Lacon- and Shrub-Decompositions

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TU Wien, Austria

Dagstuhl Seminar on Sparsity in Algorithms, Combinatorics and Logic
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Exact Characterization

Theorem (Grohe, Kreutzer, Siebertz 2011)

For graph classes $\mathcal{G}$ *closed under subgraphs*,

FO model-checking is tractable iff $\mathcal{G}$ is nowhere dense.

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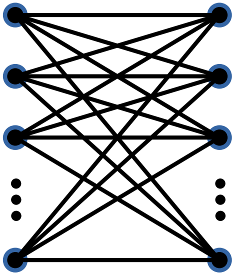
Theorem (Grohe, Kreutzer, Siebertz 2011)

For graph classes $\mathcal{G}$ *closed under subgraphs*,
FO model-checking is tractable iff $\mathcal{G}$ is nowhere dense.

Question

For what graph classes that are *closed under induced subgraphs*
is FO model-checking tractable?

Example: Fully Bipartite Graphs



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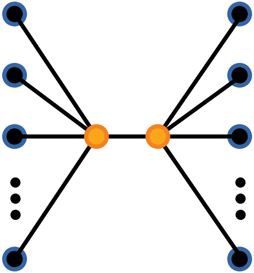

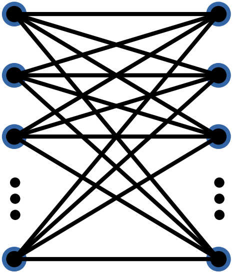

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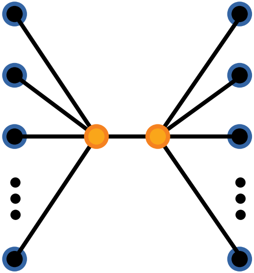
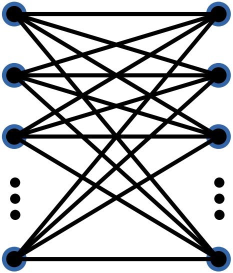


$$\mathrm{dist}(x, y) = 3$$



$$x \sim y$$

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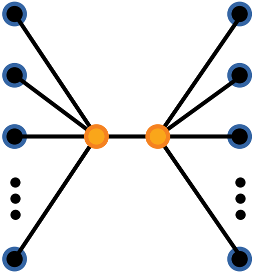
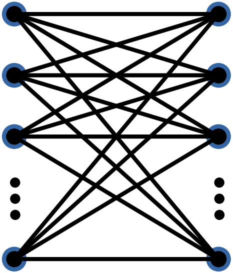


$$\mathrm{dist}(x, y) = 3$$

$$\exists x \ \mathrm{blue}(x) \wedge \varphi$$



$$x \sim y$$

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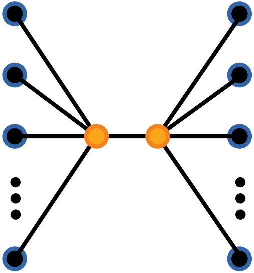
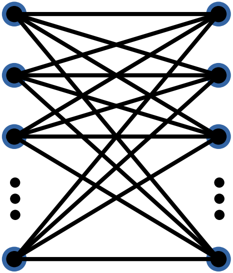
Example: Fully Bipartite Graphs



$$\exists x_1 \exists x_2 \exists x_3$$
$$\text{blue}(x_1) \wedge \text{blue}(x_2) \wedge \text{blue}(x_3)$$
$$\text{dist}(x_1, x_2) = 3 \wedge \text{dist}(x_2, x_3) = 3$$
$$\wedge \text{dist}(x_1, x_3) = 3$$



$$\exists x_1 \exists x_2 \exists x_3$$
$$x_1 \sim x_2 \wedge x_2 \sim x_3$$
$$\wedge x_1 \sim x_3$$

Interpretations

$$I = (\nu(x), \mu(x, y))$$

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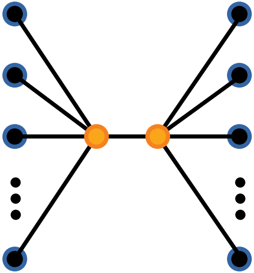
G $I(G)$

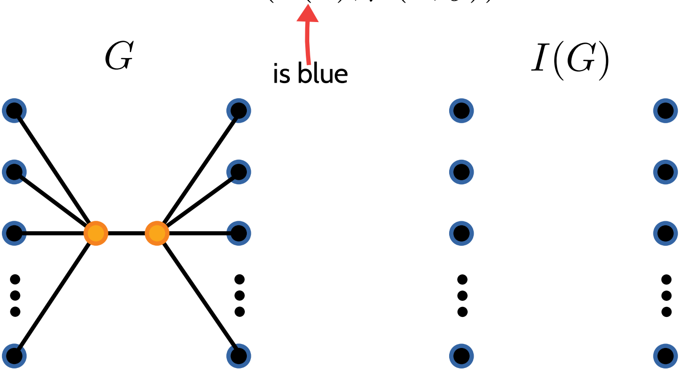


vertices: $\{v \mid G \models \nu(v)\}$

edges: $\{uv \mid G \models \mu(u, v)\}$

Interpretations

$$I = (\nu(x), \mu(x,y))$$



vertices: $\{v \mid G \models \nu(v)\}$

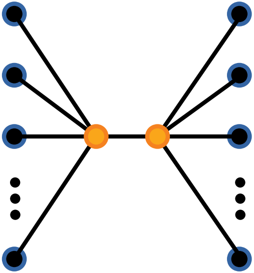
edges: $\{uv \mid G \models \mu(u,v)\}$

$$I = (\nu(x), \mu(x, y))$$

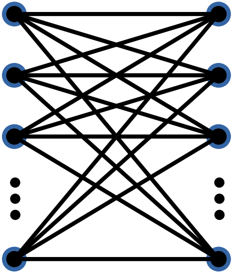
have distance three

is blue

G



$I(G)$



vertices: $\{v \mid G \models \nu(v)\}$

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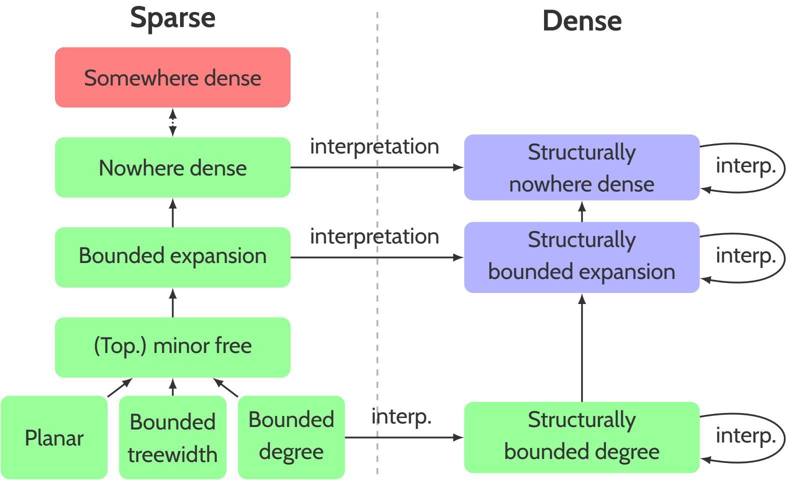
Model-Checking in Sparse and Dense Classes



Nowhere Dense: Grohe, Kreutzer, Siebertz 2011
Structurally Bounded Degree: Gajarský, Hlineným, Obdržálek, Lokshtanov, Ramanujan 2016

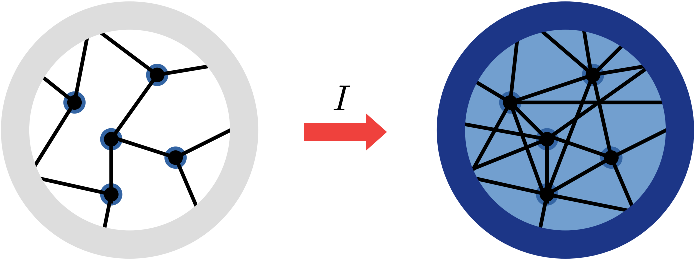
Is There A Simpler Interpretation?

Bounded Expansion

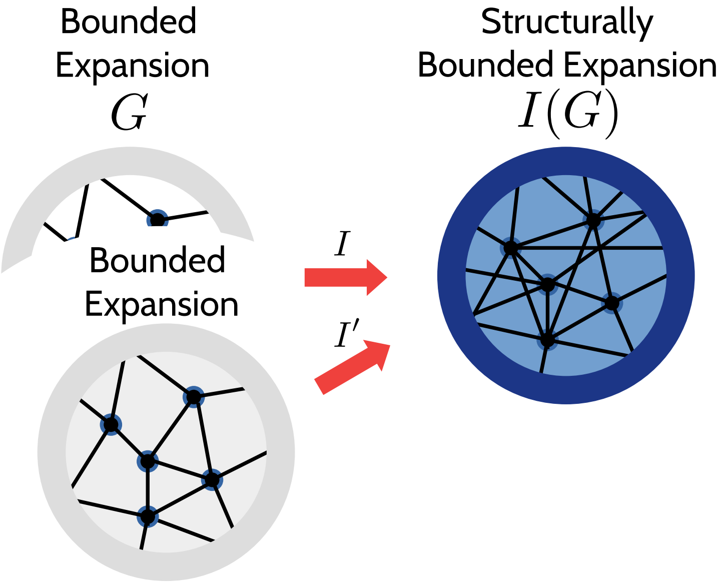
G

Structurally Bounded Expansion

$I(G)$



Is There A Simpler Interpretation?



Is There A Simpler Interpretation?

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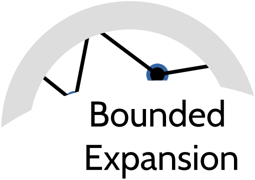
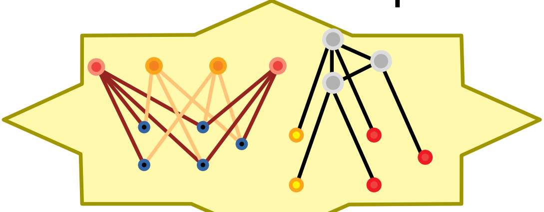


I

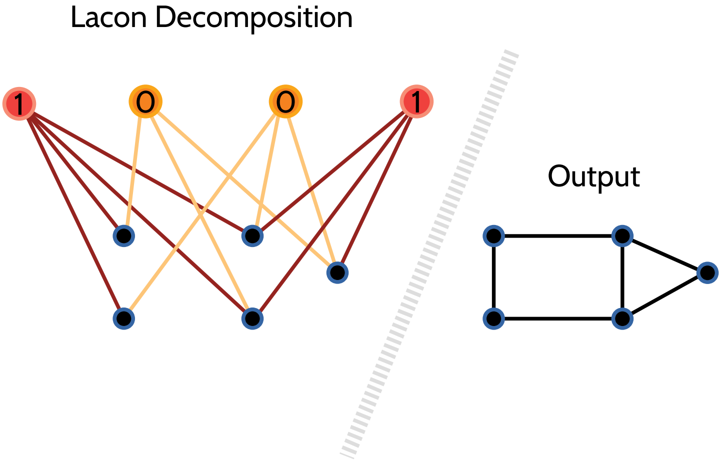
Structurally Bounded Expansion

$I(G)$

Lacon- or Shrub Decomposition



Lacon Decompositions



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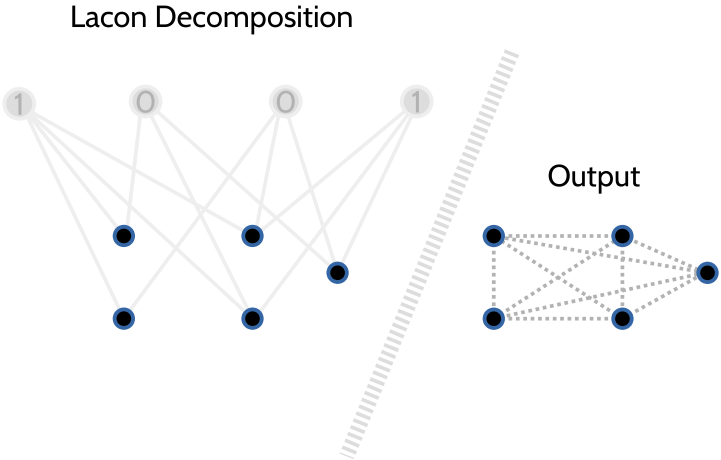
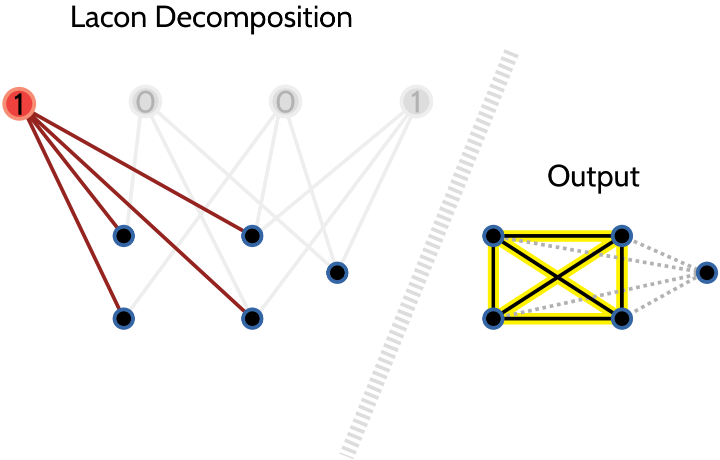
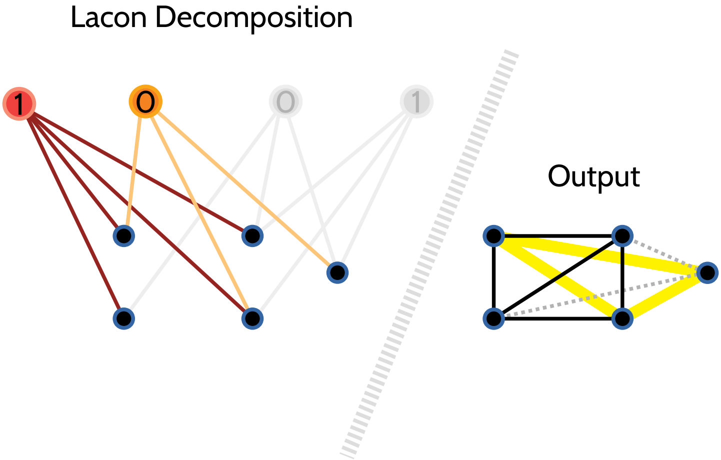



Lacon Decomposition
1
0
0
1
Output

Lacon Decompositions



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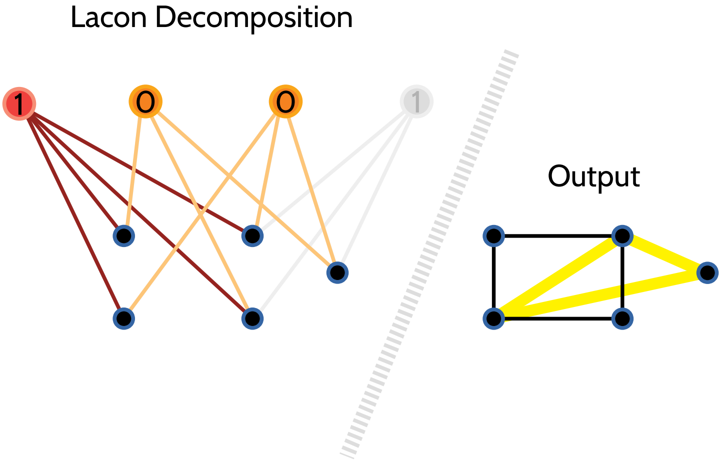

Lacon Decompositions

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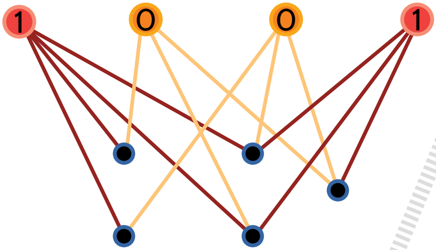

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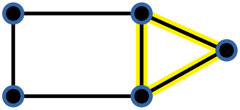

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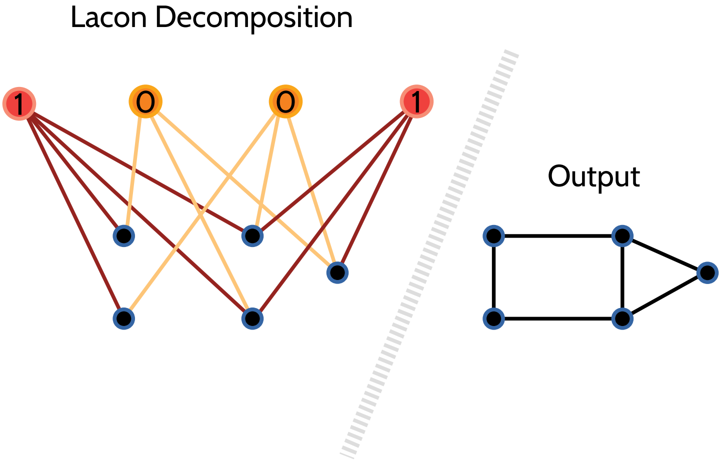

Lacon Decompositions

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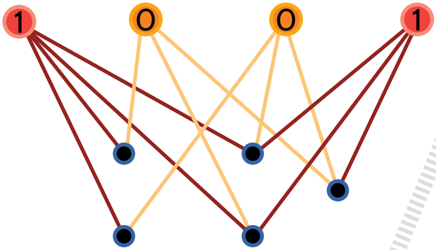
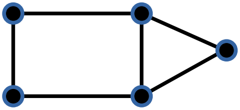
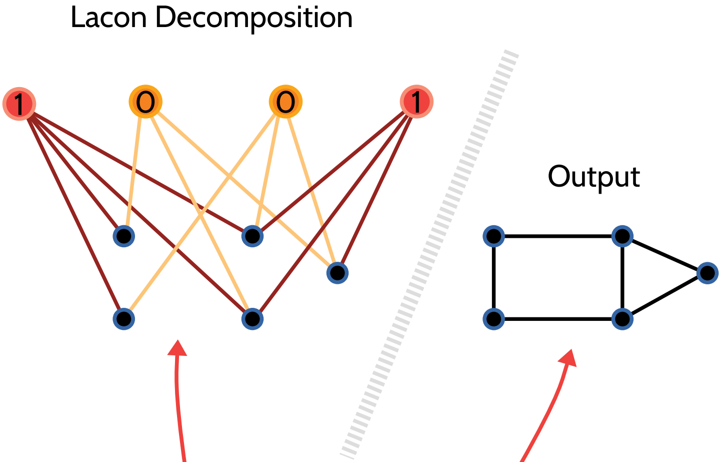


Output



has lacon decomposition with property X $\Longrightarrow$ structurally property X

Lacon Decompositions



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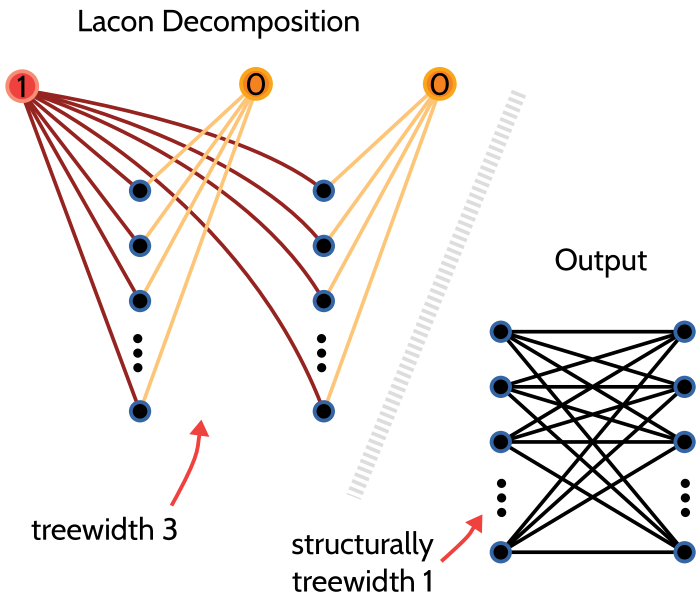

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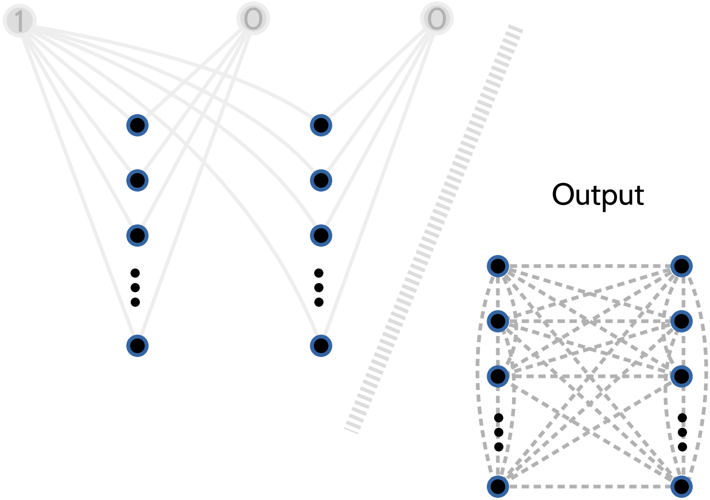

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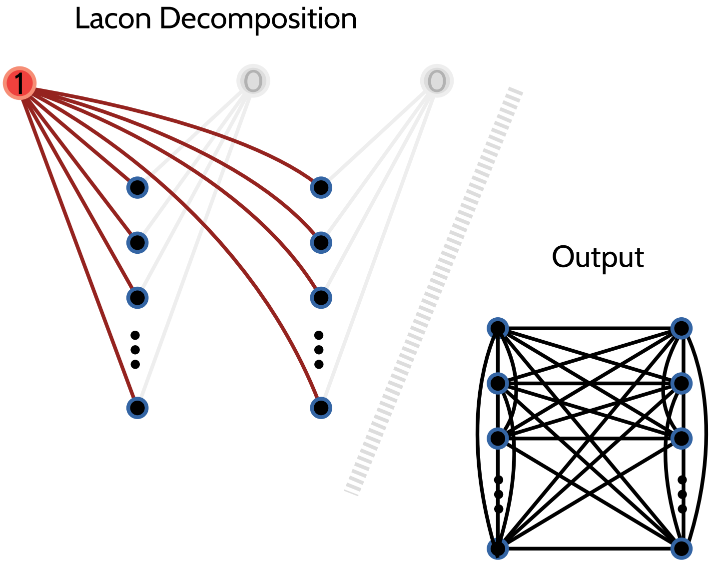
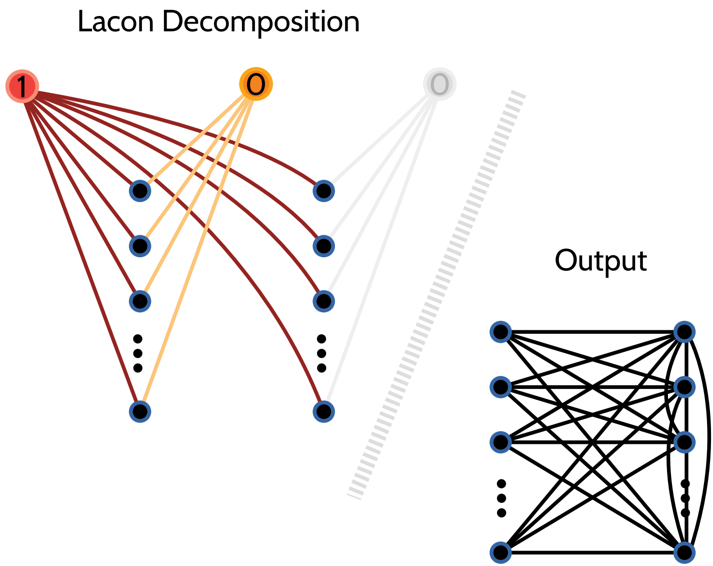
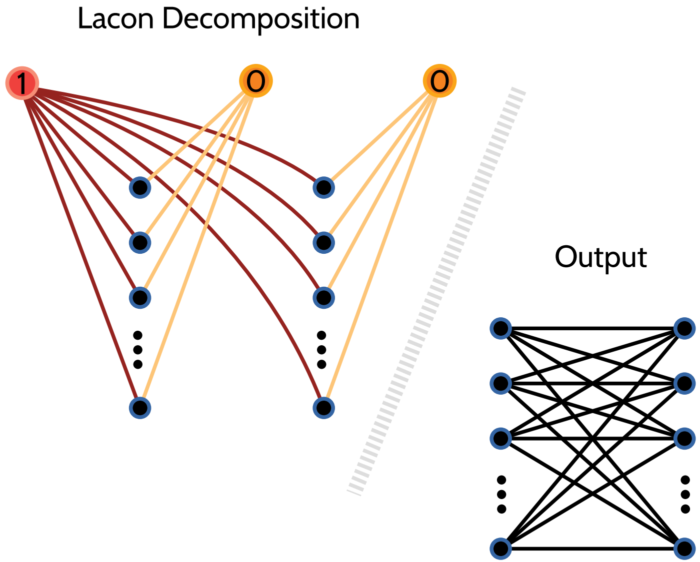



Lacon Decomposition
1
0
0
Output

Lacon Decompositions



Result

Theorem

Let $\mathcal{G}$ be a graph class. The following statements are equivalent.

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- $\mathcal{G}$ has **structurally bounded expansion.**

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Theorem

Let $\mathcal{G}$ be a graph class. The following statements are equivalent.

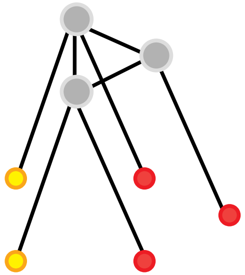
- $\mathcal{G}$ has **structurally bounded expansion.**
- $\mathcal{G}$ has **lacon decompositions**
  - **from a class with bounded expansion,**
  - **and bounded target vertex degree.**

Shrub Decompositions

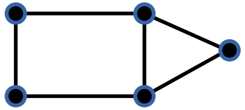
connect vertices with...

- distance 2
- distance 3 and same color

Shrub Decomposition



Output

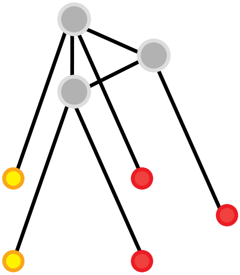


Shrub Decompositions

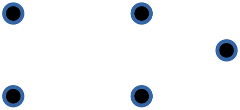
connect vertices with...

-
-

Shrub Decomposition



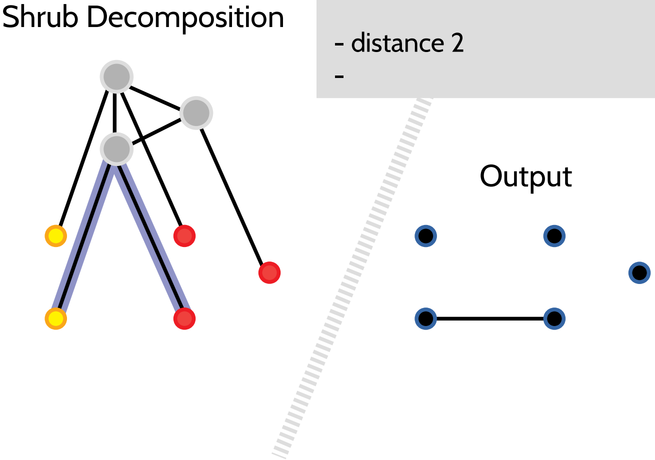
Output



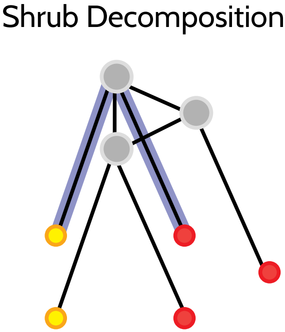
Shrub Decompositions

connect vertices with...

- distance 2
-



Shrub Decompositions



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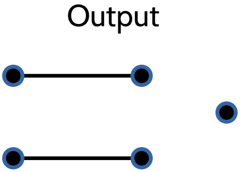
-



Shrub Decompositions

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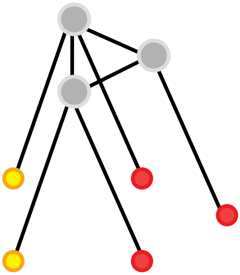
Shrub Decomposition



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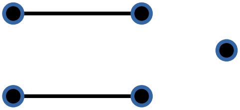
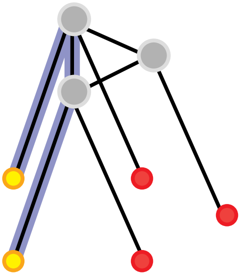


Shrub Decompositions

Shrub Decomposition



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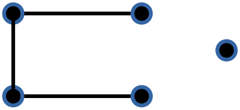
Output



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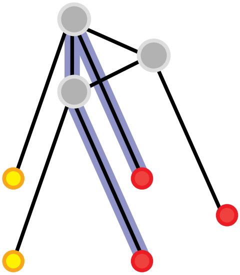

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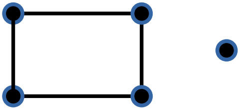

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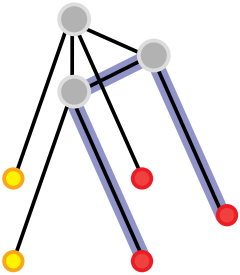

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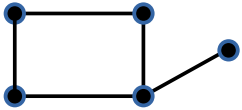

Shrub Decompositions

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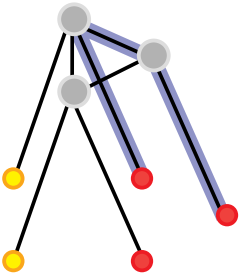

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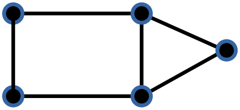

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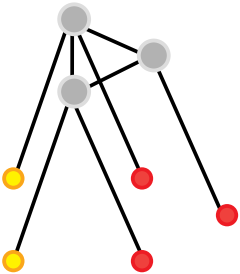
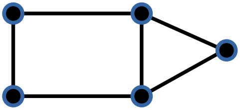


Output



has shrub decomposition with property X $\iff$ structurally property X

Result

Theorem

Let $\mathcal{G}$ be a graph class. The following statements are equivalent.

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Theorem

Let $\mathcal{G}$ be a graph class. The following statements are equivalent.

- $\mathcal{G}$ has **structurally bounded expansion.**
- $\mathcal{G}$ has **lacon decompositions** from a class with
  - **bounded expansion,**
  - **bounded target vertex degree.**

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Theorem

Let $\mathcal{G}$ be a graph class. The following statements are equivalent.

- $\mathcal{G}$ has **structurally bounded expansion.**
- $\mathcal{G}$ has **lacon decompositions** from a class with
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- $\mathcal{G}$ has **shrub decompositions** from a class with
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  - **and bounded diameter.**

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- $\mathcal{G}$ has **structurally bounded expansion.**
- $\mathcal{G}$ has **lacon decompositions** from a class with
  - **bounded expansion,**
  - **bounded target vertex degree.**
- $\mathcal{G}$ has **shrub decompositions** from a class with
  - **bounded expansion,**
  - **bounded number of colors,**
  - **and bounded diameter.**
- $\mathcal{G}$ has **low shrubdepth covers** [1].

[1] Gajarský, Kreuzer, Nešetřil, Ossona de Mendez, Siebertz, Toruńczyk 2018

Comparison to Low Shrubdepth Covers

Lacon- and Shrub Decompositions

Low Shrubdepth Covers

Comparison to Low Shrubdepth Covers

Lacon- and Shrub Decompositions	**Low Shrubdepth Covers**
○ global	○ local

Comparison to Low Shrubdepth Covers

Lacon- and Shrub Decompositions

- global
- first-order types

Low Shrubdepth Covers

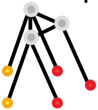
- local
- quantifier alternation

Proof Ideas

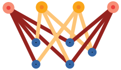
A Cycle of Implications

structurally
bounded
expansion

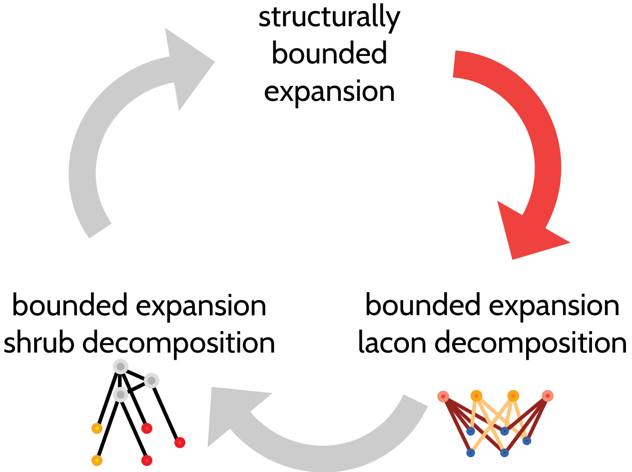
bounded expansion
shrub decomposition



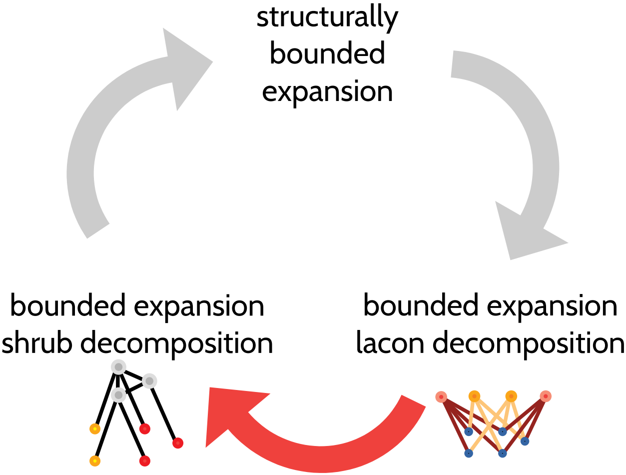
bounded expansion
lacon decomposition



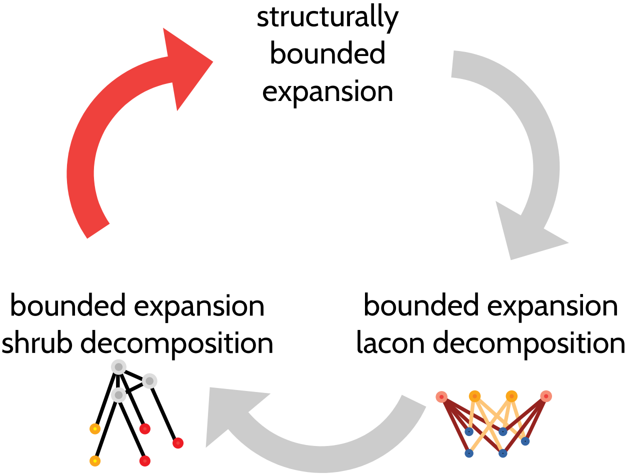
A Cycle of Implications



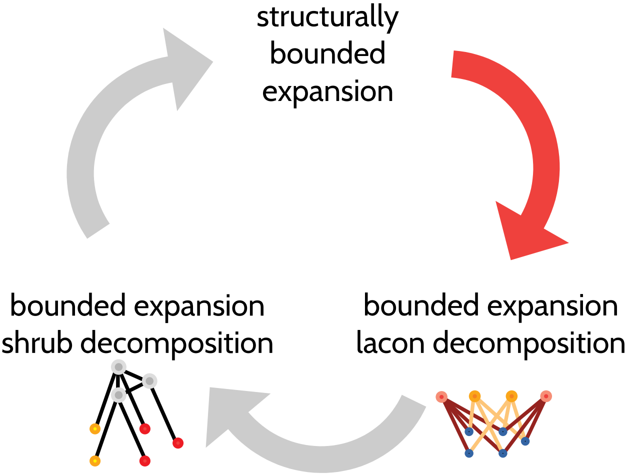
A Cycle of Implications



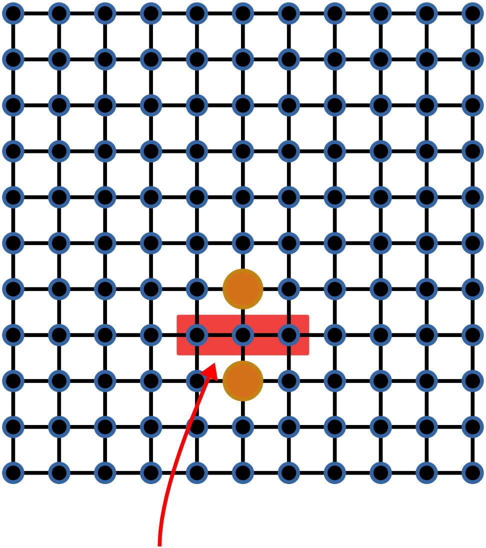
A Cycle of Implications



Step 1

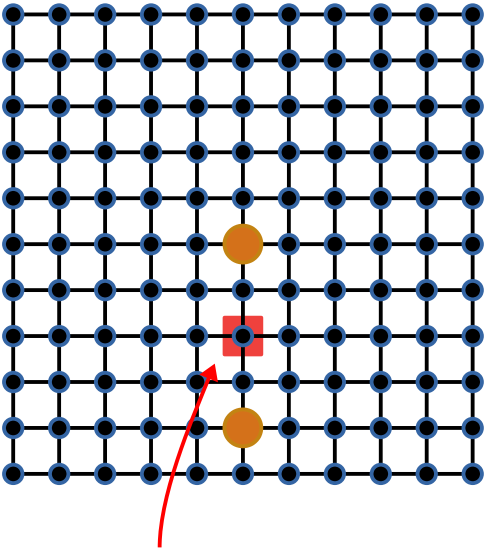

structurally
bounded
expansion
bounded expansion
lacon decomposition
bounded expansion
shrub decomposition

Local Separators



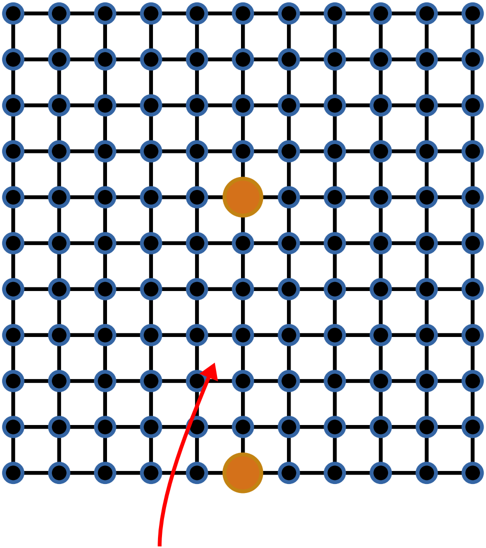
radius 5 local separator

Local Separators



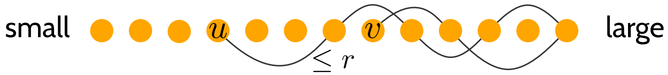
radius 5 local separator

Local Separators



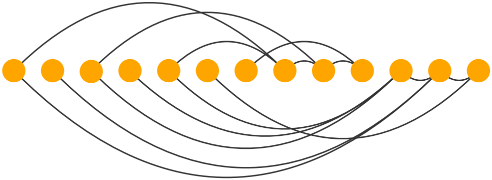
radius 5 local separator

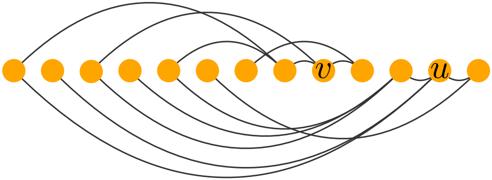
Generalized Coloring Numbers



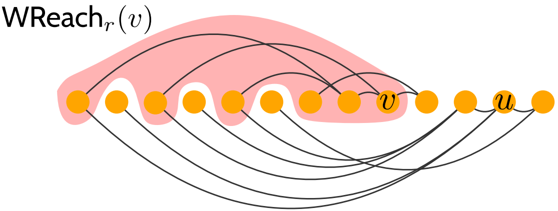
$u \in \mathsf{WReach}_r(v)$

Generalized Coloring Numbers and Separators

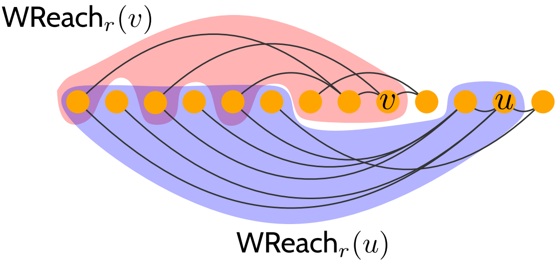




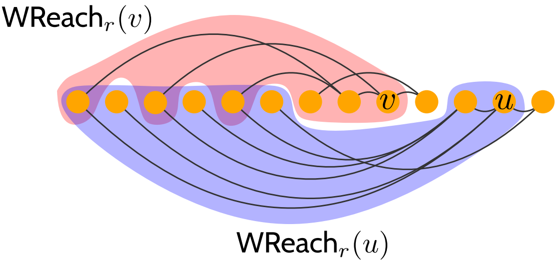
v
u


WReach$_r(v)$
v
u

Generalized Coloring Numbers and Separators



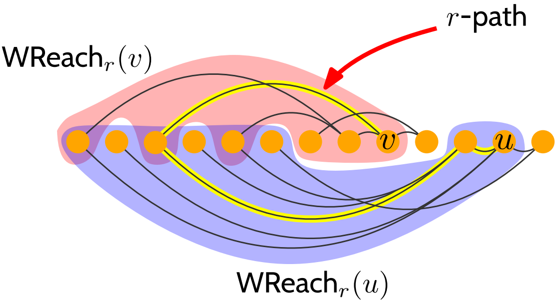
Generalized Coloring Numbers and Separators



Observations:

$\mathrm{WReach}_r(u) \cap \mathrm{WReach}_r(v)$ r-separates u and v.

Generalized Coloring Numbers and Separators



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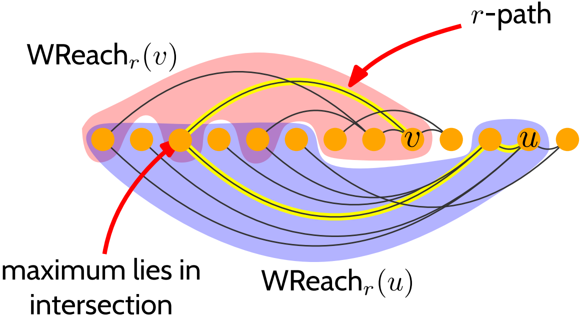

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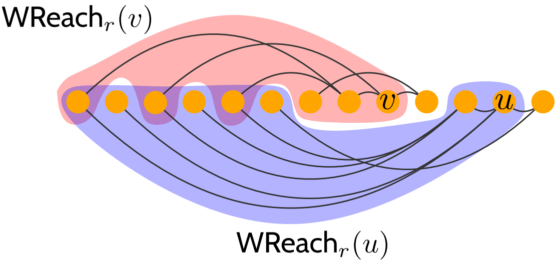

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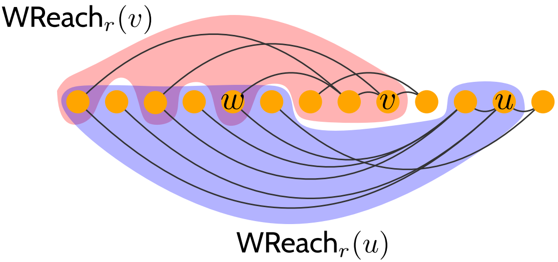

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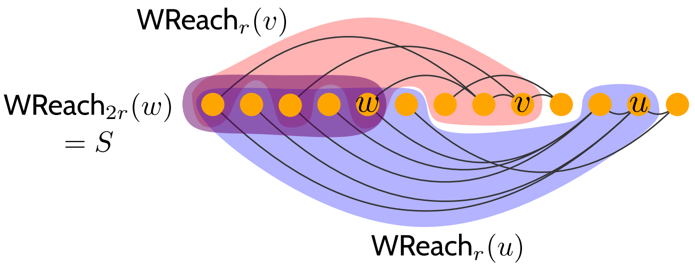
Generalized Coloring Numbers and Separators

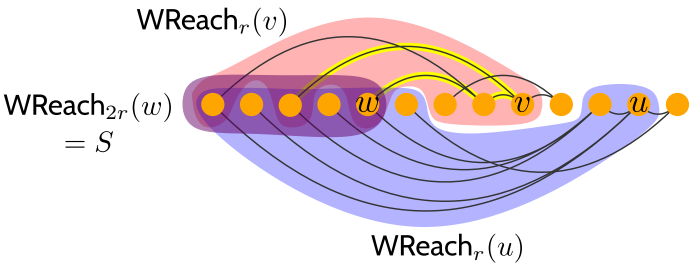


Observations:

$\mathrm{WReach}_r(u) \cap \mathrm{WReach}_r(v)$ r-separates u and v.

$\mathrm{WReach}_{2r}(x)$ r-separates u and v.

Generalized Coloring Numbers and Separators

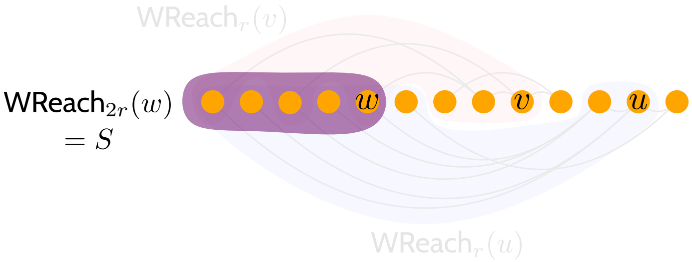


Observations:

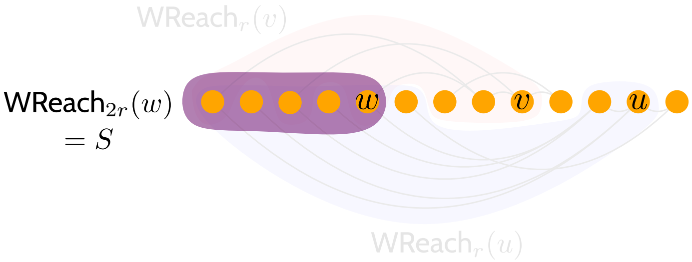
$\mathrm{WReach}_r(u) \cap \mathrm{WReach}_r(v)$ r-separates u and v.

$\mathrm{WReach}_{2r}(x)$ r-separates u and v.

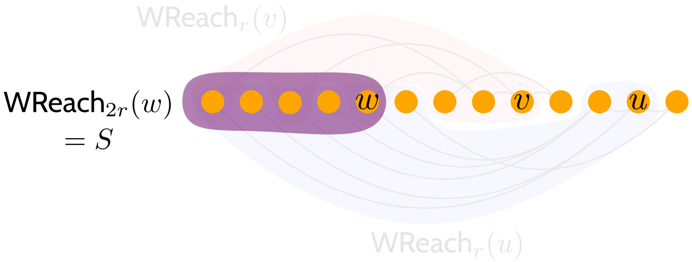
Generalized Coloring Numbers and Separators



Lacon-Decomposition: Edge between u and v if $G \models \varphi(u, v)$

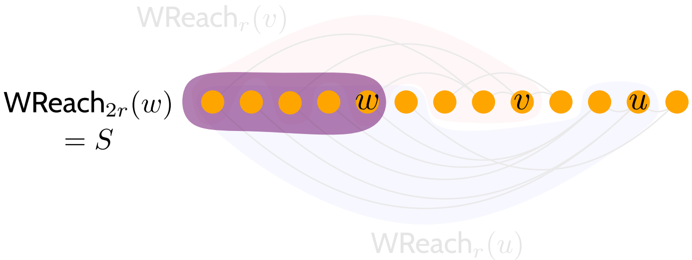


Lacon-Decomposition: Edge between u and v if $G \models \varphi(u, v)$



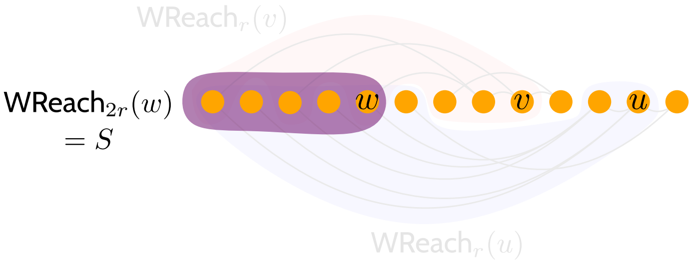
$$\tau_u = \mathsf{type}_q(u, S)$$

Lacon-Decomposition: Edge between u and v if $G \models \varphi(u, v)$



$$\tau_u = \text{type}_q(u, S)$$
$$\tau_v = \text{type}_q(v, S)$$

Lacon-Decomposition: Edge between u and v if $G \models \varphi(u, v)$

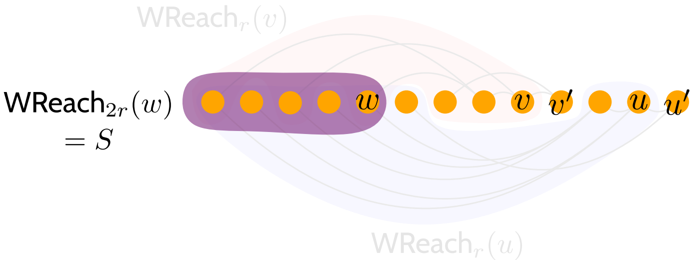


$$\tau_u = \text{type}_q(u, S)$$
$$\tau_v = \text{type}_q(v, S)$$

local FV: $\varphi(u, v)$ is completely determined by τ_u and τ_v

(for large enough r, q)

Lacon-Decomposition: Edge between u and v if $G \models \varphi(u,v)$



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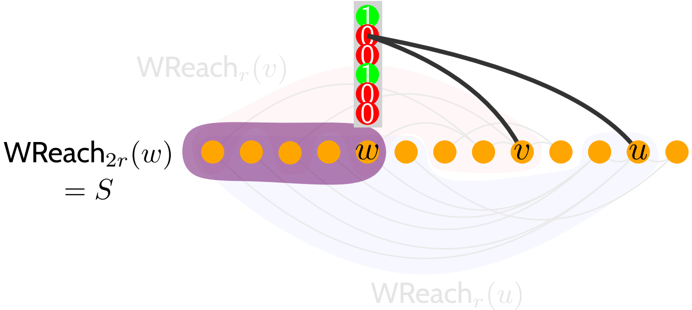
Lacon-Decomposition: Edge between u and v if $G \models \varphi(u, v)$



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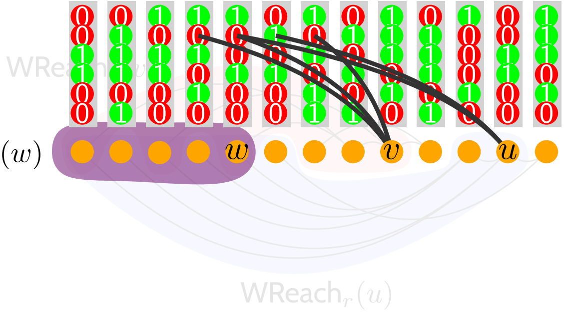
local FV: $\varphi(u, v)$ is completely determined by τ_u and τ_v

(for large enough r, q)

Generalized Coloring Numbers and Separators

Lacon-Decomposition: Edge between u and v if $G \models \varphi(u, v)$



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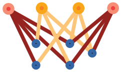
local FV: $\varphi(u, v)$ is completely determined by τ_u and τ_v

(for large enough r, q)

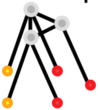
A Cycle of Implications

structurally
bounded
expansion

bounded expansion
lacon decomposition



bounded expansion
shrub decomposition

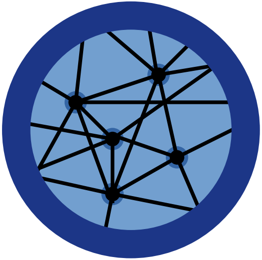


Takeaway

- First-order transductions have “limited power” on sparse graphs. They can’t do more than a lacon or shrub model.
- Local separators are a powerful tool to understand structurally sparse graphs.

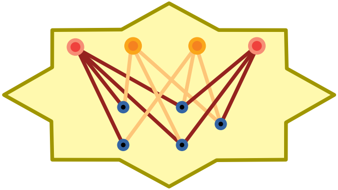
Big Question

Structurally
Bounded Expansion

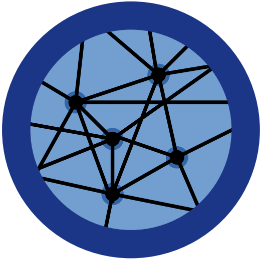


Big Question

has Decomposition
with Bounded Expansion



Structurally
Bounded Expansion



Big Question

has Decomposition
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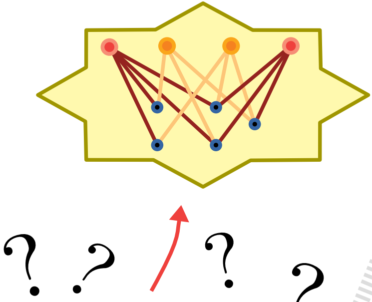
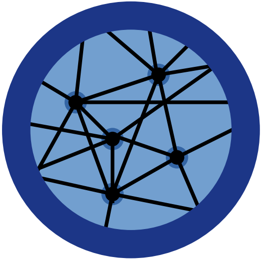


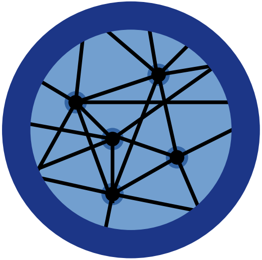
Can we compute it?

Structurally
Bounded Expansion



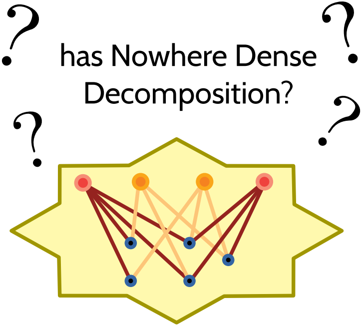
Big Question

Structurally
Nowhere Dense

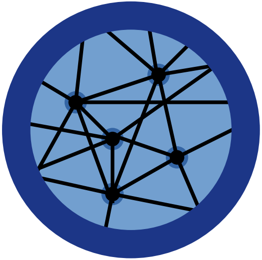


Big Question

has Nowhere Dense Decomposition?



Structurally Nowhere Dense



End

Thanks!