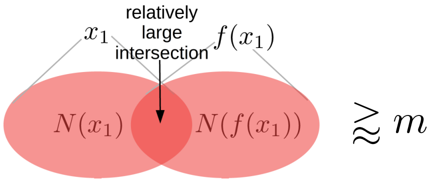
$\gtrsim m$

many x_2 with large intersection	$\Longrightarrow$	large 1-subdivided clique	$\Longrightarrow$	not bounded expansion



We assume (for simplicity) x_1 has only one x_2 with a large intersection.

Proof Sketch – Large Intersection



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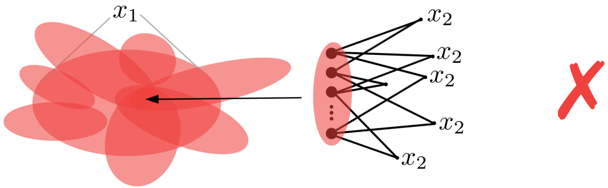
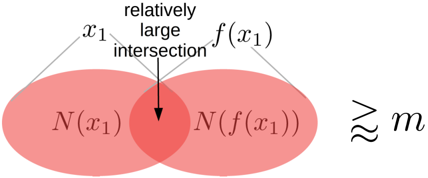


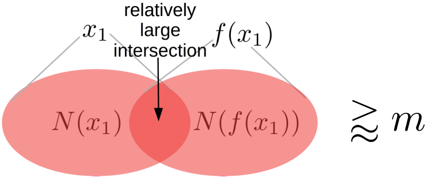
We assume (for simplicity) x_1 has only one x_2 with a large intersection.
We call it $f(x_1)$.

Proof Sketch – Large Intersection



$Q_f(x)$ true
iff $|N(x) \cup N(f(x))| \geq m$

Proof Sketch – Large Intersection



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Final Formula:

$$\Big(x_2 = f(x_1) \wedge Q_f(x_1)\Big)$$

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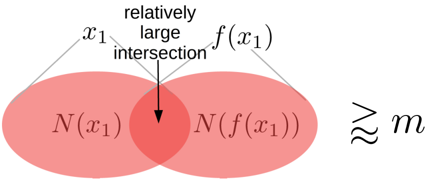

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Final Formula:

$$\Big(x_2 = f(x_1) \wedge Q_f(x_1)\Big) \vee$$
$$\Big(x_2 \neq f(x_1) \wedge \varphi_{\text{small}}(x_1, x_2)\Big)$$

Proof Sketch – Quantifier Elimination

We want to gradually simplify this formula.

$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \left(\underbrace{m_3 \leq \#x_3 \quad \overbrace{\varphi(x_1 x_2 x_3)}^{\text{quantifer-free FO}}}_{\text{replace with quantifier-free FO}}\right)\right)$$

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$$\underbrace{\varphi'''}_{\text{quantifier-free FO}}$$

First-Order Model-Checking in Random Graphs and Complex Networks

Motivation

Motivation

friendship networks
coauthor networks
internet backbone
world wide web
power-grid
biological processes
...

real world networks

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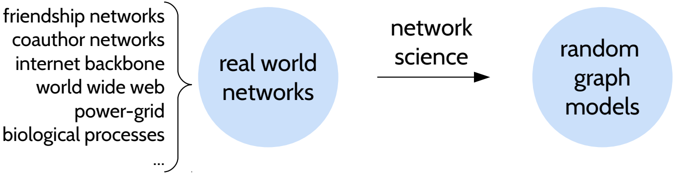
friendship networks
coauthor networks
internet backbone
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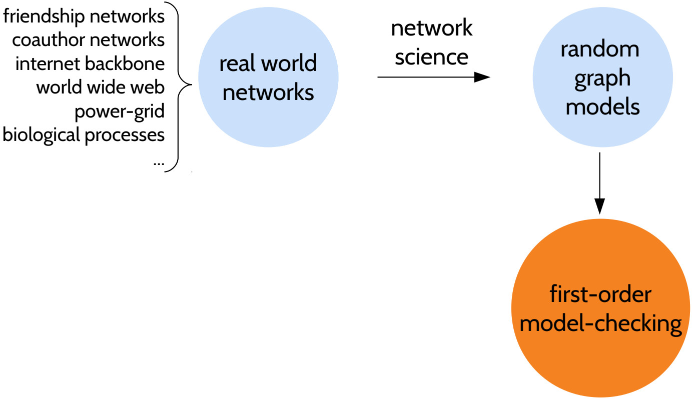
network science
random graph models

Motivation


friendship networks
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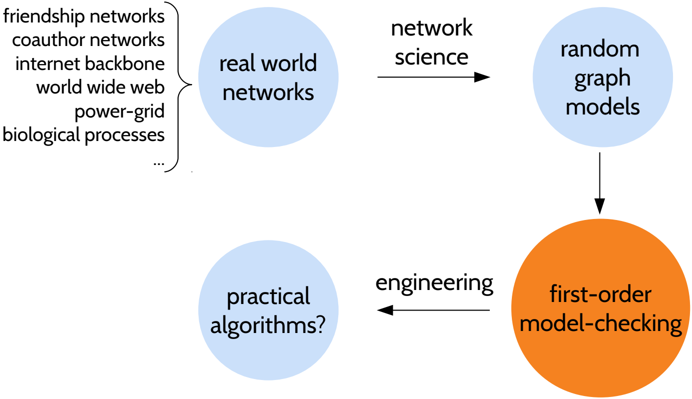
first-order model-checking

Motivation



friendship networks
coauthor networks
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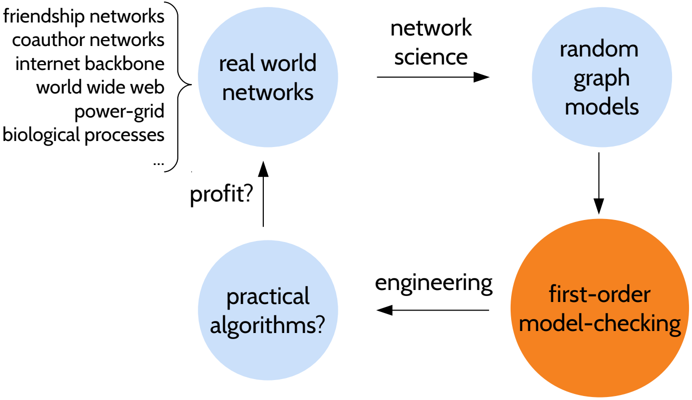
first-order model-checking
engineering
practical algorithms?

Motivation


friendship networks
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biological processes
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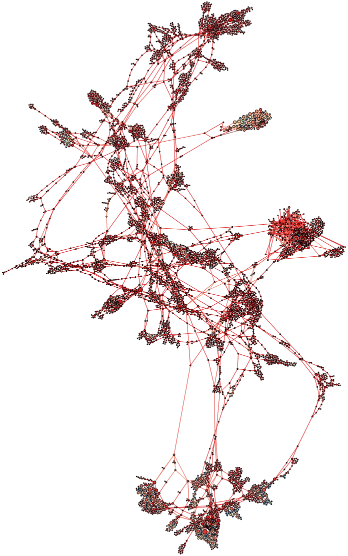
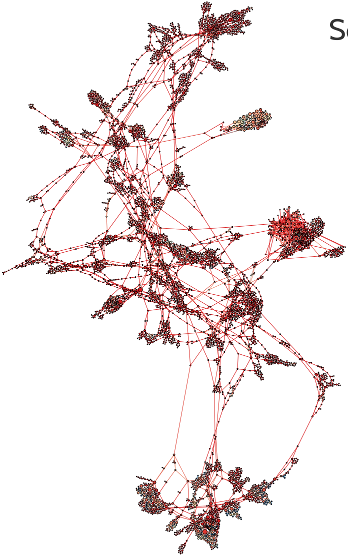
engineering
practical algorithms?
profit?

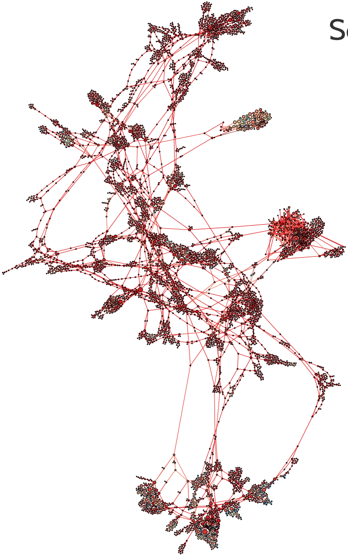
The Real World





Some central properties:

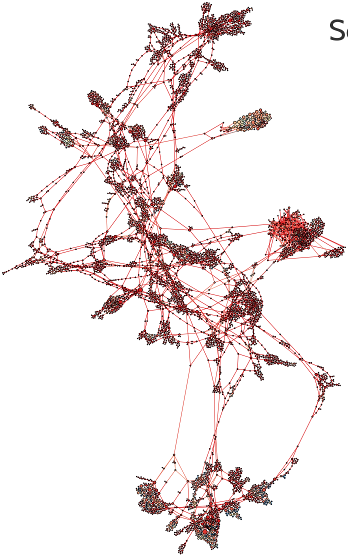
The Real World



Some central properties:

- **Skewed degree distribution**
  Fraction of vertices with degree k proportional to $k^{-\alpha}$ with $2 \leq \alpha \leq 3$?

The Real World



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- **Clustered**
  If we have a common friend we are likely friends as well

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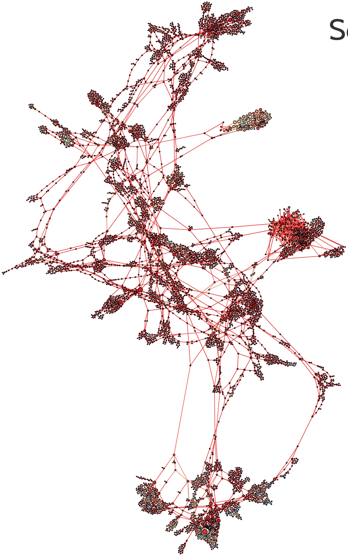
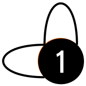


Some central properties:

- **Skewed degree distribution**
  Fraction of vertices with degree k proportional to $k^{-\alpha}$ with $2 \leq \alpha \leq 3$?
- **Clustered**
  If we have a common friend we are likely friends as well
- **Small-world property**
  Everyone is close to everyone

Example: Preferential Attachment Model

Introduced by Barabási and Albert in 1999 to explain the structure of the world wide web.



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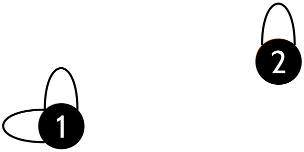

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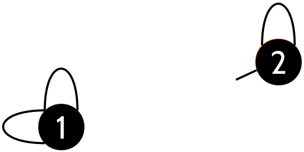

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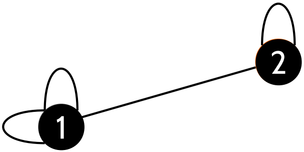

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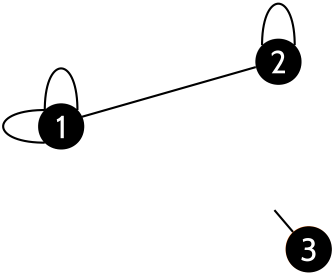

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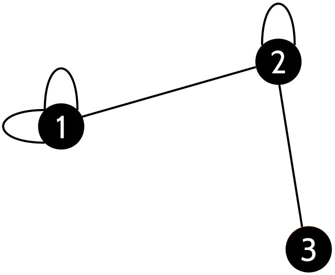

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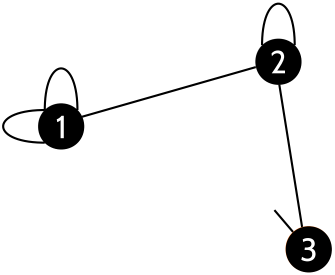

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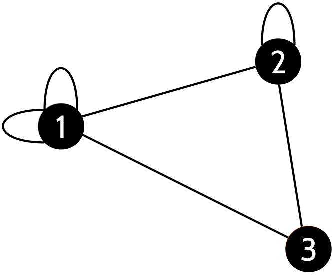

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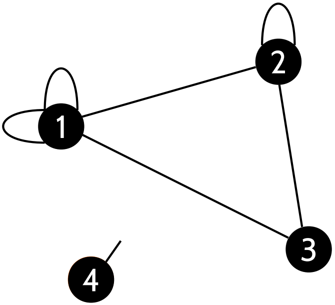

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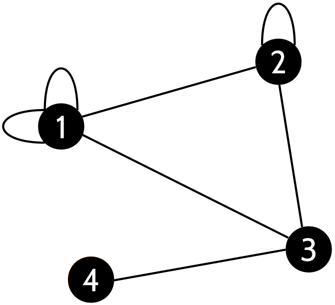

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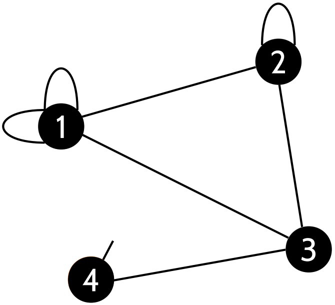

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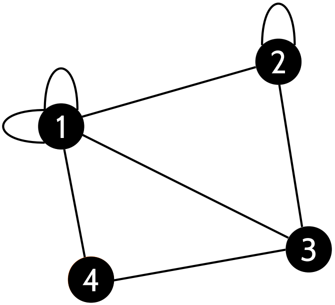

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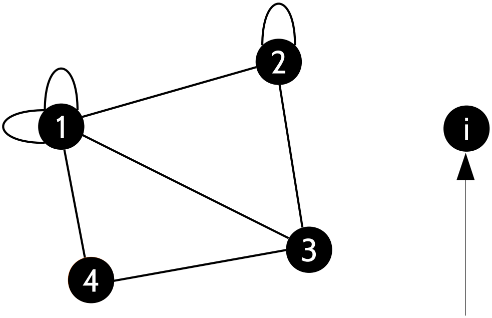
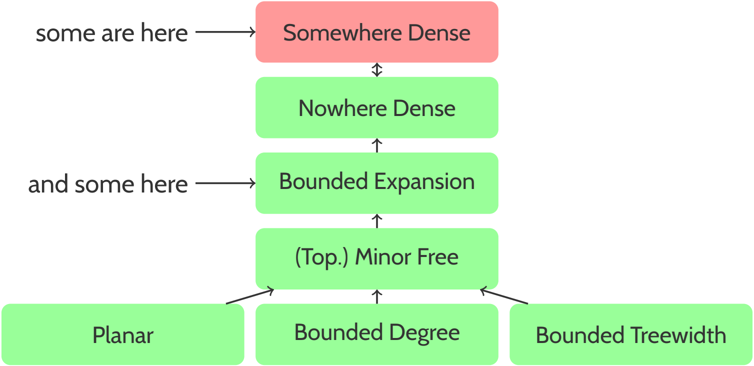


$$\text{expected degree} \approx \sqrt{\frac{n}{i}}$$

Previous Results



[Grohe 2001], [Farrell et. al. 2015], [Demaine et. al. 2019], [Dreier et. al. 2020]

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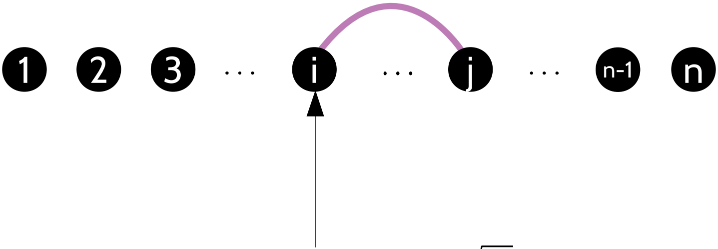
Big Question: model-checking on clustered models?

Example: Chung–Lu Model

A more direct way to get a desirable degree distribution.

$$\frac{1}{\sqrt{i \cdot j}}$$



$$\text{expected degree} \approx \sqrt{\frac{n}{i}}$$

α-power-law-boundedness

A random graph model with vertices $1, \ldots, n$ is 3-*power-law-bounded* if the probability that some subset of edges $E \subseteq \binom{1,\ldots,n}{2}$ is present is at most

$$\log(n)^{O(|E|^2)} \prod_{ij \in E} \frac{1}{\sqrt{i \cdot j}}.$$

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✓

- Preferential attachment model
- Chung–Lu model
- Erdös–Rényi model
- Configuration model
- …

✗

- Hyperbolic random graph model
- random intersection model
- Watts–Strogatz model
- Kleinberg model
- …

Asymptotic Structure of 3-power-law-bounded models

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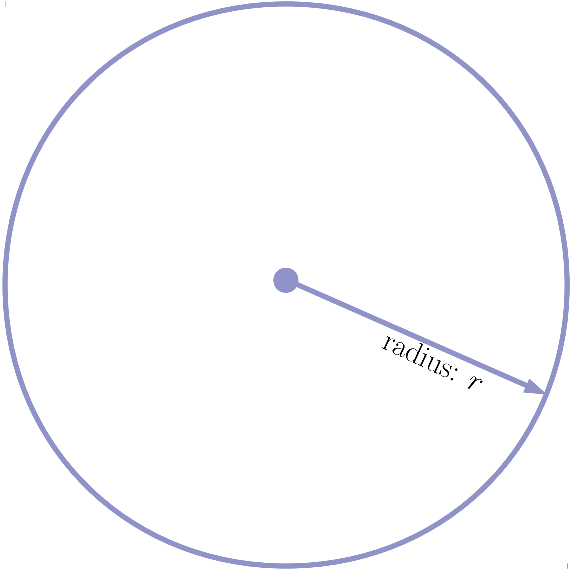

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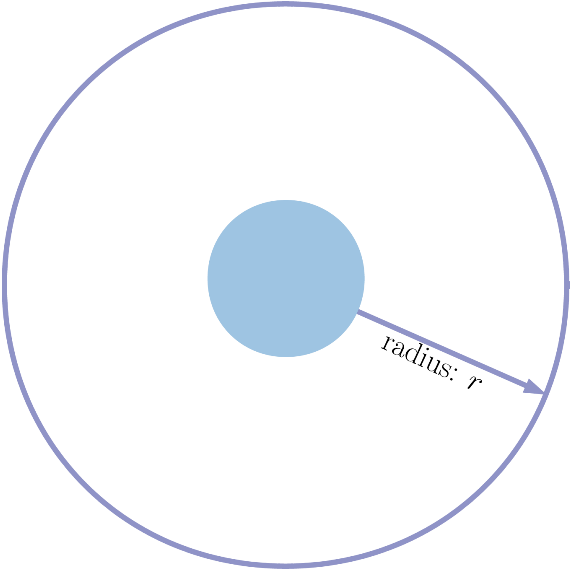

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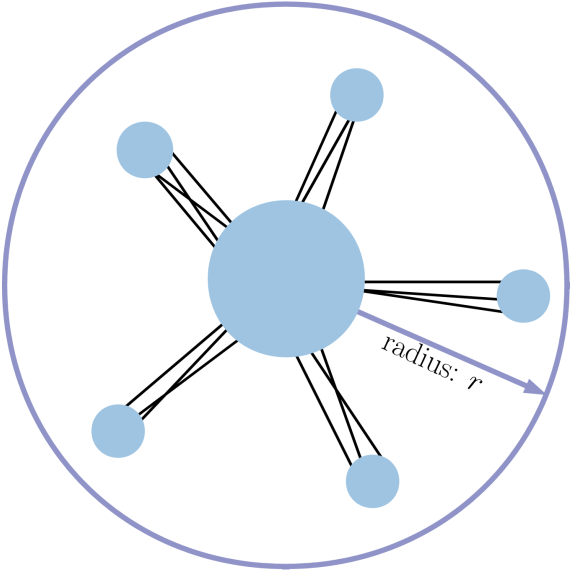

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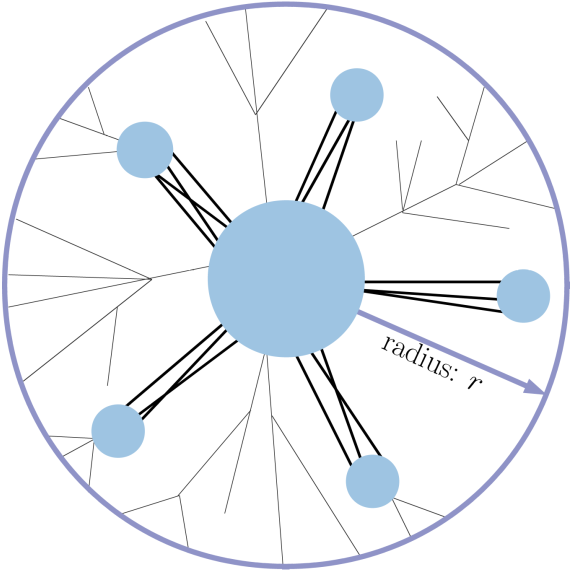
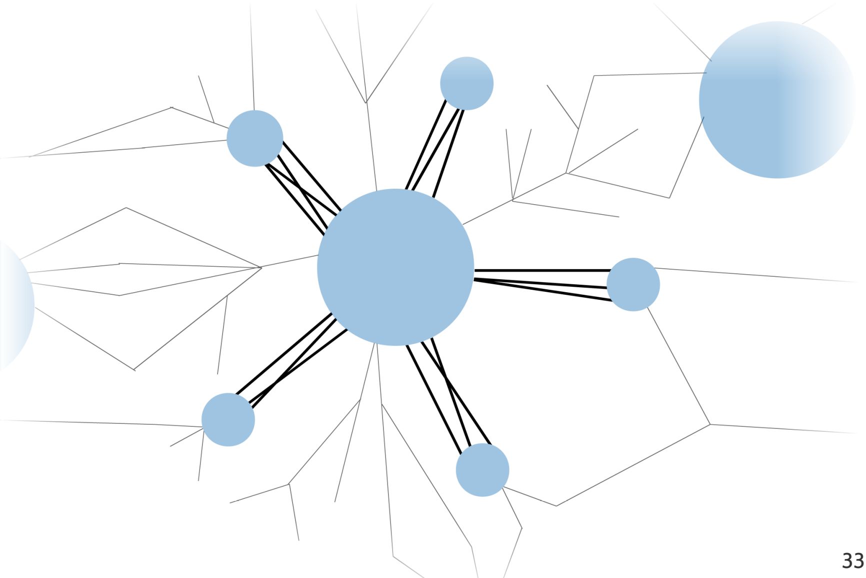


Algorithm

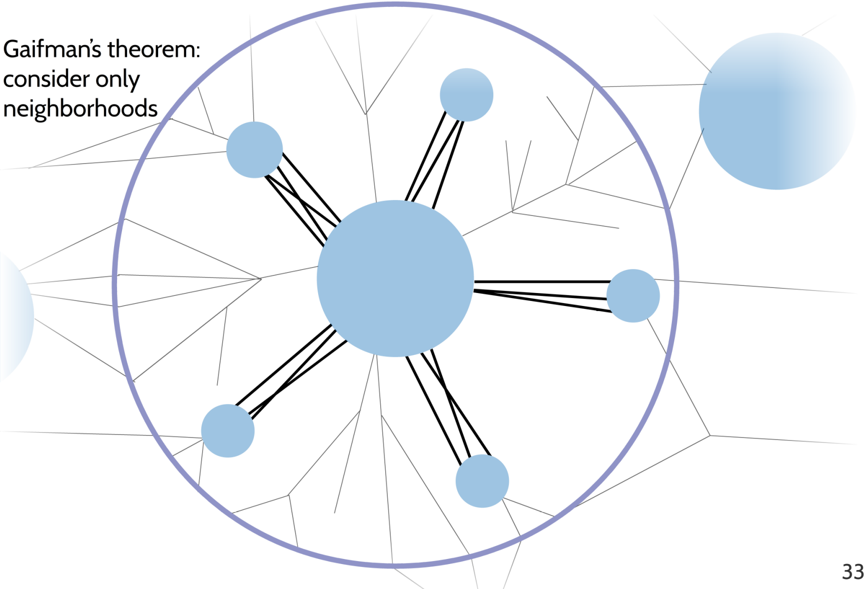
Input: graph sampled from 3-power-law-bounded model

Algorithm



Algorithm

Gaifman's theorem:
consider only
neighborhoods



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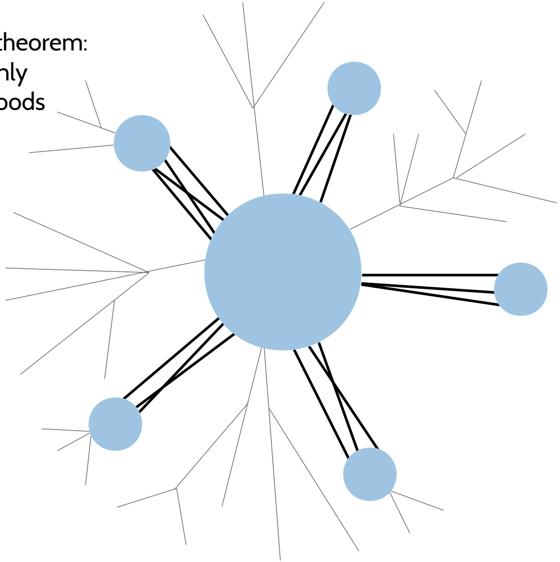
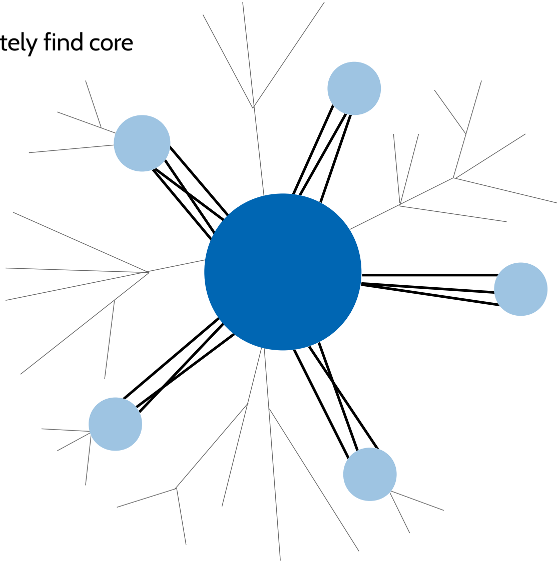


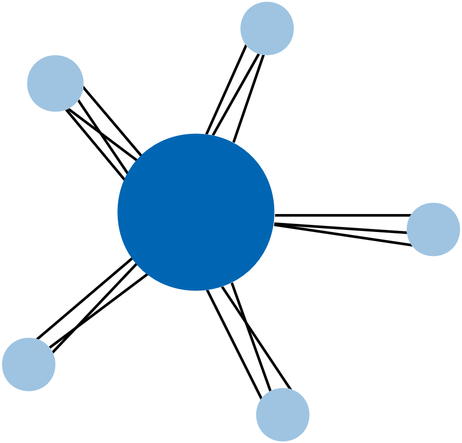
Algorithm

approximately find core



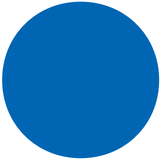
Algorithm

prune trees



Algorithm

prune protrusions

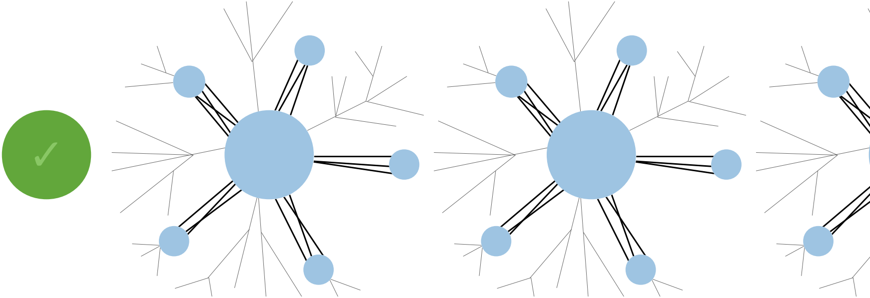


Algorithm

use brute force on core

Algorithm

repeat for every neighborhood



Runtime Analysis

Runtime: $n^{O(1)}$.

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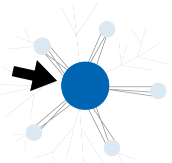
Runtime: $n^{O(1)} \cdot \sum_{x=1}^{n} \Pr[$  $= x] \cdot$

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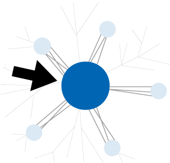
Runtime: $n^{O(1)} \cdot \sum_{x=1}^{n} \Pr[$  $= x] \cdot x^{|\varphi|}$

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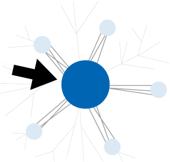
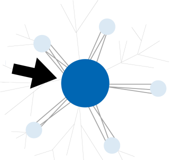
Runtime: $n^{O(1)} \cdot \sum_{x=1}^{n} \Pr[$  $= x] \cdot x^{|\varphi|}$

To get a run time of $f(|\varphi|)n^{O(1)}$ we bound

$\Pr[$  $\geq x]$ for every x.

Summary

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Theorem

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Big Question: model-checking on clustered models?

Let $\alpha > 2$. A random graph model $\mathcal{G}$ is *α-power-law-bounded* if for every $n \in \mathbf{N}$ there exists an ordering $v_1, \ldots, v_n$ of $V(\mathcal{G}_n)$ such that for all $E \subseteq \binom{\{v_1,\ldots,v_n\}}{2}$

$$\Pr\big[E \subseteq E(\mathcal{G}_n)\big] \leq$$

$$\prod_{v_i v_j \in E} \frac{(n/i)^{1/(\alpha-1)}(n/j)^{1/(\alpha-1)}}{n} \cdot \begin{cases} 2^{O(|E|^2)} & \text{if } \alpha > 3 \\ \log(n)^{O(|E|^2)} & \text{if } \alpha = 3 \\ O(n^{\varepsilon})^{|E|^2} \text{ for every } \varepsilon > 0 & \text{if } \alpha < 3. \end{cases}$$

A graph H is an r-*shallow topological minor* of a graph G if a graph obtained from H by subdividing every edge up to $2r$ times is isomorphic to a subgraph of G. The set of all r-shallow topological minors of a graph G is denoted by $G \triangledown r$.

A graph class $\mathcal{C}$ has *bounded expansion* if there exists a function $f \colon \mathbf{N} \to \mathbf{N}$ such that for all $r \in \mathbf{N}$ and all $G \in \mathcal{C}$

$$\max_{H \in G \triangledown r} \frac{||H||}{|H|} \leq f(r).$$