Algorithmic Meta-Theorems for Exact and Approximate Counting in Sparse Graph Classes

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"All problems expressible in a certain logic can be solved efficiently on certain graphs."

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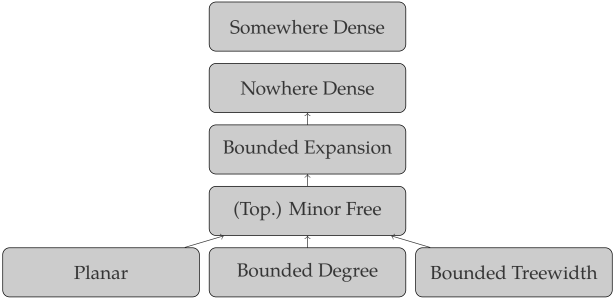

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a counting first-order logic

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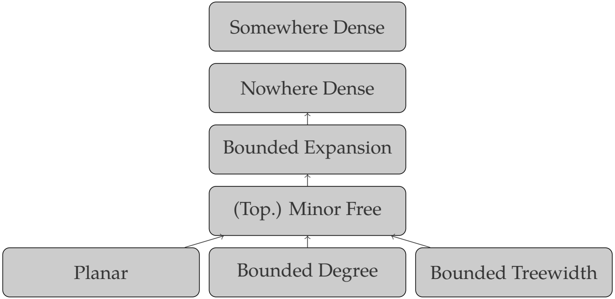
bounded expansion classes



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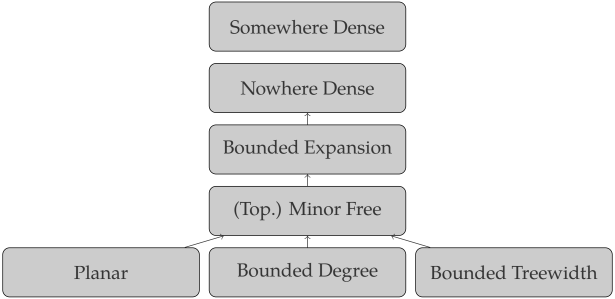

Expressibility

A parameterized graph problem P is *expressible* in a logic L if for every $k \in \mathbf{N}$ there exists $\varphi_k \in \mathrm{L}$ such that for all graphs G we have $G \models \varphi$ iff $\langle G, k \rangle \in P$.

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- p-IndependentSet is expressible in FO:

$$\varphi_k \equiv \exists x_1 \dots \exists x_k \bigwedge_{i,j} x_i \not\sim x_j$$

- p-DominatingSet is expressible in FO:

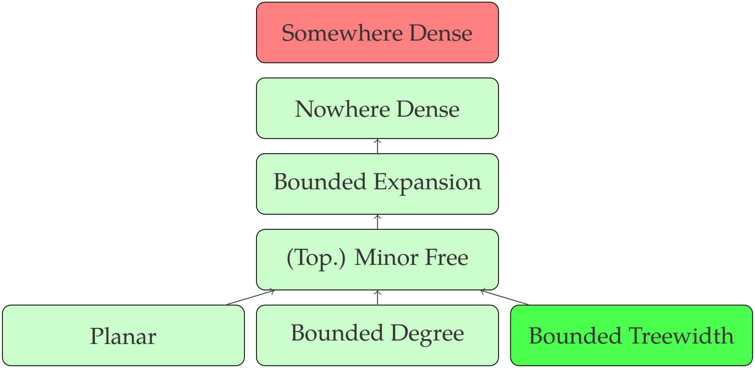
$$\varphi_k \equiv \exists x_1 \dots \exists x_k \forall y \bigvee_{i} y \sim x_i \wedge y = x_i$$

The Parameterized Model-Checking Problem

p-MC$(\mathcal{G}, \mathrm{L})$	
Input:	A graph $G \in \mathcal{G}$ and a sentence $\varphi \in \mathrm{L}$
Parameter:	$\lvert\varphi\rvert$
Problem:	$G \models \varphi$?

If p-MC$(\mathcal{G}, \mathrm{L}) \in$ FPT then every problem expressible in L is also in FPT when restricted to $\mathcal{G}$.

Previous Results without Counting



If $\mathcal{G}$ has bounded treewidth then $p\text{-MC}(\mathcal{G}, \text{MSO}) \in \text{FPT}$.

[Courcelle 1990]

If $\mathcal{G}$ is nowhere dense then $p\text{-MC}(\mathcal{G}, \text{FO}) \in \text{FPT}$.

[Kreutzer, Grohe 2011]

Counting Problems

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[Amini, Fomin, Saurabh, 2008]

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Can be solved on apex-minor-free graphs in time $2^{\sqrt{k}} n^{O(1)}$.

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Is $W[1]$-hard for 2-degenerate graphs.

[Golovach, Villanger 2008]

p-HalfDominatingSet	
Input:	A graph G and $k \in \mathbf{N}$
Parameter:	k
Problem:	Are there k vertices dominating $n/2$ vertices?

Can be expressed in a counting extension of FO called FOC:

$$\varphi_k \equiv \exists x_1 \ldots \exists x_k \, \#y \left(\bigvee_i y \sim x_i \wedge y = x_i \right) \geq n/2$$

A Counting Extension of FO

The counting logic FOC:
[Kuske, Schweikardt 2017]

- the rules of FO
- $t_1 \diamondsuit t_2$ for counting terms t_1, t_2 and $\diamondsuit \in \{\leq, \geq, =, \equiv_2, \ldots\}$.

Counting terms:

- $t \in \mathbf{N}$
- $\#\bar{y}\,\varphi$ for every FOC formula φ
- $(t_1 + t_2)$ and $(t_1 \cdot t_2)$ for counting terms t_1, t_2

Tractability of FOC Model-Checking

If $\mathcal{G}$ has bounded degree then p-MC($\mathcal{G}$, FOC) $\in$ FPT.
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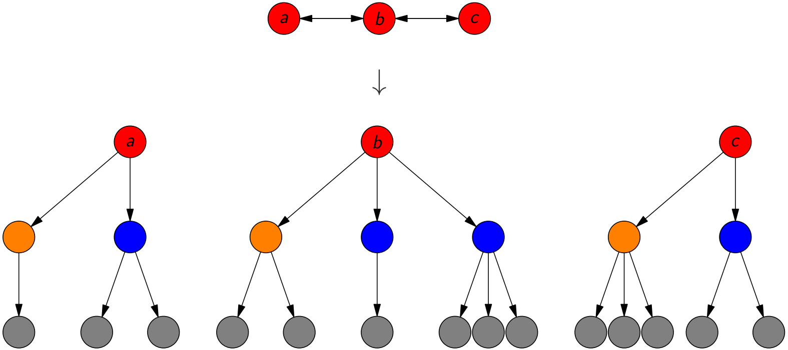
FOC seems too powerful.

Hardness of FOC Model-Checking

Reduce p-CLIQUE to p-MC($\mathcal{G}$, FOC) on forests of depth three.



$$\varphi(ab) \equiv \exists x\ \exists y\ a \sim x \wedge b \sim y \wedge \left(\#z\ x \sim z = \#z\ y \sim z\right)$$

$$k\text{-clique} \Leftrightarrow \exists x_1 \ldots \exists x_k \bigwedge_{i,j} \varphi(x_i x_j)$$

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- compute $\#\bar{x}\ \varphi(\bar{x})$ for $\varphi \in \text{FO}$,
- evaluate the FOC fragment with

$$\underbrace{[\dots] \leq [\dots]}_{\text{at most one free variable}}.$$

How about $\exists x_1 \dots \exists x_k \, \#y \left(\bigvee_i y \sim x_i \wedge y = x_i \right) \geq n/2$?

The Big Question

$$\mathrm{FO} \subset \quad \textbf{?} \quad \subset \mathrm{FOC}$$

The Big Question

$$\mathrm{FO} \subset \mathrm{FOX} \subset \mathrm{FOC}$$

FOX

The counting logic FOX:

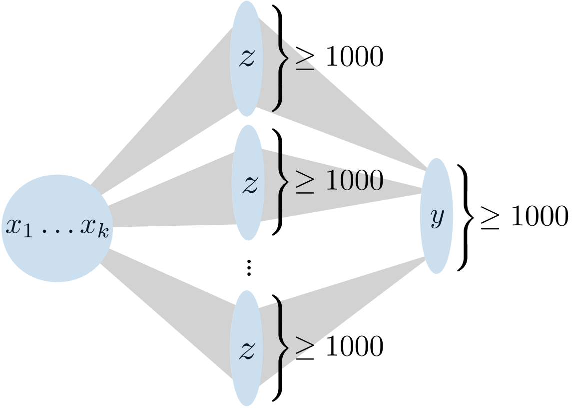
- the rules of FO
- $\#y\ \varphi \geq m$ for every $m \in \mathbf{N}$ and FOX formula φ

Contains the p-HalfDominatingSet formula

$$\exists x_1 \ldots \exists x_k\ \#y\ \Big(\bigvee_i y \sim x_i \wedge y = x_i\Big) \geq n/2$$

Nested Counting in FOX

$$\exists x_1 \ldots \exists x_k \, \#y \left(\#z \left(\bigvee_i z \sim x_i \wedge z \sim y \right) \geq 1000 \right) \geq 1000$$



Bad News

p-MC($\mathcal{G}$, FOX) is AW[$*$]-hard on trees.

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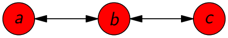
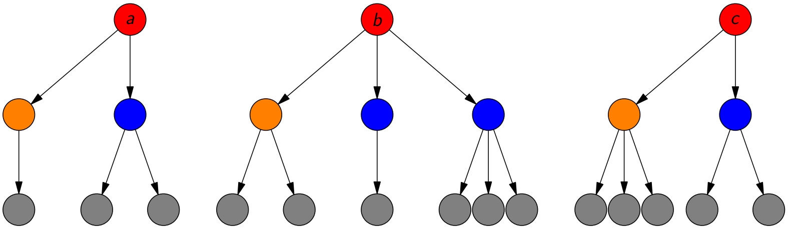
Reduce p-Clique to p-MC($\mathcal{G}$, FOC) on forests of depth three.



$$\downarrow$$



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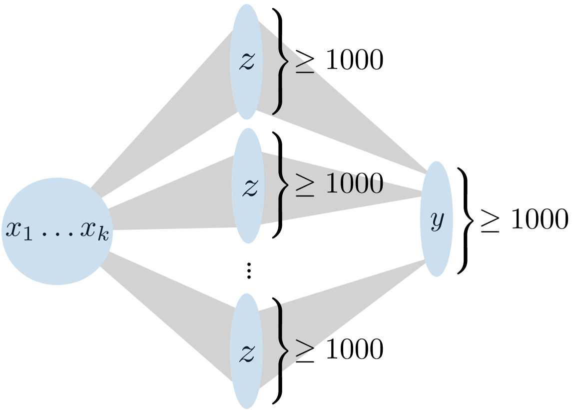

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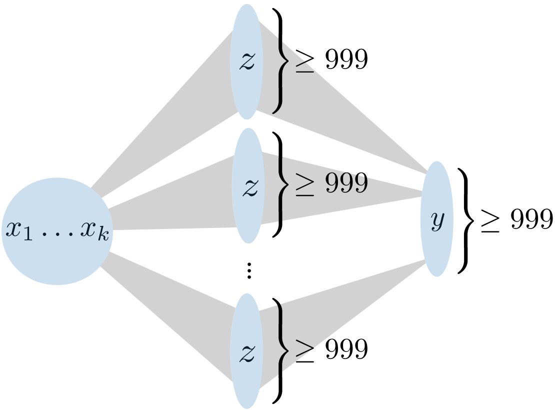

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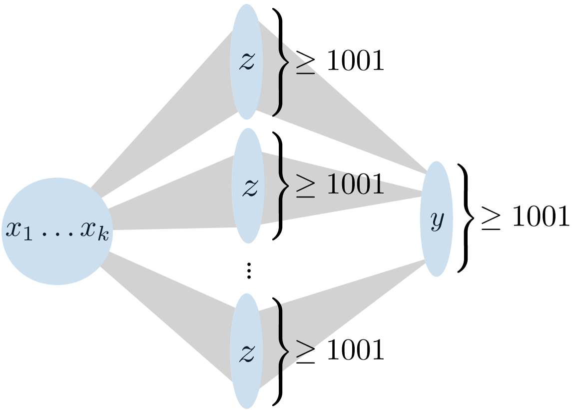

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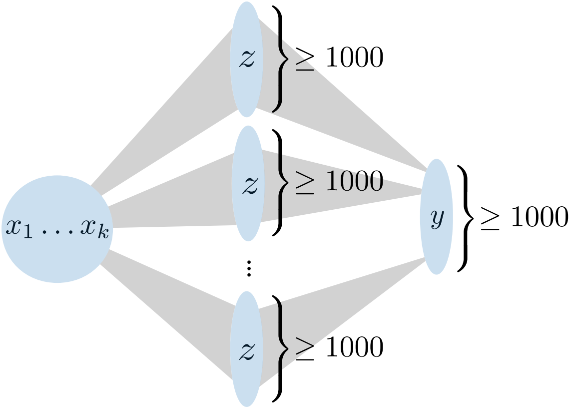

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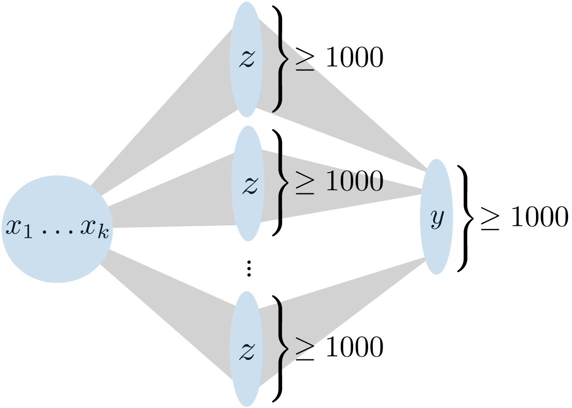
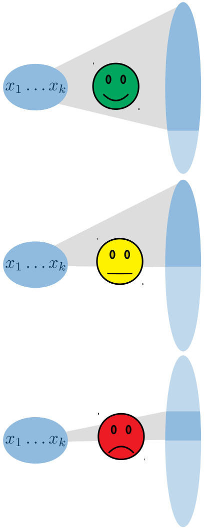


A formula φ is ε-*stable* on a graph G if scaling the counting literals by $(1 \pm \varepsilon)$ does not change whether $G \models \varphi$.

Stability

p-HalfDominatingSet: $\exists x_1 \ldots \exists x_k \,\#y \left(\bigvee_i y \sim x_i \wedge y = x_i\right) \geq n/2$



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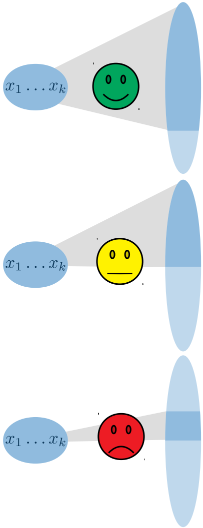

There exists a set dominating $\geq (1+\varepsilon)n/2$ vertices.

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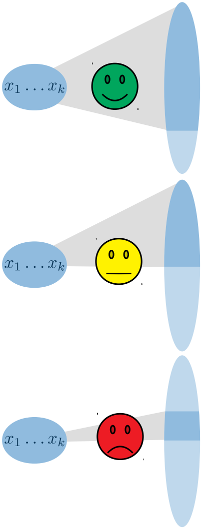

There exists a set dominating $\geq (1 + \varepsilon)n/2$ vertices.

All sets dominate $< (1 - \varepsilon)n/2$ vertices.

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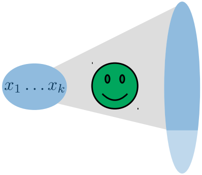

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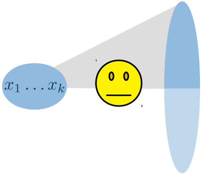
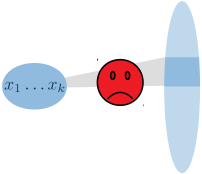


All sets dominate $< (1+\varepsilon)n/2$ vertices and there exists a set dominating $\geq (1-\varepsilon)n/2$ vertices.



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Main Result: Approximate Model-Checking

Theorem

Let $\mathcal{G}$ be a graph class with bounded expansion and $\varepsilon > 0$.

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Let $\mathcal{G}$ be a graph class with bounded expansion and $\varepsilon > 0$. There exists an algorithm which takes $G \in \mathcal{G}$, $\varphi \in$ FOX, runs in time $f(|\varphi|)n$ and returns , , or .

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- If  then $G \models \varphi$.
- If  then $G \not\models \varphi$.
- If  then φ is ε-unstable on G.

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Proof Idea: Quantifier Elimination

$$m_1 \leq \#x_1 \left(m_2 \leq \#x_2 \left(m_3 \leq \#x_3\ \varphi(x_1 x_2 x_3)\right)\right)$$

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$$\sum_{i=1}^{k} d(x_i)\varepsilon m \geq m \text{ with } d(x_i) \in \{0, \dots, 1/\varepsilon\}$$

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Add unary predicates $D_j(v)$ indicating that $d(v) = j$.

$\varphi'(x_1 \dots x_k)$: *"predicates of $x_1 \dots x_k$ represent sum $\geq m$."*

Further Results

How about extensions of FOX?

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Approximate Model-Checking becomes hard if FOX also allows

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How about extensions of FOX?

FOX allows $\#y\ \varphi \geq m$.

Approximate Model-Checking becomes hard if FOX also allows

- comparing $\#y$ and $\#z$,
- counting tuples $\#yz$,
- multiplication of counting terms,
- subtraction of counting terms.

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p-PartialDominatingSet can be solved in time $f(k)n$ on graph classes with bounded expansion.

This holds for all problems of the form

$$\exists x_1 \dots \exists x_k \# y \; \underbrace{\varphi_k(y x_1 \dots x_k)}_{\in \text{FO}}.$$

Modulo Counting

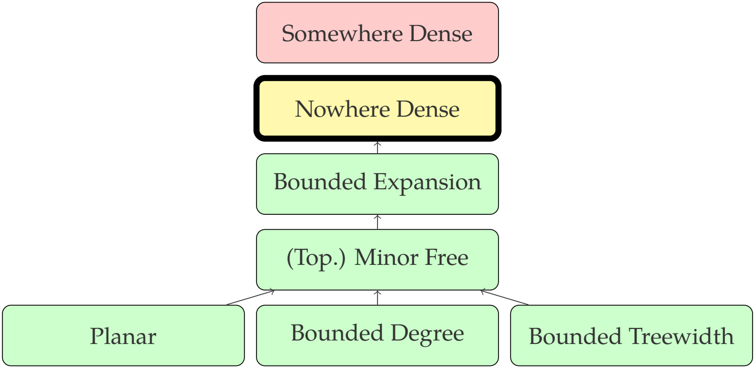
Theorem

One can do model-checking for Modulo-FOC in bounded expansion graph classes in time $f(|\varphi|)n$.

Modulo-FOC:

- Even number of odd degrees: $\#x\ (\#y\ x \sim y \equiv_2 1) \equiv_2 0$
- Euler circuit: $\forall x\ \#y\ x \sim y \equiv_2 0$
- Remove k to get Euler circuit: $\exists z_1 \ldots \exists z_k (\forall x\ x \neq z_i \vee \#y\ y \neq z_i \wedge x \sim y \equiv_2 0)$

Big Question



Can we generalize our results no nowhere dense graph classes?

Thank you!