Computing Treedepth

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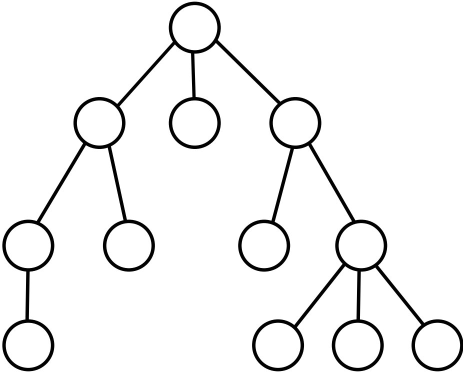
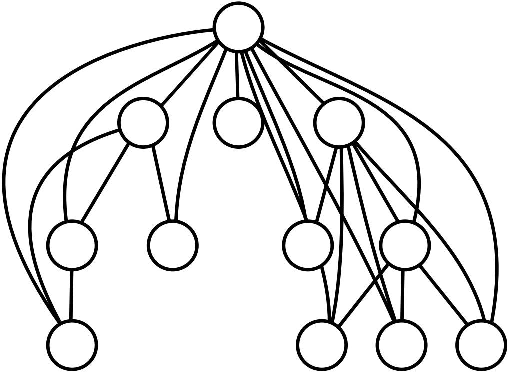
Somnath Sikdar

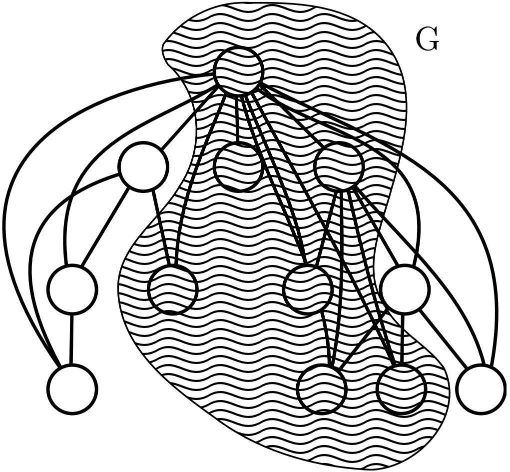
RWTH Aachen

March 14, 2014

Treedepth is a width measure.



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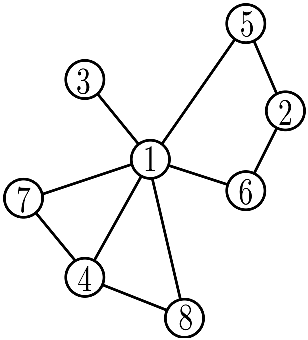
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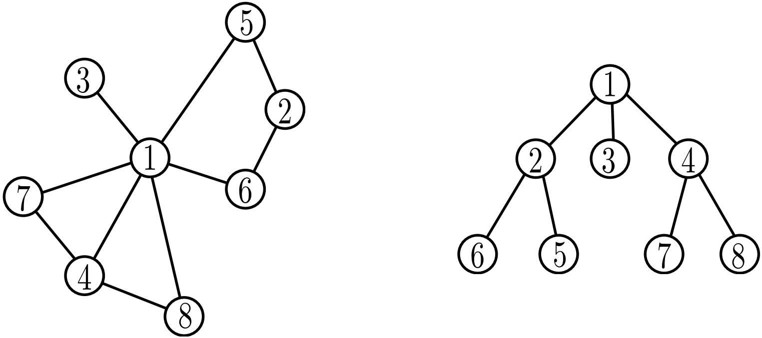
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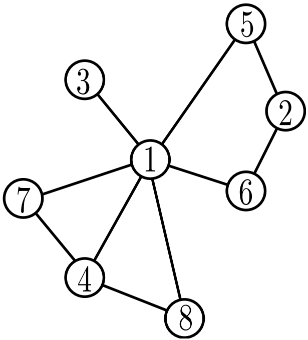
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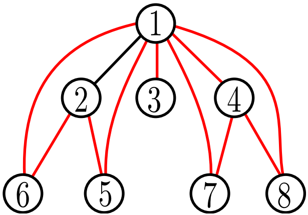
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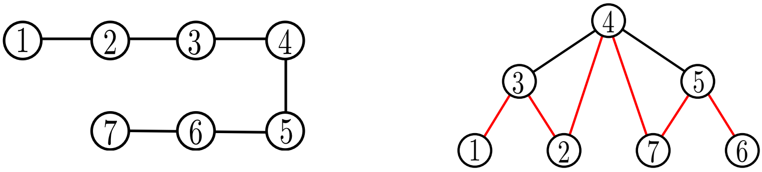

Treedepth $t \rightarrow$ Maximal path length $2^t - 1$.

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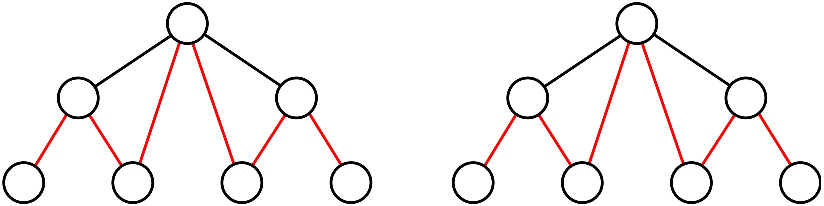

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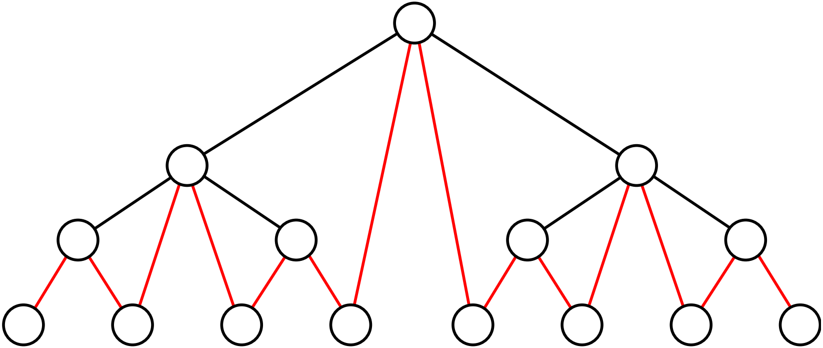


Definition (Treedepth decomposition)

A *treedepth decomposition* of a graph G is a rooted forest F such that $V(G) \subseteq V(F)$ and $E(G) \subseteq E(clos(F))$.

Definition (Treedepth)

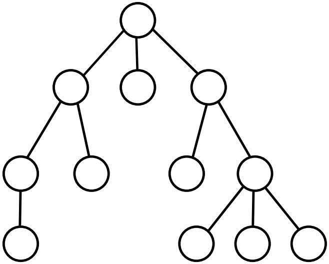
The *treedepth* $\mathbf{td}(G)$ of a graph G is the minimum height of any treedepth decomposition of G.

Equivalent notions

A graph G has treedepth at most t if and only if:

- G is a subgraph the closure of a tree (forest) of height t
- G is the subgraph of a trivially perfect graph with clique size at most t
- G has a centered coloring with t colors
- G has a ranked coloring with t colors

A DFS is a Treedepth decomposition



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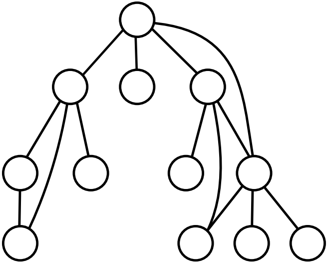

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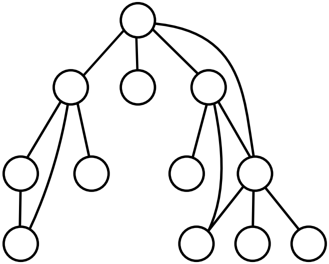
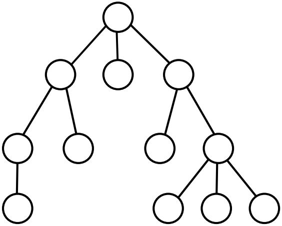
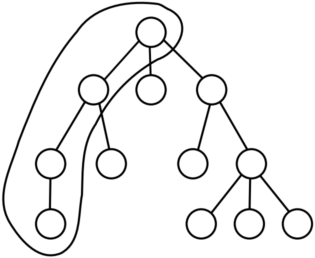


Treedepth $t \Rightarrow$ Maximal path length $2^t - 1 \Rightarrow 2^t$-approximation

Treedepth to pathwidth



Treedepth to pathwidth



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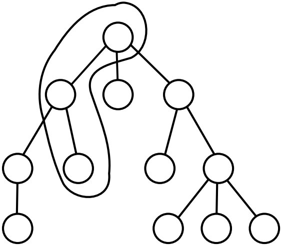

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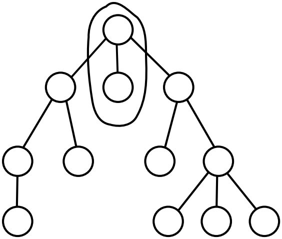

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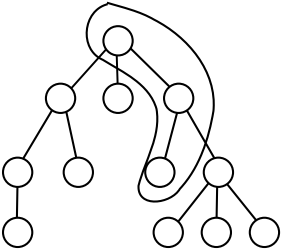

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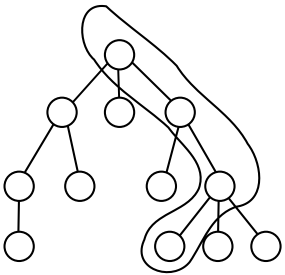

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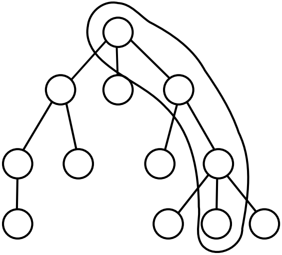

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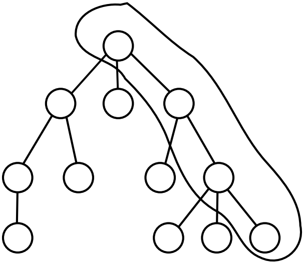

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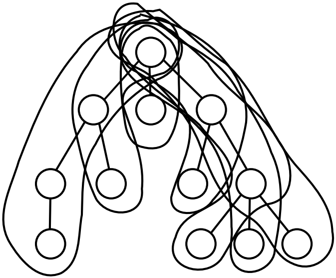
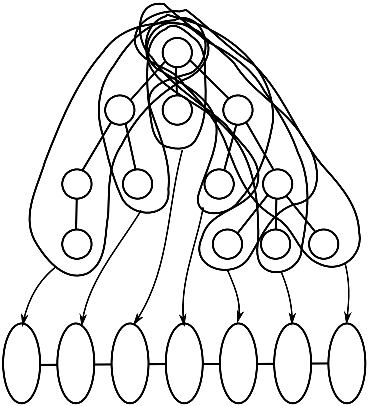


Treedepth to pathwidth



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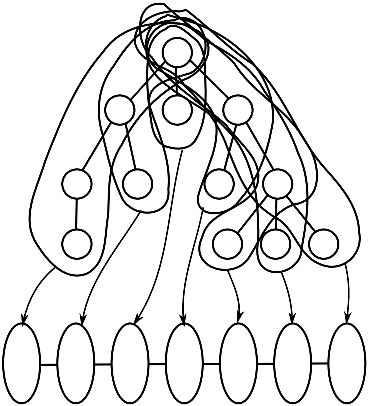

$$\mathbf{tw}(G) \leq \mathbf{pw}(G) \leq \mathbf{td}(G) - 1$$

Treedepth $t \Rightarrow$ Path decomposition of width $2^t - 2$

Known ways to compute the treedepth of a graph:

- In $f(t) \cdot n^3$ time by Robertson and Seymour.
- $\mathbf{tw}(G) \leq \mathbf{td}(G) - 1 \Rightarrow$ By Courcelle's Theorem $2^{2^{2^{\cdot^{\cdot^{\cdot^{t}}}}}} \cdot n$.
- Algorithm by Bodlaender et. al. with running time $2^{O(w^2 t)} \cdot n^2$.

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Problem

Is there a simple linear time algorithm to check $\mathbf{td}(G) \leq t$ *for fixed* t*?*

Our results:

- A (relatively) simple direct algorithm in time $2^{2^{O(t)}} \cdot n$.
- A fast algorithm in time $2^{O(t^2)} \cdot n$.

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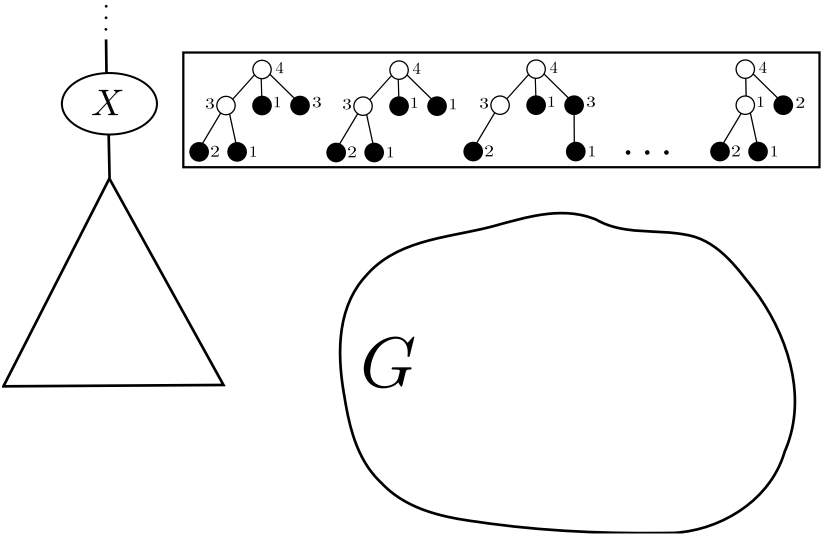
Both results follow from an algorithm on tree decompositions which runs in time $2^{O(wt)} \cdot n$.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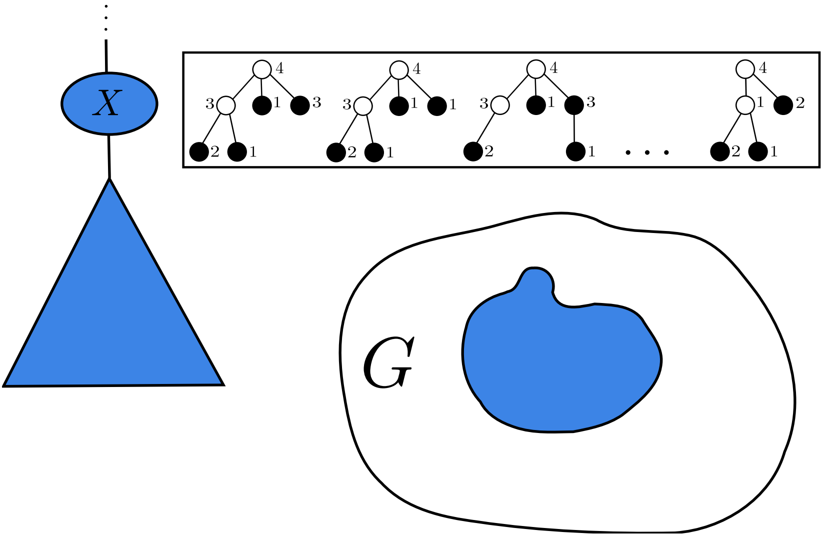
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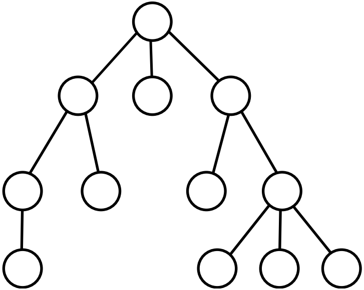

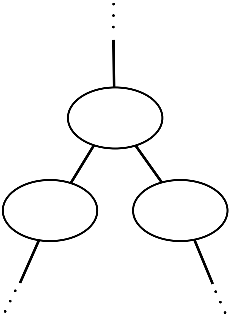

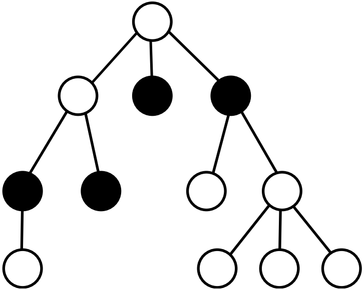

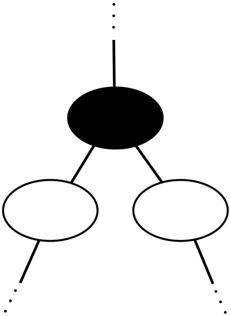

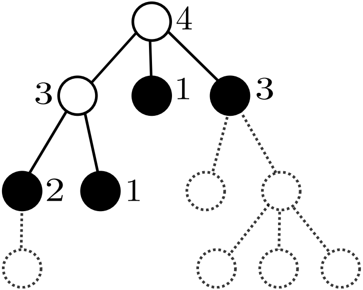
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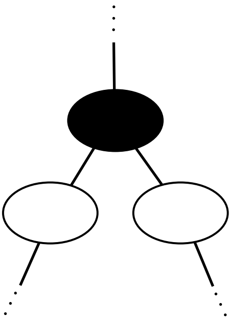

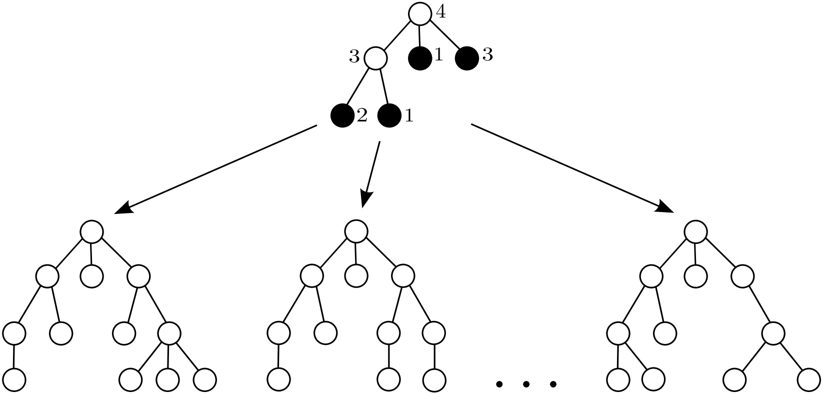
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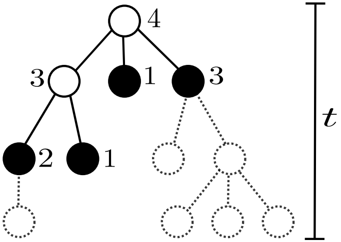
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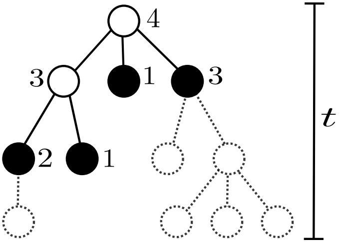
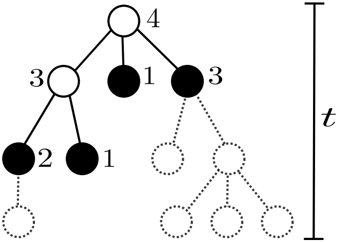


Any partial decomposition has less than *wt* nodes.

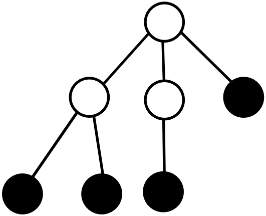


Any partial decomposition has less than *wt* nodes.

Counting

- How many trees with labeled leaves?
- How many height labelings per tree?

How many trees with labeled leaves?



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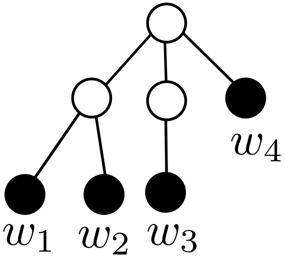

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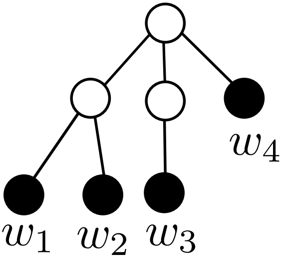

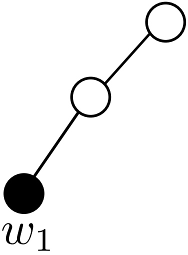

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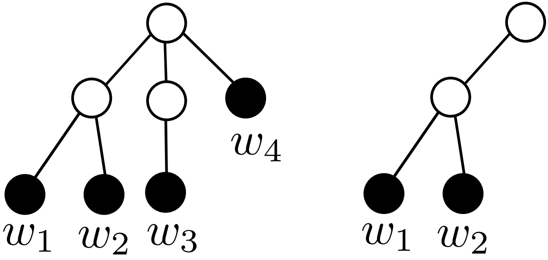

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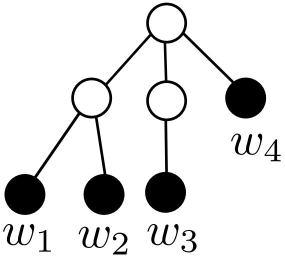

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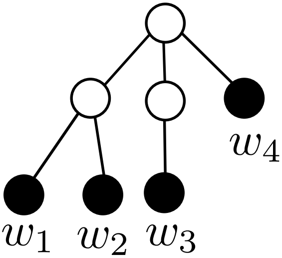

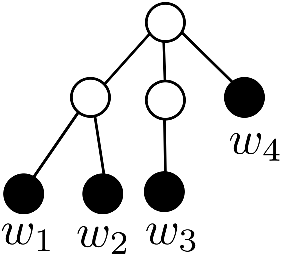

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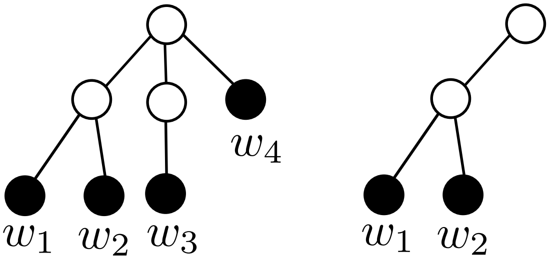

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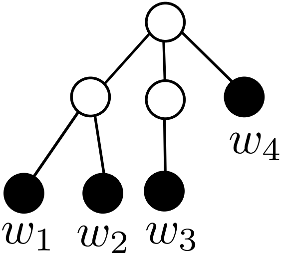

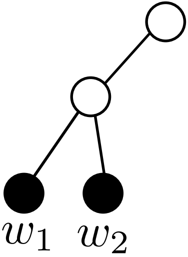

$$\prod_{i=1}^{w} it \cdot t$$

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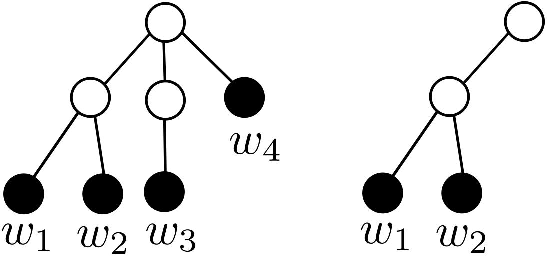

$$\prod_{i=1}^{w} it \cdot t \leq w! \cdot t^{2w}$$

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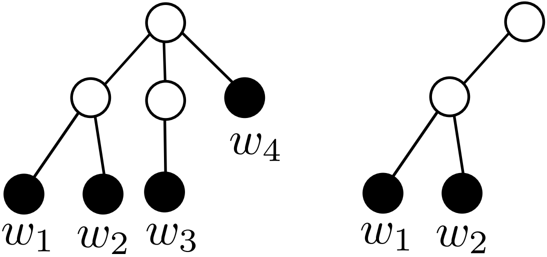

$$\prod_{i=1}^{w} it \cdot t \leq w! \cdot t^{2w} \leq 2^{2w \log t + w \log w}$$

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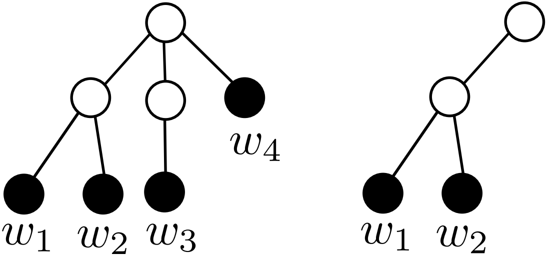
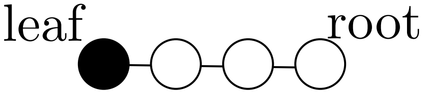


$$\prod_{i=1}^{w} it \cdot t \leq w! \cdot t^{2w} \leq 2^{2w \log t + w \log w} = 2^{O(wt)}$$

How many height labelings per tree?

How many labeling can a path have?



$$(1, 2, 3, 4, 5, 6, \ldots, t)$$

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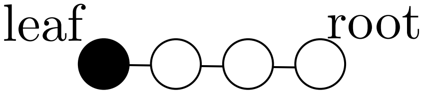

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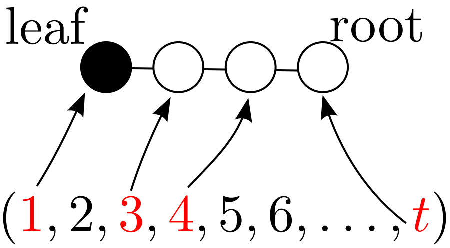

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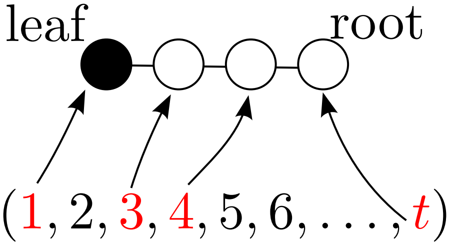
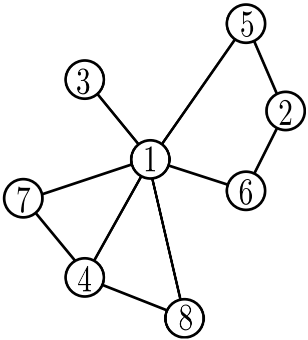
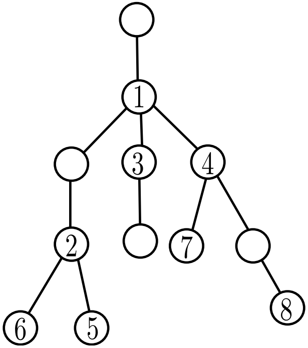


$$(2^t)^w = 2^{O(wt)}$$

The total number of labeled trees is $2^{O(wt)} \cdot 2^{O(wt)} = 2^{O(wt)}$.

Non-trivially improvable treedepth decompositions





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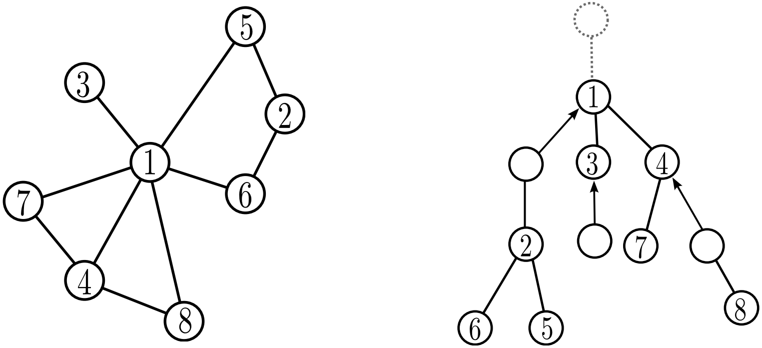

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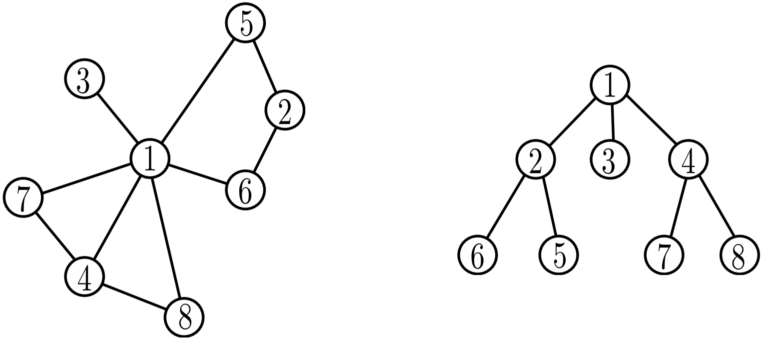

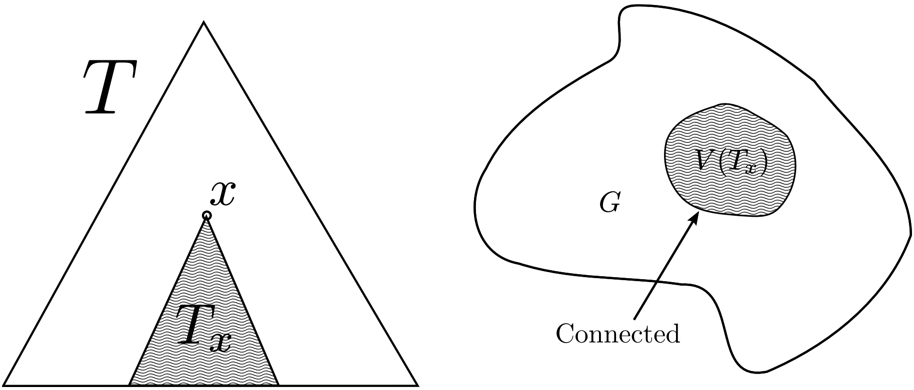

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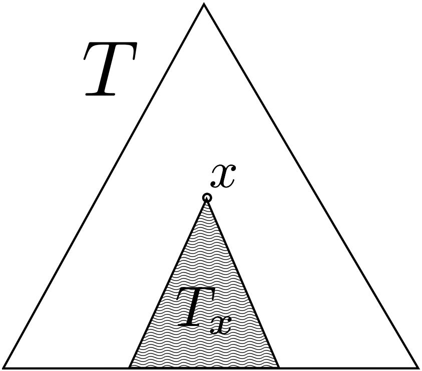
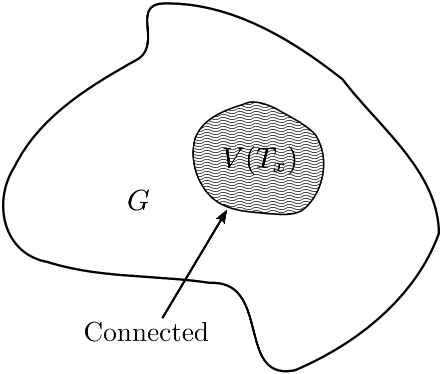
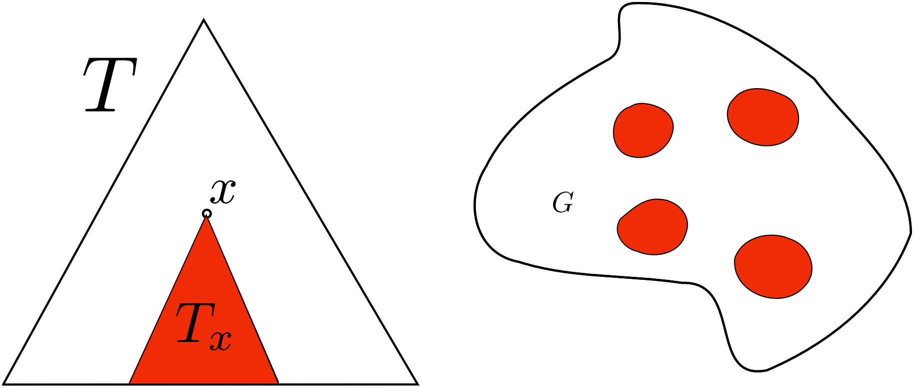
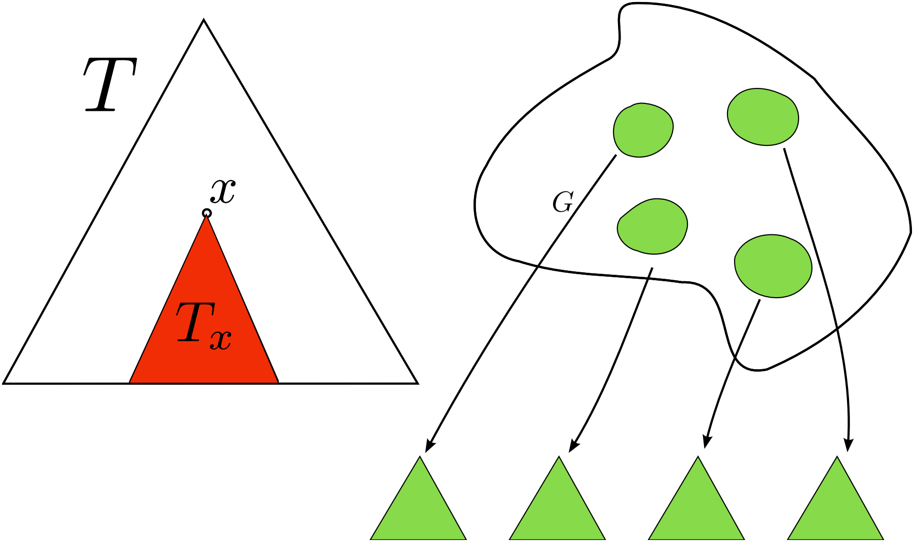




Definition (Nice treedepth decomposition)

Consider a treedepth decomposition T of G that is not trivially improvable. We say that T is *nice* if for every vertex $x \in V(T)$, the subgraph of G induced by the vertices in T_x is connected.


T
x
T_x
G



T
x
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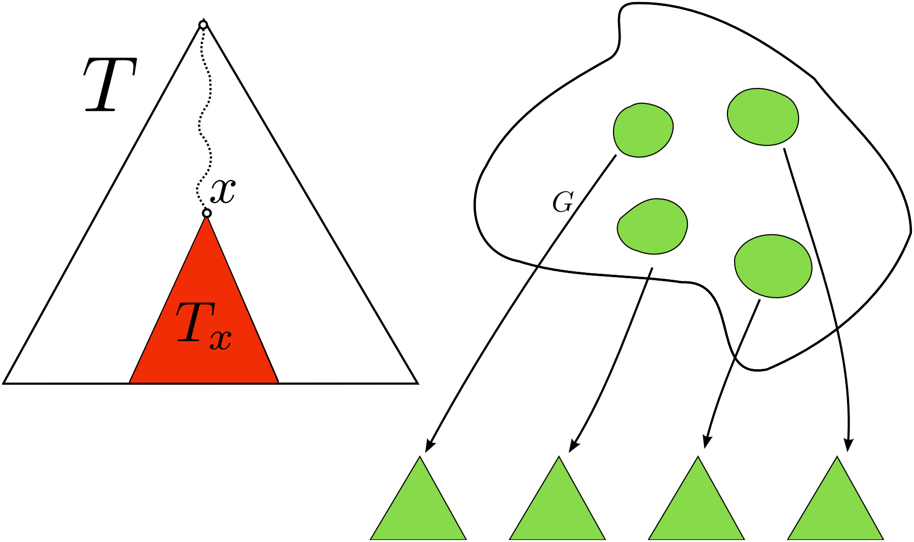
T
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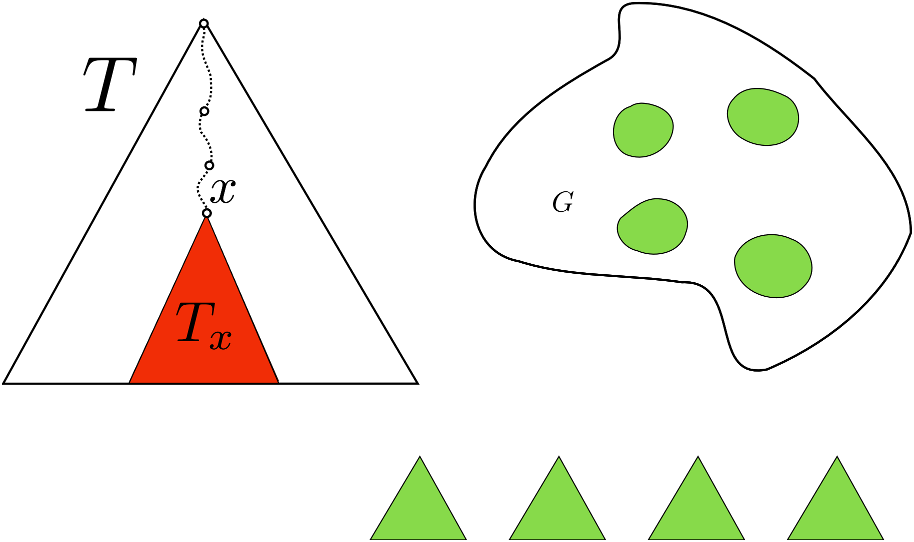
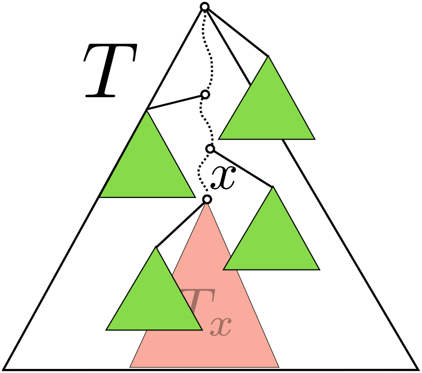
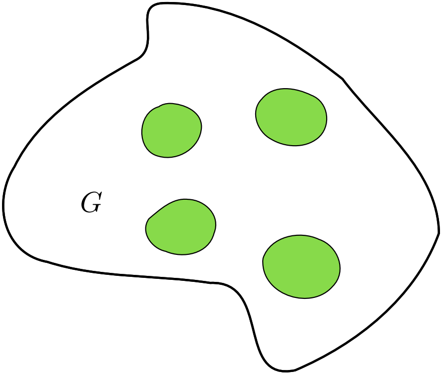


T
x
T_x
G

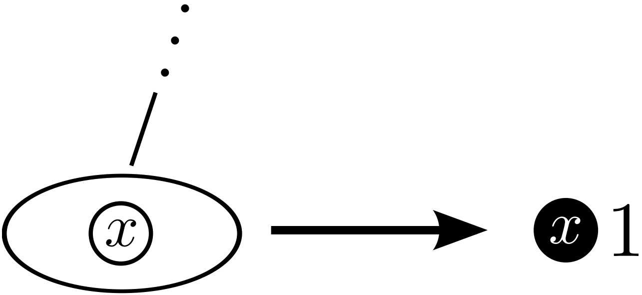

T
x
T_x


G

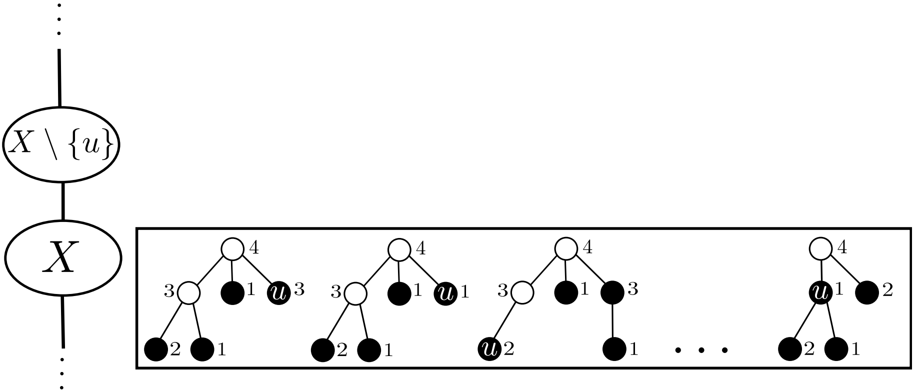
Lemma

For any graph there exists a treedepth decomposition of minimal depth which is nice and non-trivially improvable.

Leaf



Forget



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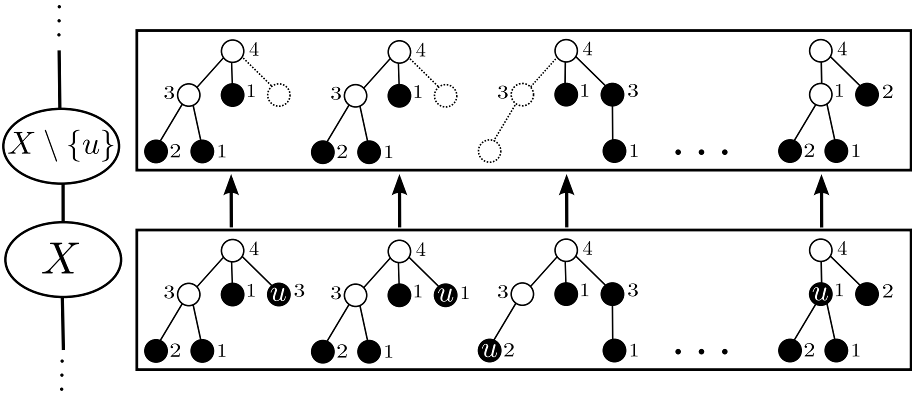
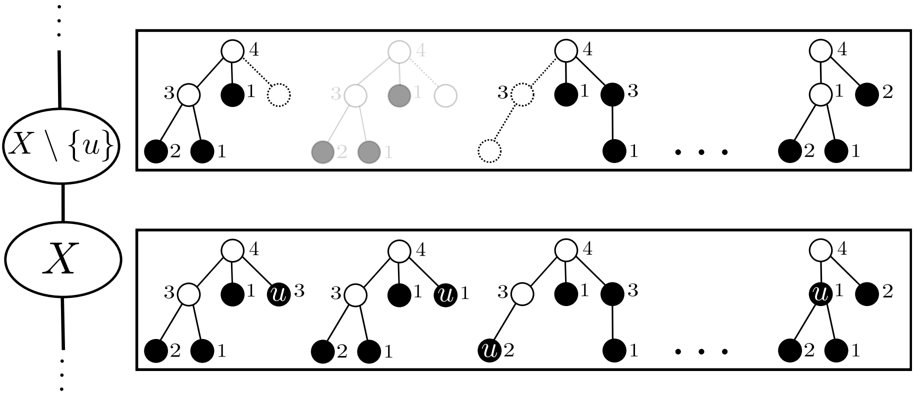
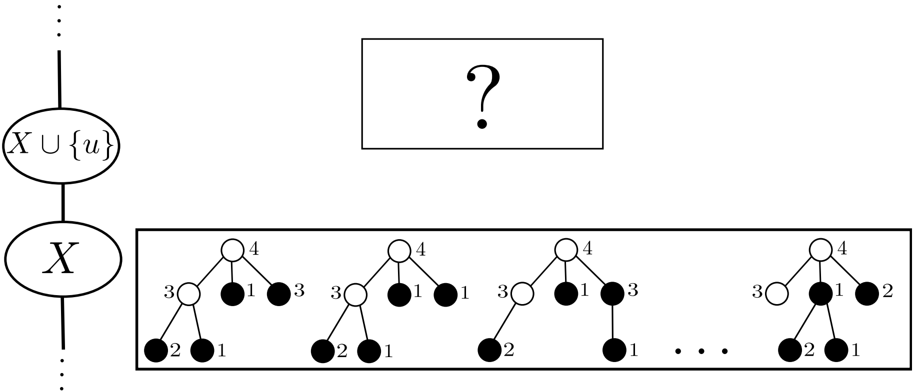


Forget



Introduce

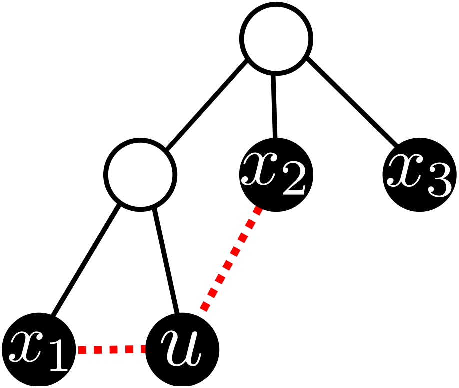


Introduce

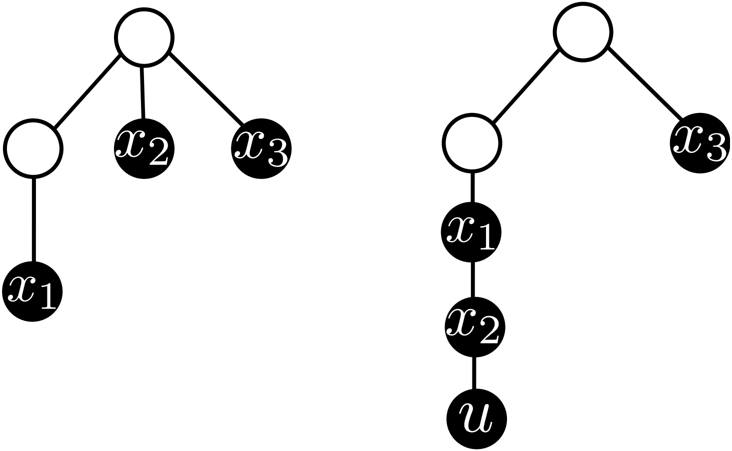
1. We guess every tree T with the following conditions:
   - $X \cup \{u\} \subseteq V(T)$,
   - All leafs are in $X \cup \{u\}$,
   - Height at most t.
2. We keep some, dismiss others.

Introduce

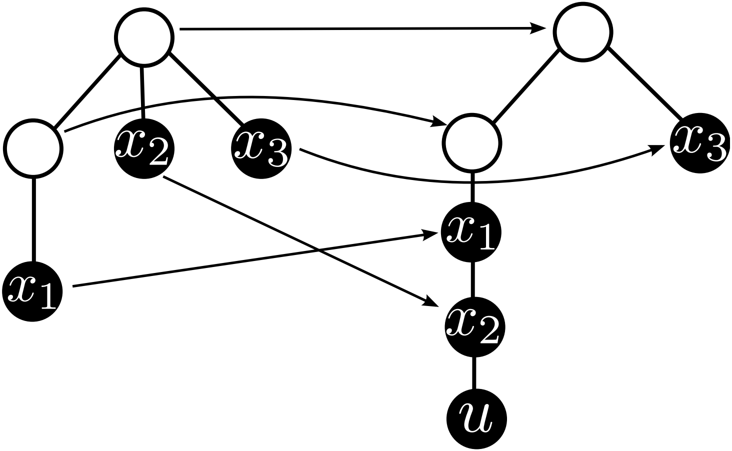
Throw away if edges of X are missing



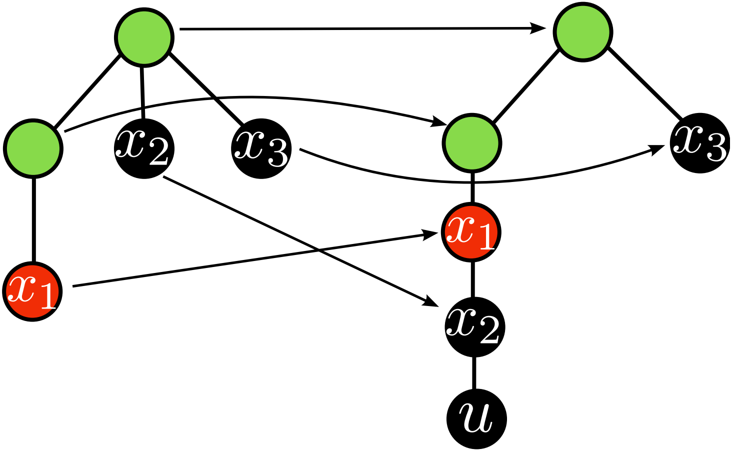
Topological generalization



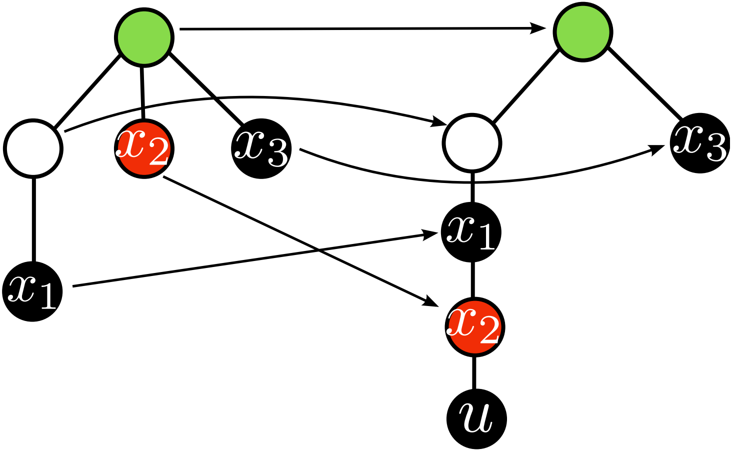
Topological generalization



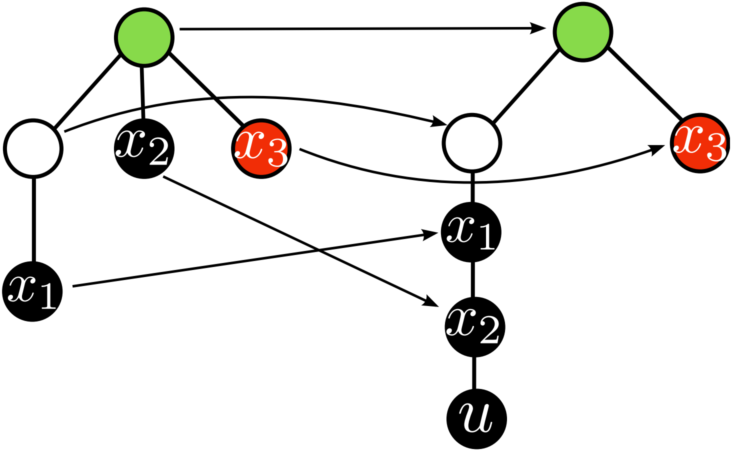
Topological generalization



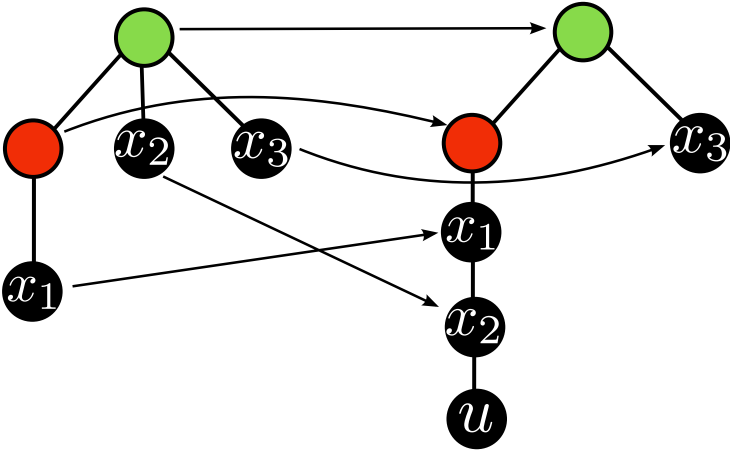
Topological generalization



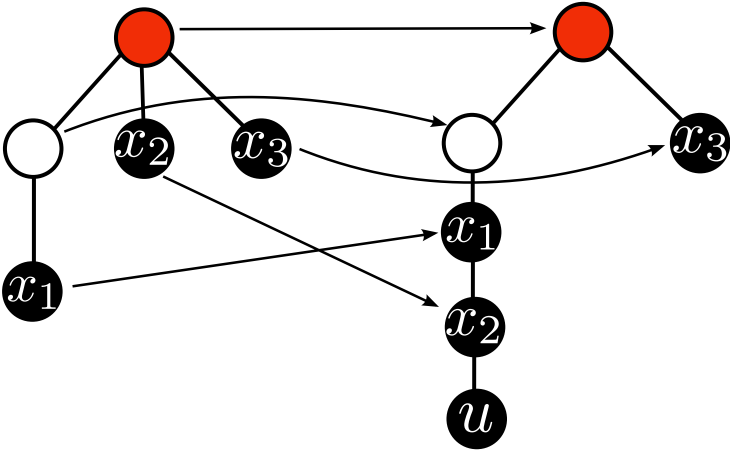
Topological generalization



Topological generalization



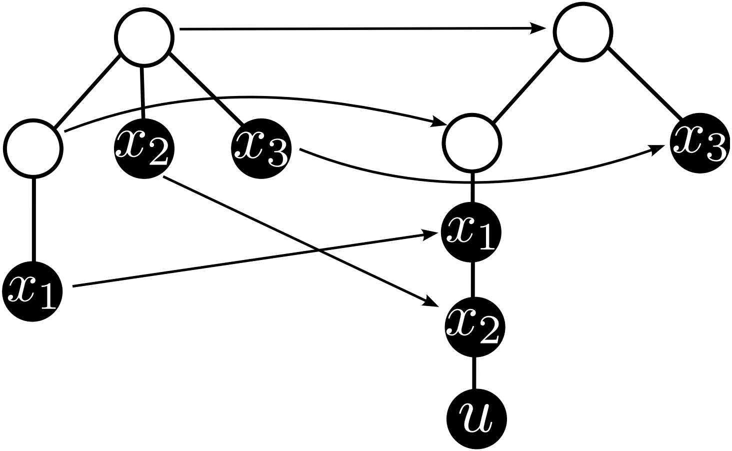
Topological generalization



Introduce

1. Guess tree with certain conditions:
   - $X \cup \{u\} \subseteq V(T)$,
   - All leafs are in $X \cup \{u\}$,
   - Height at most t.
2. If edge missing between nodes of $X \cup \{u\}$, discard.
3. If not a topological generalization of an element in old table, discard.

Why should this work?



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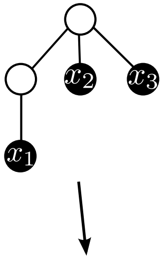

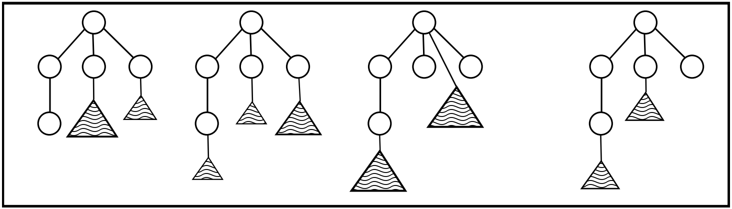

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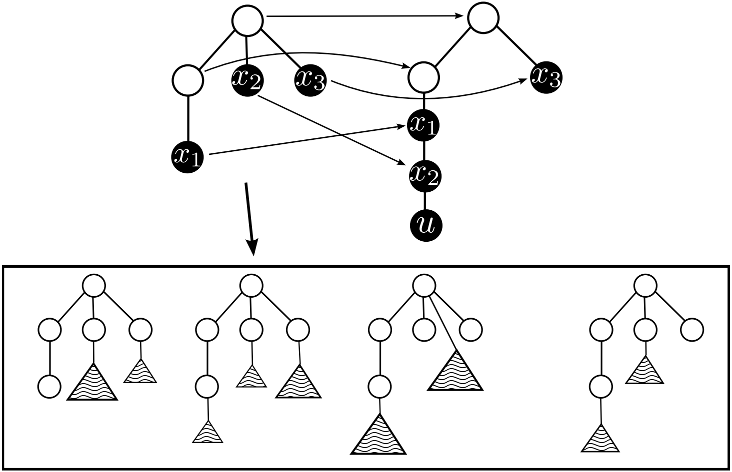

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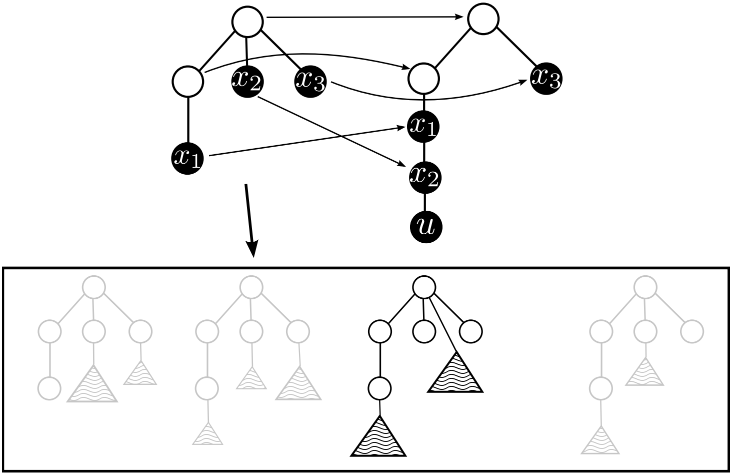

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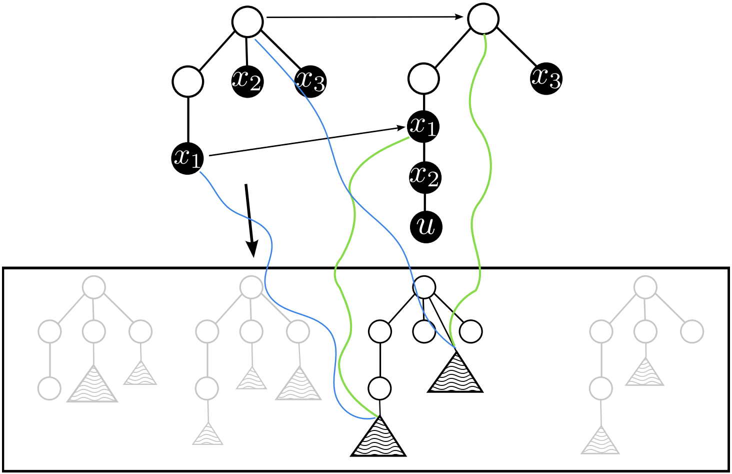

Theorem

Given a graph G with n nodes and a tree decomposition of G of width w the treedepth t of G can be decided in time $2^{O(wt)} \cdot n$.

Simple algorithm

1. Depth-first-search to construct treedepth decomposition T.
2. If depth greater than $2^t - 1$ say NO.
3. Construct path decomposition $\mathcal{P}$ from T of width 2^t.
4. Run algorithm on $\mathcal{P}$.

Theorem

There is a (simple) algorithm to decide if a graph G with n nodes has treedepth t which runs in time $2^{2^{O(t)}} \cdot n$.

Fast algorithm

1. Use single exponential 5-approximation for treewidth[1].
2. Remember $\mathbf{tw}(G) \leq \mathbf{pw}(G) \leq \mathbf{td}(G) - 1$.
3. If width is greater than $5t$ say NO.
4. Else run algorithm on tree decomposition.

Theorem

There is a algorithm to decide if a graph G with n nodes has treedepth t which runs in time $2^{O(t^2)} \cdot n$.

[1] Very useful result by Hans Bodlaender, Pål G. Drange, Markus S. Dregi, Fedor V. Fomin, Daniel Lokshtanov and Michał Pilipczuk

We have seen:

- An algorithm on tree decompositions which runs in time $2^{O(wt)} \cdot n$.
- A (relatively) simple direct algorithm in time $2^{2^{O(t)}} \cdot n$.
- A fast algorithm in time $2^{O(t^2)} \cdot n$.

Related Questions

- Is it sensible to implement?
- Can we approximate treedepth in time $2^{O(t)} \cdot n$.
- Can we find new algorithms on treedepth which are better than working in the path decomposition given by a treedepth decomposition?

Thank you for listening.

Questions?