The Parameterized Complexity of the Rectangle Stabbing Problem and its Variants*

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Abstract. We study the parameterized complexity of an NP-complete geometric covering problem called d-DIMENSIONAL RECTANGLE STABBING where we are given a set of axis-parallel d-dimensional hyperrectangles, a set of axis-parallel $(d-1)$-dimensional hyperplanes and a positive integer k; the question is whether one can select at most k hyperplanes so that every hyperrectangle is intersected by at least one of them. This problem is well-studied from the approximation point of view, while its parameterized complexity remained unexplored so far. We show that the case $d \geq 3$ is W[1]-hard with respect to the parameter k. The case $d = 2$ is still open and we investigate several natural restrictions of this case and show them to be fixed-parameter tractable.

1 Introduction

A geometric covering problem, in the broadest sense, consists of a set of geometric objects and a set of "resources"; the goal is to find a small set of resources that "covers" all objects. Geometric covering problems arise in many practical applications and are subject of intensive research (see [6, 7, 11]). In this paper we consider a geometric covering problem known as d-DIMENSIONAL RECTANGLE STABBING. Here, the input consists of a set R of axis-parallel d-dimensional hyperrectangles, a set L of axis-parallel $(d-1)$-dimensional hyperplanes, and a positive integer k; the question is whether there is a set $L' \subseteq L$ with $|L'| \leq k$ such that every hyperrectangle from R is intersected by at least one hyperplane from L'. In the special case of $d = 2$, the set R consists of axis-parallel rectangles in the plane, and L consists of vertical and horizontal lines. In the approximation setting, the optimization version of d-DIMENSIONAL RECTANGLE STABBING is considered, which asks for a *minimum-cardinality* set $L' \subseteq L$ to cover all rectangles from R.

The literature provides a bunch of results concerning the approximability of d-DIMENSIONAL RECTANGLE STABBING. Hassin and Megiddo [8] described

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a factor-$d2^{d-1}$ approximation algorithm for the problem variant where L consists of lines instead of hyperplanes and all hyperrectangles in R are identical. Gaur et al. [6] gave a factor-2 approximation algorithm for the case $d = 2$ and extended this result to the problem d-DIMENSIONAL RECTANGLE STABBING, for which they provided a factor-d approximation algorithm. Moreover, Mecke et al. [12] gave a factor-d approximation algorithm for a problem called d-C1P-SET COVER, which is a generalization of d-DIMENSIONAL RECTANGLE STABBING. A restricted version of 2-DIMENSIONAL RECTANGLE STABBING, where for every rectangle the number of horizontal lines intersecting it is bounded from above by one, is known as INTERVAL STABBING. This problem was considered by Kovaleva and Spieksma [9, 10], leading to constant-factor approximation algorithms for several variants of the problem. Hassin and Megiddo [8] gave approximation algorithms for the more general variant of INTERVAL STABBING where for every rectangle the number of horizontal lines or the number of vertical lines intersecting it is bounded from above by one. Weighted and capacitated versions of 2-DIMENSIONAL RECTANGLE STABBING have been considered by Even et al. [2].

Here, we consider d-DIMENSIONAL RECTANGLE STABBING from the point of view of parameterized complexity. More specifically, we investigate whether d-DIMENSIONAL RECTANGLE STABBING is fixed-parameter tractable with respect to the parameter "solution size" k, that is, if there exists an algorithm running in $O(f(k) \cdot |R \cup L|^{O(1)})$ time with f depending only on k.

On the one hand, we show in Section 3 that for $d \geq 3$ the problem is W[1]-hard, meaning that it is unlikely that there exists such an algorithm. On the other hand, in Section 4 we consider several natural restrictions of the case $d = 2$ and show them to be fixed-parameter tractable. The parameterized complexity for the case $d = 2$ without further restrictions remains open.

2 Preliminaries

A parameterized problem is a subset of $\Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet and $\mathbb{N}$ is the set of natural numbers. An instance of a parameterized problem is therefore a pair (I, k), where k is called the parameter. In the framework of parameterized complexity [1, 4, 13], the running time of an algorithm is viewed as a function of two quantities: the size of the problem instance *and* the parameter. A parameterized problem is said to be *fixed parameter tractable (FPT)* if there exists an algorithm for the problem running in $f(k) \cdot |I|^{O(1)}$ time, where f is a computable function only depending on k.

A common tool in the development of fixed-parameter algorithms is to use a set of *data reduction rules* to obtain what is called a *problem kernel*. A *data reduction rule* is a polynomial-time algorithm which takes a problem instance (I, k) and either decides whether (I, k) is a YES-instance or outputs an instance (I', k') such that $|I'| \leq |I|$, $k' \leq k$, and (I', k') is a YES-instances iff (I, k) is a YES-instance. An instance to which none of a given set of data reduction rules applies is called *reduced* with respect to these rules. A reduced instance (I', k') is called a *problem kernel* if its size is bounded from above by a function f depending only on k. If a parameterized problem has a kernel, then it is clearly fixed-parameter tractable.

A parameterized problem π_1 is *fixed-parameter reducible* to a parameterized problem π_2 if there are two computable functions $f, g : \mathbb{N} \to \mathbb{N}$ and an algorithm Φ which transforms an instance (I, k) of π_1 into an instance $(I', f(k))$ of π_2 in $g(k) \cdot |I|^{O(1)}$ time such that $(I', f(k))$ is a YES-instance for π_2 iff (I, k) is a YES-instance for π_1. The basic complexity class for fixed-parameter intractability is $W[1]$, as there is strong evidence that $W[1]$-hard problems are not fixed-parameter tractable [1, 4, 13]. To show that a problem is $W[1]$-hard, one needs to exhibit a fixed-parameter reduction from a known $W[1]$-hard problem to the problem at hand.

A graph $G = (V, E)$ is called *k-colorable* if there is a function $c : V \to \{1, \ldots, k\}$ satisfying $\forall \{u, v\} \in E : c(u) \neq c(v)$; the function c is then called a *proper vertex k-coloring* for G.

To achieve our our hardness result, we consider d-DIMENSIONAL RECTANGLE STABBING as a covering problem on binary matrices, which is a restriction of the following, very general matrix problem:

> **Set Cover**
> **Given:** A binary matrix M and a positive integer k.
> **Question:** Is there a set of at most k columns of M such that the submatrix M' of M that is induced by these columns has at least one 1 in every row?

To define restricted versions of SET COVER, we need the following definitions.

Definition 1 1. Given a binary matrix M, a *block of 1's* in a row of M is a maximal set of consecutive 1-entries in this row.
2. A binary matrix M has the *d-consecutive ones property (d-C1P)* if every row of M has at most d blocks of 1's.
3. A binary matrix M with columns $c_1, \ldots, c_n$ has the *separated d-consecutive ones property (d-SC1P)* if the columns of M can be partitioned into d sets of consecutive columns $C^1 = \{c_1, \ldots, c_{n_1}\}$, $C^2 = \{c_{n_1+1}, \ldots, c_{n_2}\}$, $\ldots$, $C^d = \{c_{n_{d-1}+1}, \ldots, c_n\}$ such that for every $p \in \{1, \ldots, d\}$ the submatrix of M induced by C^p has at most one block of 1's per row.

If SET COVER is restricted by demanding that the input matrix M must have the d-C1P, then we call the resulting problem d-C1P-SET COVER; if M must have the d-SC1P, then we call the resulting problem d-SC1P-SET COVER.

Observation 1 *The problems d-DIMENSIONAL RECTANGLE STABBING and d-SC1P-SET COVER are equivalent.*

This observation is easy to see—the ith dimension in a d-DIMENSIONAL RECTANGLE STABBING instance can be represented by the column set C^i in a d-SC1P-SET COVER instance and vice versa.

For some of our FPT algorithms, we make use of the following well-known fact: Given a set of axis-parallel rectangles and a set of vertical (horizontal) lines, the task of finding a minimum-cardinality subset of these vertical (horizontal) lines that intersects all rectangles is polynomial-time solvable[3]: Order the rectangles with respect to their right (bottom) end. Then, repeatedly take the first rectangle r in this order, include the rightmost vertical (bottommost horizontal) line l that intersects r into the solution, and delete all rectangles intersected by l, until all rectangles are deleted. The solution obtained is a minimum-size set of vertical (horizontal) lines that are required to intersect all rectangles. Moreover, all rectangles r together that are selected by the algorithm form a "certificate" in the sense that they cannot be intersected by a set of vertical (horizontal) lines that is smaller than the solution found by the algorithm. The pseudocode of this algorithm is displayed in Figure 2.

3 W[1]-Hardness Proof for d-Dimensional Rectangle Stabbing with $d \geq 3$

In this section we prove that d-DIMENSIONAL RECTANGLE STABBING with parameter k is W[1]-hard for every $d \geq 3$. To this end, we exhibit a parameterized reduction from MULTICOLORED CLIQUE, which is defined as follows, to 3-SC1P-SET COVER.

> **Multicolored Clique**
> **Given:** An undirected k-colorable graph $G = (V, E)$, a positive integer k, and a proper vertex k-coloring $c : V \to \{1, \ldots, k\}$ for G.
> **Question:** Is there a size-k clique in G?

MULTICOLORED CLIQUE is W[1]-complete [3][4]. Fellows et al. [3] give a parameterized reduction from MULTICOLORED CLIQUE to 3-C1P-SET COVER, which proves the W[1]-hardness of the latter problem. However, this reduction does not show the W[1]-hardness of 3-SC1P-SET COVER because of its more restricted nature.

The basic scheme of the reduction. The basic scheme of our reduction is fairly similar to the one used in the hardness proof for 3-C1P-SET COVER given by Fellows et al. [3]. The key idea is using an alternative, equivalent formulation of MULTICOLORED CLIQUE: Given an undirected k-colorable graph $G = (V, E)$, a positive integer k, and a proper vertex k-coloring $c : V \to \{1, \ldots, k\}$ for G, find a set $E' \subseteq E$ with $|E'| = \binom{k}{2}$ and a set $V' \subseteq V$ with $|V'| = k$ that satisfy the following constraints:

1. For every unordered pair $\{a, b\}$ of colors from $\{1, \ldots, k\}$, the edge set E' contains an edge whose endpoints are colored with a and b.

[3] This problem is equivalent to CLIQUE COVER on interval graphs.
[4] For a proof sketch see the Appendix.

2. For every color from $\{1, \ldots, k\}$, the vertex set V' contains a vertex of this color.
3. If E' contains an edge $\{u, v\}$, then V' contains the vertices u and v.

Given an instance (G, k, c) of Multicolored Clique, we construct an equivalent instance (M, k') of 3-SC1P-Set Cover based on this alternative formulation. To this end, define the *color of an edge* $\{u, v\}$, denoted with $d(\{u, v\})$, as $d(\{u, v\}) = \{c(u), c(v)\}$. We order the edges $E = \{e_1, \ldots, e_{|E|}\}$ and vertices $V = \{v_1, \ldots, v_{|V|}\}$ of G such that edges and vertices of the same color appear consecutively. (That is, for every pair $p_1, p_2 \in \{1, \ldots, |E|\}$ with $p_1 < p_2$ it holds that if $d(e_{p_1}) = d(e_{p_2})$ then $\forall p_3 \in \{p_1 + 1, \ldots, p_2 - 1\} : d(e_{p_3}) = d(e_{p_1}) = d(e_{p_2})$; for every pair $q_1, q_2 \in \{1, \ldots, |V|\}$ with $q_1 < q_2$ it holds that if $c(v_{q_1}) = c(v_{q_2})$ then $\forall q_3 \in \{q_1 + 1, \ldots, q_2 - 1\} : c(v_{q_3}) = c(v_{q_1}) = c(v_{q_2})$.)

The idea of the reduction is that every column of M corresponds to an edge or a vertex of the given graph G; the rows of M are constructed in such a way that any column subset of M that is a solution for 3-SC1P-Set Cover on (M, k') corresponds to a solution (E', V') for Multicolored Clique on (G, k, c). To this end, the rows of M must enforce that the three constraints for Multicolored Clique mentioned above are satisfied. In order to obtain a matrix that has the 3-SC1P, we need not only one, but *two* columns in M for every edge e in G. Hence an instance (G, k, c) of Multicolored Clique is mapped to an instance (M, k'), where $k' = 2 \cdot \binom{k}{2} + k$. We next describe the construction of M.

The columns of M. The matrix M has $2 \cdot |E| + |V|$ columns, partitioned into three sets $C^1 = \{c^1_1, \ldots, c^1_{|E|}\}$, $C^2 = \{c^2_1, \ldots, c^2_{|E|}\}$, and $C^3 = \{c^3_1, \ldots, c^3_{|V|}\}$ and ordered as follows: $c^1_1, \ldots, c^1_{|E|}, c^2_1, \ldots, c^2_{|E|}, c^3_1, \ldots, c^3_{|V|}$.

The rows of M. The rows of M have to ensure that every solution for 3-SC1P-Set Cover on $(M, k' = 2 \cdot \binom{k}{2} + k)$ corresponds to a subset of edges and vertices of G satisfying the three constraints mentioned above. Because there are two columns in M for every edge in G, we need *four* types of rows: Rows of Type 1 and 2 ensure that any set of k' columns that forms a solution for 3-SC1P-Set Cover contains exactly $\binom{k}{2}$ columns from C^1—one of each edge color—, $\binom{k}{2}$ columns from C^2—one of each edge color—, and k columns from C^3—one of each vertex color. Type 3 rows ensure that the columns chosen from C^1 and C^2 are consistent: if a solution contains the column c^1_j then it must contain c^2_j and vice versa. Finally, Type 4 rows ensure that if a solution contains a column c^1_j corresponding to the edge $\{u, v\}$ then it also contains the columns corresponding to the vertices u and v. See Fig. 1 for an illustration.

Type 1 rows. For every edge color $\{a, b\}$, add two rows $r^1_{\{a,b\},C^1}$ and $r^1_{\{a,b\},C^2}$ to M. For $i = 1, 2$, the row $r^1_{\{a,b\},C^i}$ has a 1 in every column $c^i_j \in C^i$ with $d(e_j) = \{a, b\}$, and 0's in all other columns.

Type 2 rows. For every vertex color $a \in \{1, \ldots, k\}$, add a row r^2_a to M which has a 1 in every column $c^3_j \in C^3$ with $c(v_j) = a$, and 0's in all other columns.

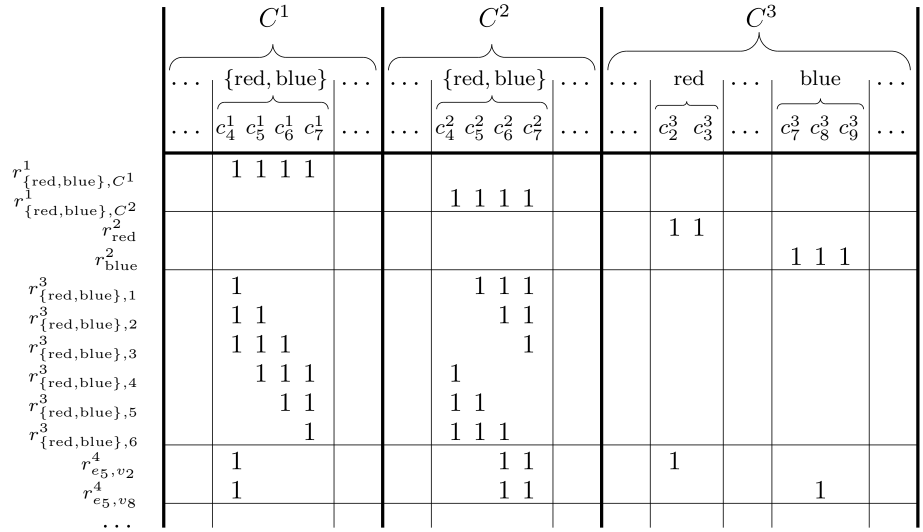

Fig. 1. Example for the construction of M. We assume that in G there are exactly two red vertices v_2, v_3 and exactly three blue vertices v_7, v_8, v_9, among vertices of other colors. The only edges between red and blue vertices are e_4, e_5, e_6, e_7 with $e_5 = \{v_2, v_8\}$. Note that $\text{first}(\{\text{red}, \text{blue}\}) = 4$.

Type 3 rows. For every edge color $\{a, b\}$, define

$$E_{\{a,b\}} := \{e \in E \mid d(e) = \{a, b\}\},$$
$$\text{first}(\{a, b\}) := \min\{p \in \{1, \dots, |E|\} \mid d(e_p) = \{a, b\}\}.$$

Now, for every edge color $\{a, b\}$, add a set of $2 \cdot (|E_{\{a,b\}}| - 1)$ rows $r^3_{\{a,b\},i}$ where $1 \leq i \leq 2 \cdot (|E_{\{a,b\}}| - 1)$. A row $r^3_{\{a,b\},i}$, $i \in \{1, \dots, |E_{\{a,b\}}| - 1\}$, has a 1 in

- every column $c^1_j \in C^1$ with $d(e_j) = \{a, b\}$ and $j < \text{first}(\{a, b\}) + i$ and
- every column $c^2_j \in C^2$ with $d(e_j) = \{a, b\}$ and $j \geq \text{first}(\{a, b\}) + i$,

and 0's in all other columns. A row $r^3_{\{a,b\},i}$ with $i \in \{|E_{\{a,b\}}|, \dots, 2 \cdot (|E_{\{a,b\}}| - 1)\}$ has a 1 in

- every column $c^1_j \in C^1$ with $d(e_j) = \{a, b\}$ and $j \geq \text{first}(\{a, b\}) + i - (|E_{\{a,b\}}| - 1)$ and
- every column $c^2_j \in C^2$ with $d(e_j) = \{a, b\}$ and $j < \text{first}(\{a, b\}) + i - (|E_{\{a,b\}}| - 1)$,

and 0's in all other columns.

Type 4 rows. For every edge $e_p = \{v_{q_1}, v_{q_2}\} \in E$, add two rows $r^4_{e_p,v_{q_1}}$ and $r^4_{e_p,v_{q_2}}$ to M. For $i = 1, 2$, the row $r^4_{e_p,v_{q_i}}$ has a 1 in

- every column $c_j^1 \in C^1$ with $j < p$,
- every column $c_j^2 \in C^2$ with $j > p$, and
- the column $c_{q_i}^3 \in C^3$,

and 0's in all other columns.

Lemma 1. *Let (G, k, c) be an instance of* Multicolored Clique *and let (M, k') be the instance of* 3-SC1P-Set Cover *obtained by the above construction. Then G has a clique of size k if and only if there exists a set of $k' = 2 \cdot \binom{k}{2} + k$ columns in M that hits a 1 in every row.*

Proof. Omitted due to lack of space. □

Theorem 1. *For every $d \geq 3$, d-*dimensional Rectangle Stabbing *is* W[1]-*hard with respect to the parameter k.*

By adding some additional columns to the above construction, we get the following result.

Theorem 2. *For every $d \geq 3$, the restricted variant of d-*dimensional Rectangle Stabbing *where every hyperrectangle in R is a hypercube is* W[1]-*hard with respect to the parameter k.*

With a very similar reduction from Multicolored Clique we can also show the W[1]-hardness of the following problem: Given a set R of axis-parallel d-dimensional hyperrectangles, a set L of axis-parallel lines, and a positive integer k, is there a set $L' \subseteq L$ with $|L'| \leq k$ such that every hyperrectangle from R is intersected by at least one line from L'?

Theorem 3. *For every $d \geq 3$, the problem of stabbing hyperrectangles with lines as described above is* W[1]-*hard with respect to the parameter k.*

4 FPT Algorithms for Restrictions of 2-Dimensional Rectangle Stabbing

In the previous section we have shown that d-dimensional Rectangle Stabbing with parameter k is W[1]-hard for $d \geq 3$. However, the parameterized complexity of 2-dimensional Rectangle Stabbing, where a set R of axis-parallel rectangles has to be stabbed with at most k lines chosen from a given set L of vertical and horizontal lines, is still open. In this section we consider some natural restrictions of this problem and show them to be fixed-parameter tractable.

For an instance (R, L, k) of 2-dimensional Rectangle Stabbing, let $L = V \uplus H$, where $V = \{v_1, \ldots, v_n\}$ are the vertical lines ordered from left to right and $H = \{h_1, \ldots, h_m\}$ are the horizontal lines ordered from top to bottom. For a rectangle $r \in R$, let $\mathrm{l}(r), \mathrm{r}(r), \mathrm{t}(r), \mathrm{b}(r)$ be the index of the leftmost, rightmost, topmost and bottommost line intersecting r. Define the width $\mathrm{w}(r) := \mathrm{r}(r) -$

$\mathrm{l}(r) + 1$ and the height $\mathrm{h}(r) := \mathrm{b}(r) - \mathrm{t}(r) + 1$ as the number of vertical and horizontal lines, respectively, intersecting r.

We start with some well-known reduction rules for 2-dimensional Rectangle Stabbing, whose correctness is obvious.

1. If there is a rectangle that is intersected by no line from L, the given instance is a no-instance.
2. If there is a rectangle that is intersected by exactly one line $l \in L$, then delete l, delete all rectangles that are intersected by l, and decrease k by one.
3. If there are two lines $l_1, l_2 \in L$ such that $\forall r \in R$: (l_2 intersects r) $\Rightarrow$ (l_1 intersects r), then delete l_2.
4. If there are two rectangles r_1, r_2 such that $\forall l \in L$: (l intersects r_1) $\Rightarrow$ (l intersects r_2), then delete r_2.

The following observation is an immediate consequence of Reduction Rule 3.

Observation 2 *In a reduced problem instance, for every vertical line $v_j \in V$ there exist rectangles $r, r' \in R$ with $\mathrm{l}(r) = j$ and $\mathrm{r}(r') = j$. For every horizontal line $h_i \in H$ there exist rectangles $r, r' \in R$ with $\mathrm{t}(r) = i$ and $\mathrm{b}(r') = i$.*

In particular, Observation 2 implies that in a reduced problem instance there exist rectangles $r, r' \in R$ such that $\mathrm{w}(r) = 1$ and $\mathrm{h}(r') = 1$.

4.1 Case 1: Rectangles Have Bounded Height

We first consider the case where the height of every rectangle in R is bounded by a number b. The special case $b = 1$ where every rectangle is a horizontal segment is NP-complete; Hassin and Megiddo [8] and Kovaleva and Spieksma [9, 10] gave approximation algorithms for this case and some of its variants.

For our FPT considerations, we use a simple search-tree algorithm using Observation 2. At every step, apply the reduction rules until the current instance is reduced, search for a rectangle r with $\mathrm{r}(r) = 1$, and branch as follows: either select the single vertical line that intersects r or select one of the at most b horizontal lines that intersect r.

Theorem 4. *The restricted variant of* 2-dimensional Rectangle Stabbing *where the height $\mathrm{h}(r)$ of every rectangle $r \in R$ is bounded from above by a number b can be solved in $O((b+1)^k \cdot n^{O(1)})$ time and is therefore fixed-parameter-tractable with respect to the combined parameters k and b.*

This algorithm can be modified to work for the weighted version of the problem where every line has a weight that is bounded from below by 1 and from above by a number b'. The reduction rules need modification for this problem version; the running time of the algorithm is $O((b + b')^k \cdot n^{O(1)})$.

4.2 Case 2: Rectangles Have Bounded Width or Height

Next, we consider a generalization of Case 1: Here, for every rectangle r in R the *width* $\mathrm{w}(r)$ *or the height* $\mathrm{h}(r)$ is bounded from above by a number b. The special case $b = 1$, where every rectangle is either a horizontal or a vertical segment, is NP-complete and was already considered by Hassin and Megiddo [8] from the approximation point of view.

The approach outlined in Section 4.1 does not work anymore since in a reduced instance the height of every rectangle r with $\mathrm{r}(r) = 1$ may be unbounded. However, there is again a search-tree algorithm. Let $R_h \subseteq R$ be the set of rectangles with bounded height and let $R_v \subseteq R$ be the set of rectangles with bounded width. Now, we write k as a sum $k_h + k_v$ in all possible ways, where k_h and k_v denote the number of horizontal and vertical lines, respectively, allowed to be chosen into the solution. For every split of k into k_h and k_v, we run a branching algorithm, which performs in every step the following actions.

First, compute the minimum number of vertical lines required to intersect the rectangles in R_h. This is polynomial-time doable, and the simple greedy algorithm in Figure 2 obtains such a set of vertical lines. If R_h cannot be stabbed with a set of at most k_v vertical lines, then the algorithm in Figure 2 outputs a set $R_h^0 \subseteq R_h$ of size $k_v + 1$ such that the optimum number of vertical lines needed to intersect all rectangles in R_h^0 is exactly $k_v + 1$. Any solution for d-dimensional Rectangle Stabbing on (R, L, k) consisting of at most k_v vertical and at most k_h horizontal lines must intersect at least one rectangle in R_h^0 by a horizontal line. Hence, branch on the $(k_v + 1) \cdot b$ possibilities to do so.

If, however, all rectangles in R_h can be intersected with k_v vertical lines, we use the greedy algorithm to check whether the rectangles in R_v can be intersected with k_h horizontal lines. If not, we branch on $(k_h + 1) \cdot b$ possibilities in analogy to the branching for R_h^0 described above; otherwise, we return the union of the solutions returned by the two calls to the greedy algorithm. Figure 3 shows a pseudocode for this algorithm. The branching number is at most bk, which leads to the following theorem.

Theorem 5. *The restricted variant of* 2-dimensional Rectangle Stabbing *where the width or the height of every rectangle in R is bounded from above by a number b can be solved in $O((bk)^k \cdot n^{O(1)})$ time and is therefore fixed-parameter-tractable with respect to the combined parameters k and b.*

4.3 Case 3: Bounded Intersection

In this subsection we consider a restriction of 2-dimensional Rectangle Stabbing in which every horizontal line intersects at most b rectangles from R; this restriction was already considered by Kovaleva and Spieksma [9, 10] from the approximation point of view. For $b = 1$, this problem is clearly polynomial-time solvable since the horizontal lines can just be ignored. For $b = 2$ the problem is NP-complete[5], but there is an easy $O(k^k \cdot n^{O(1)})$-time branching algorithm.

[5] For a proof see the Appendix.

```
function greedy(R, L, k) {
    // Input:   R: a set of rectangles,
    //          L: a set of lines that are either all vertical or all horizontal,
    //          k: a nonnegative integer.
    // Output: Either L′ ⊆ L or R⁰ ⊆ R.
    //          If all rectangles from R can be stabbed with a set L′ of at most k lines
    //          from L, then such a set L′ is returned.
    //          Otherwise, a set R⁰ of k + 1 rectangles from R is returned that cannot be
    //          stabbed with at most k lines from L.
    R′ := R;   R⁰ := ∅;   L′ := ∅;
    while R′ ≠ ∅: {
        if L contains only vertical lines: {
            r := a rectangle from R′ with minimum r(r);   l := v_{r(r)};   }
        else { // L contains only horizontal lines
            r := a rectangle from R′ with minimum b(r);   l := h_{b(r)};   }
        R⁰ := R⁰ ∪ {r};   L′ := L′ ∪ {l};
        delete all rectangles from R′ that are intersected by l;
        if |R⁰| = k + 1: return R⁰;   }
    return L′;   }
```

Fig. 2. Greedy algorithm for stabbing a set R of rectangles with at most k lines chosen from a given set L of vertical lines or horizontal lines.

However, this algorithm cannot be generalized for the case $b \geq 3$. In this subsection, we show that this restriction of 2-DIMENSIONAL RECTANGLE STABBING is fixed-parameter tractable with respect to the combined parameters k and b by developing a problem kernel.

First, in addition to the reduction rules mentioned previously, we use the following reduction rule:

5. If there are $bk + 2$ rectangles $r_1, \ldots, r_{bk+2} \in R$ such that for each $i \in \{1, \ldots, bk + 1\}$ it holds that every vertical line that intersects r_i also intersects r_{bk+2}, then delete r_{bk+2}.

The correctness of this reduction rule follows from the fact that k horizontal lines cannot intersect all rectangles $r_1, \ldots, r_{bk+1}$. Hence, if the instance with r_{bk+2} deleted is a YES-instance, every solution must contain a vertical line stabbing some of the rectangles $r_1, \ldots, r_{bk+1}$, and this line also stabs r_{bk+2} in the original instance, which, therefore, is also a YES-instance.

The following two observations are immediate consequences of Reduction Rule 5.

Observation 3 *For every rectangle r in a reduced instance there are at most bk rectangles $r' \neq r$ with $\mathrm{l}(r') \geq \mathrm{l}(r)$ and $\mathrm{r}(r') \leq \mathrm{r}(r)$.*

Observation 4 *In a reduced instance, for every $j \in \{1, \ldots, n\}$ there are at most $bk + 1$ rectangles r with $\mathrm{l}(r) = j$.*

Lemma 2. *For every rectangle $r \in R$ in a reduced instance it holds that $\mathrm{r}(r) \leq (bk + 1) \cdot \mathrm{l}(r)$.*

```
function stab(R_h, R_v, H, V, k_h, k_v) {
        // Input:   R_h: a set of rectangles with bounded height,
        //          R_v: a set of rectangles with bounded width,
        //          H, V: a set of horizontal lines and a set of vertical lines,
        //          k_h, k_v: nonnegative integers.
        // Output: A subset of H ∪ V containing ≤ k_h lines from H and ≤ k_v lines from V
        //          that stabs all rectangles from R_h ∪ R_v, or null, if no such subset exists.
        if greedy(R_h, V, k_v) returns a set R_h^0 ⊆ R_h of rectangles: {
                if k_h = 0: return null;
                for every r ∈ R_h^0: for every h ∈ H(r): {
                        R'_h := R_h \ R_h(h);   R'_v := R_v \ R_v(h);   H' := H \ {h};
                        A := stab(R'_h, R'_v, H', V, k_h − 1, k_v);
                        if A ≠ null: return A ∪ h;   }
                return null;   }
        if greedy(R_v, H, k_h) returns a set R_v^0 ⊆ R_v of rectangles: {
                if k_v = 0: return null;
                for every r ∈ R_v^0: for every v ∈ V(r): {
                        R'_h := R_h \ R_h(v);   R'_v := R_v \ R_v(v);   V' := V \ v;
                        A := stab(R'_h, R'_v, H, V', k_h, k_v − 1);
                        if A ≠ null: return A ∪ v;   }
                return null;   }
        B := solution V' returned by greedy(R_h, V, k_v);
        C := solution H' returned by greedy(R_v, H, k_h);   return B ∪ C;   }
```

Fig. 3. Branching algorithm for stabbing a set $R_v \cup R_h$ of rectangles with at most k_v lines chosen from a given set V of vertical lines and at most k_h lines chosen from a given set H of horizontal lines.

Proof. By induction on $\mathrm{l}(r)$. By Observation 3, the lemma is true for all rectangles r with $\mathrm{l}(r) = 1$. Now assume the lemma to be true for all rectangles r with $1 \leq \mathrm{l}(r) \leq j$, and let r be a rectangle with $\mathrm{l}(r) = j + 1$. For the sake of a contradiction, assume that $\mathrm{r}(r) \geq (bk + 1) \cdot \mathrm{l}(r) + 1$. Observation 2 implies that there must be a rectangle r' with $\mathrm{r}(r') = p$ for every $p \in \{\mathrm{l}(r), \ldots, \mathrm{r}(r) - 1\}$. Due to Observation 3, at most bk of these rectangles can have $\mathrm{l}(r') \geq \mathrm{l}(r)$, and, hence, there exists $p \in \{\mathrm{r}(r) - bk - 1, \ldots, \mathrm{r}(r) - 1\}$ such that there is a rectangle r' with $\mathrm{r}(r') = p$ and $\mathrm{l}(r') < \mathrm{l}(r)$. But then, by the induction hypothesis, $\mathrm{r}(r') \leq (bk+1) \cdot (\mathrm{l}(r) - 1)$, which is a contradiction to $\mathrm{r}(r') = p \geq \mathrm{r}(r) - bk - 1$, since we assumed that $\mathrm{r}(r) \geq (bk + 1) \cdot \mathrm{l}(r) + 1$. This proves the lemma. □

Lemma 3. *In a reduced instance, for every $j \in \{1, \ldots, n - 1\}$ there is a rectangle $r \in R$ with $\mathrm{l}(r) > j$ and $\mathrm{r}(r) \leq (bk + 1) \cdot j + 1$.*

Proof. Assume for the sake of contradiction that there exists $j \in \{1, \ldots, n - 1\}$ such that for every rectangle $r \in R$ with $\mathrm{l}(r) > j$ it holds that $\mathrm{r}(r) > (bk+1) \cdot j + 1$. Consider a rectangle r' with $\mathrm{r}(r') = (bk + 1) \cdot j + 1$. Such a rectangle exists by Observation 2. Then it holds that $\mathrm{l}(r') \leq j$ due to our assumption. But by Lemma 2 we have $\mathrm{r}(r') \leq (bk + 1) \cdot j$, a contradiction. □

Corollary 1. *Let $q \leq n$, and let $\{v_{j_1}, v_{j_2}, \ldots, v_{j_q}\} \subseteq V$ with $j_1 < j_2 < \ldots < j_q$ be a set of vertical lines stabbing all rectangles from R in a reduced instance. Then for every $i \in \{1, \ldots, q\}$ it holds that $j_i \leq \frac{(bk+1)^i - 1}{bk}$.*

Proof. By induction on i. For $i = 1$, the statement holds because in any reduced instance there is a rectangle r with $\mathrm{l}(r) = \mathrm{r}(r) = 1$. Assume that the statement holds for $i-1$, that is, $j_{i-1} \leq \frac{(bk+1)^{i-1}-1}{bk}$. By Lemma 3, there is a rectangle $r \in R$ with $\mathrm{l}(r) > j_{i-1}$ and $\mathrm{r}(r) \leq (bk+1) \cdot j_{i-1} + 1$. Clearly this rectangle is not stabbed by any line from $\{v_{j_1}, \ldots, v_{j_{i-1}}\}$ and therefore, we have $j_i \leq \mathrm{r}(r) \leq (bk+1) \cdot j_{i-1} + 1 \leq \frac{(bk+1)^{i}-1}{bk}$. □

Observation 5 *If an instance of the restricted variant of* 2-DIMENSIONAL RECTANGLE STABBING *is a* YES-*instance, then there is a set $V' \subseteq V$ of at most bk vertical lines that intersect all rectangles in R.*

Proof. Replace every horizontal line h in an optimal solution by at most b vertical lines that intersect the rectangles intersected by h. □

Now we are ready to prove the existence of a problem kernel.

Theorem 6. *The restricted variant of* 2-DIMENSIONAL RECTANGLE STABBING *where every horizontal line intersects at most b rectangles has a kernel of size $O((bk+1)^{bk})$ and is therefore fixed-parameter tractable with respect to the combined parameters k and b.*

Proof. Given an instance of this restricted version, first find the optimal number of vertical lines needed to intersect all rectangles. As noted before, this is polynomial-time doable. If the optimal solution size is greater than bk, report that the given instance is a NO-instance. Otherwise, by Corollary 1, we know that every set of vertical lines $\{v_{j_1}, \ldots, v_{j_{bk}}\}$ that intersects all rectangles in R has $j_{bk} \leq \frac{(bk+1)^{bk}-1}{bk}$. If the given instance is a YES-instance, then R cannot contain any rectangle r with $\mathrm{l}(r) > j_{bk}$. For every $j \in \{1, \ldots, j_{bk}\}$, however, there are at most $bk+1$ rectangles r with $\mathrm{l}(r) = j$ due to Observation 4. Hence, if R contains more than $O((bk+1)^{bk})$ rectangles, report that the given instance is a NO-instance. □

5 Open Questions

We showed that d-DIMENSIONAL RECTANGLE STABBING with $d \geq 3$ is W[1]-hard. However, the parameterized complexity of the perhaps most interesting case $d = 2$ remains open, as well as that of 2-C1P-SET COVER. Even for the restriction of 2-DIMENSIONAL RECTANGLE STABBING where no two rectangles from R "overlap" (two rectangles r_1, r_2 overlap if there exist a vertical line v and a horizontal line h that intersect both r_1 and r_2) we do not know the parameterized complexity. For d-DIMENSIONAL RECTANGLE STABBING, we do not know if the restricted variant is W[1]-hard where all hyperrectangles in R are hypercubes of the same edge length. Another open question is whether there are faster parameterized algorithms for the restrictions of 2-DIMENSIONAL RECTANGLE STABBING considered here.

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Appendix

Proof Sketch of the W[1]-Hardness of Multicolored Clique

We describe a parameterized reduction from the W[1]-complete problem Clique [1] to Multicolored Clique.

> **Clique**
> **Given:** An undirected graph $G = (V, E)$ and a positive integer k.
> **Question:** Is there a clique of size at least k in G?

Given an instance $(G = (V, E), k)$ of Clique, construct an instance $(G' = (V'E'), k, c)$ of Multicolored Clique as follows. The vertex set V' consists of k copies of V, that is, for every vertex $v \in V$ there are k vertices $v^1, v^2, \ldots, v^k$ in V'. The edge set E' is given by

$$E' = \bigcup_{i,j \in \{1,\ldots,k\} \wedge i \neq j} \{\{u^i, v^j\} \mid \{u, v\} \in E\},$$

and the coloring c is given by $c(v^i) = i$. The correctness of this reduction is obvious.

Proof Sketch for the NP-Hardness of a Restricted Variant of 2-Dimensional Rectangle Stabbing

We show that 2-dimensional Rectangle Stabbing is NP-hard even when every horizontal line in the input intersects at most two rectangles from R. The reduction is from the NP-complete problem Vertex Cover (see [5]). Given a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, first transform it into a graph G' by replacing every edge $e = \{v_i, v_j\}$ of G by a path v_j-x_e-y_e-v_j. Then G' has a vertex cover of size $|E| + k$ iff G has a vertex cover of size k. Now the vertex cover instance $(G', |E| + k)$ can be transformed into an instance of 2-SC1P-Set Cover as follows. Let $r = |V| + 2|E|$ and $s = 3|E|$, and let M be an $s \times r$-matrix whose columns represent the vertices and whose rows represent the edges of G'. The columns of M are ordered as follows:

$$v_1, v_2, \ldots, v_n, x_{e_1}, y_{e_1}, \ldots, x_{e_m}, y_{e_m}.$$

An entry in a row i and a column j of M is 1 if and only if the edge e_i is incident to the vertex corresponding to column j. Clearly M has the 2-SC1P with $C^1 = \{v_1, v_2, \ldots, v_n\}$ and $C^2 = \{x_{e_1}, y_{e_1}, \ldots, x_{e_m}, y_{e_m}\}$. Moreover, there are exactly two 1's in every column $x_{e_1}, y_{e_1}, \ldots, x_{e_m}, y_{e_m}$. Therefore, the matrix M can be transformed into an equivalent instance of the restricted variant of 2-dimensional Rectangle Stabbing. One can easily show that G' has a vertex cover of size $|E| + k$ if and only if M has a set cover solution of size $|E| + k$.