Characterising Structural Sparseness by Neighbourhood Complexity

Or: If your neighbourhood is boring, you probably live in a sparse region.

Felix Reidl
felix.reidl@gmail.com

Joint work with
Fernando Sánchez Villaamil
& Konstantinos Stavropoulos

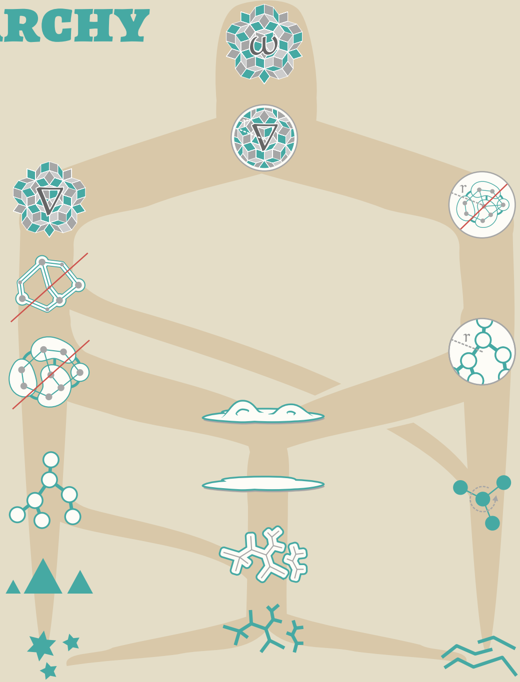
Warwick '16

SPARSE HIERARCHY

Less

Structure,
Tractability

More



Graph measures

A *graph measure* is an isomorphism invariant function that maps graphs to $\mathbb{R}^+$.

Graph measures

A *graph measure* is an isomorphism invariant function that maps graphs to $\mathbb{R}^+$.

- **Average degree** $\frac{2|E(G)|}{|V(G)|}$
- **Clique number** $\omega(G)$
- **Degeneracy** $\max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$
- **Treewidth** $\mathbf{tw}(G)$

GRAPH MEASURES

A *graph measure* is an isomorphism invariant function that maps graphs to $\mathbb{R}^+$.

- **Average degree** $\frac{2|E(G)|}{|V(G)|}$
- **Clique number** $\omega(G)$
- **Degeneracy** $\max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$
- **Treewidth** $\mathrm{tw}(G)$

Density measures (Average degree, Clique number, Degeneracy)

Width measures (Treewidth)

Graph measures

A *graph measure* is an isomorphism invariant function that maps graphs to $\mathbb{R}^+$.

- **Average degree** $\frac{2|E(G)|}{|V(G)|}$
- **Clique number** $\omega(G)$
- **Degeneracy** $\max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$

Density measures

- **Treewidth** $\mathrm{tw}(G)$

Width measures

Parametrised graph measures

A *parametrised graph measure* is a family of graph measures $(f_r)_{r \in \mathbb{N}_0}$.

Parametrised graph measures

A *parametrised graph measure* is a family of graph measures $(f_r)_{r \in \mathbb{N}_0}$.

Graph measure + Depth parametre = Notion *of* structural sparseness

Parametrised graph measures

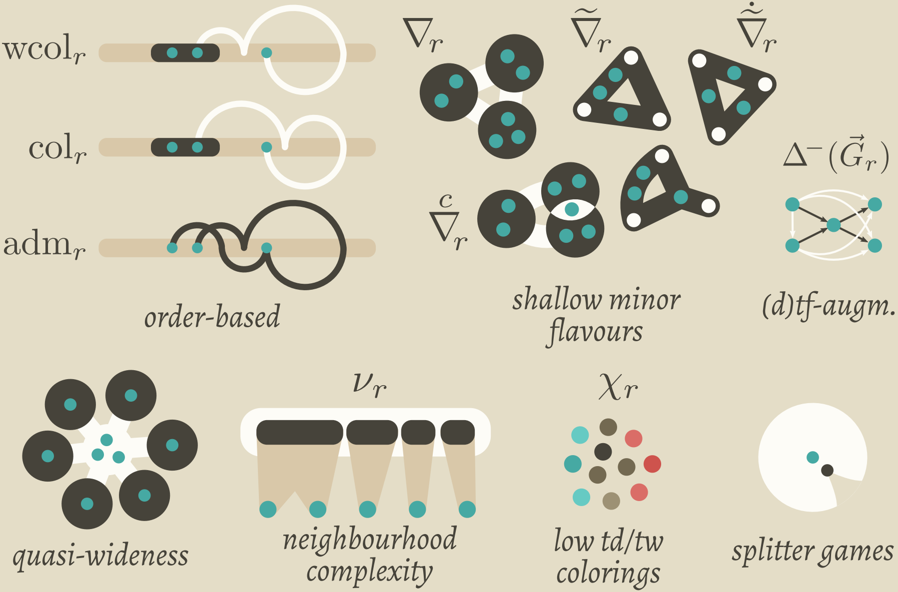
A *parametrised graph measure* is a family of graph measures $(f_r)_{r \in \mathbb{N}_0}$.

Graph measure + **Depth parametre** = **Notion *of* structural sparseness**

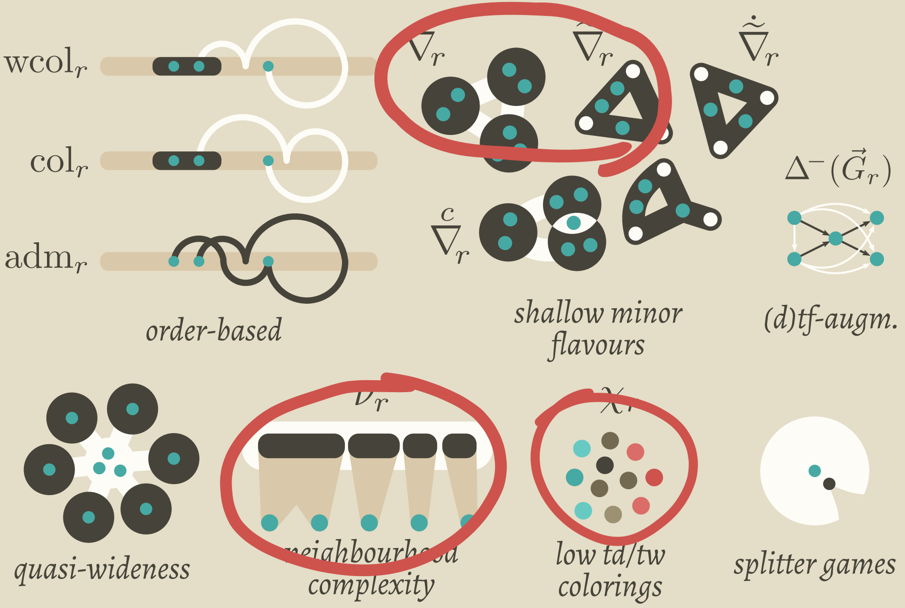
A graph class $\mathcal{G}$ is f_r-bounded if there exists g s.t.

$$f_r(\mathcal{G}) = \sup_{G \in \mathcal{G}} f_r(G) \leqslant g(r) \quad \text{for all } r.$$

Choose your poison

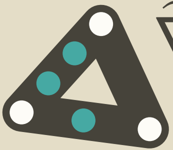


Choose your poison



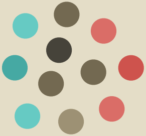
Characterising Structural Sparseness

Shallow Topological Minors *by* r-centred Colouring Numbers &

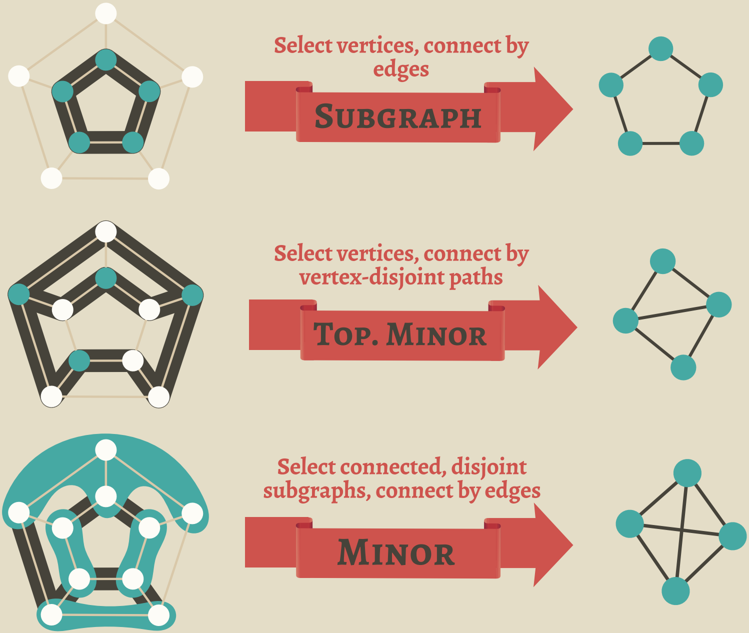


$\widetilde{\nabla}_r$

χ_r



Substructures



Forbidden Substructures

H does not appear as a subgraph.

H does not appear as a topological minor.

H does not appear as a minor.

Forbidden Substructures

 does not appear as a subgraph.

= Triangle-free graphs

 does not appear as a topological minor.

= Forests

 does not appear as a minor.

= Forests

Forbidden Substructures

 does not appear as a subgraph.

= Triangle-free graphs

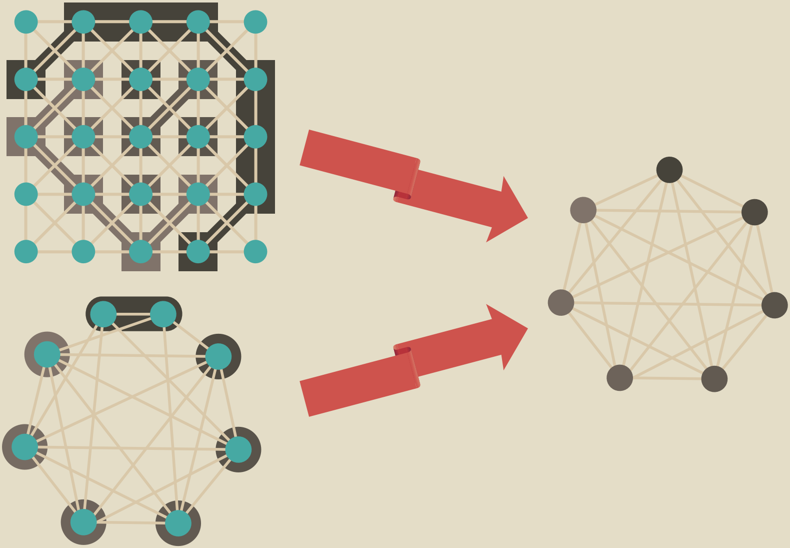
 does not appear as a topological minor.

= Forests

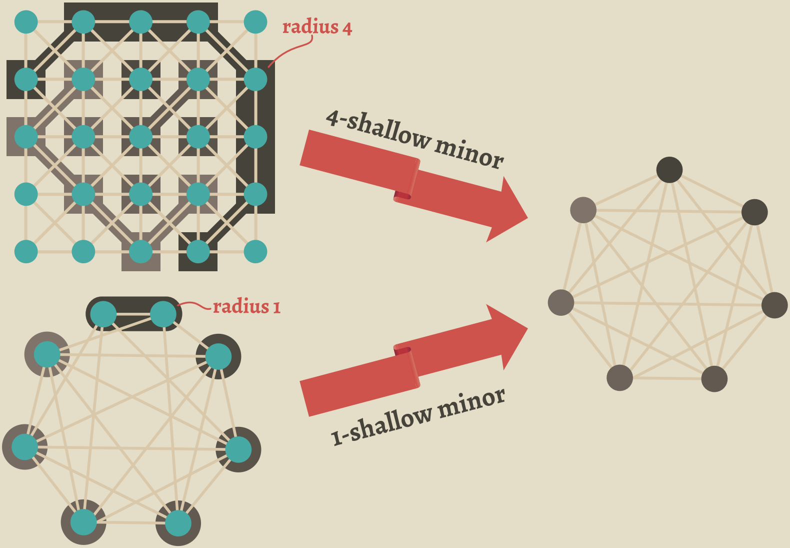
 does not appear as a minor.

= Forests

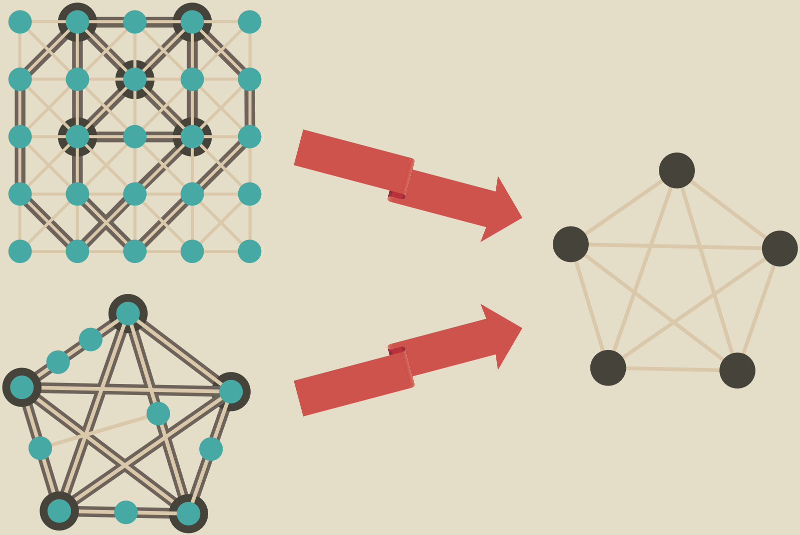
Not all minors are equal!



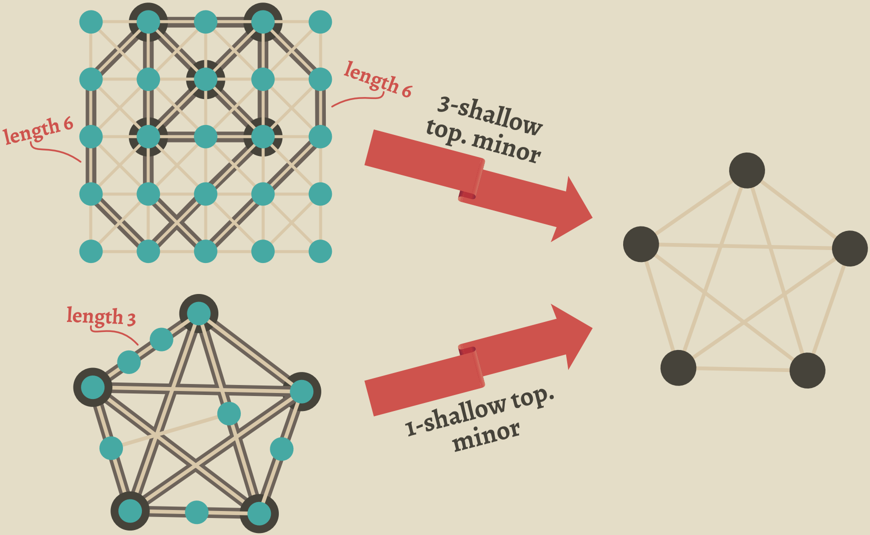
Not all minors are equal!



Not all top. minors are equal!



Not all top. minors are equal!



The magic formula

Graph measure + Depth parametre = notion *of* Structural sparseness

The magic formula

Graph measure	+	**Depth parametre**	=	**Notion *of* structural sparseness**
$\frac{\|H\|}{|H|}$	+	**minor depth** r	=	

The magic formula

Graph measure + **Depth parametre** = **Notion *of* Structural sparseness**

$\frac{\|H\|}{|H|}$ + **minor depth** r = $\nabla_r(G) = \max_{H \preccurlyeq^r_{\mathsf{m}} G} \frac{\|H\|}{|H|}$

The magic formula

Graph measure + **Depth parametre** = **Notion *of* structural sparseness**

$\frac{\|H\|}{|H|}$ + **minor depth** r = $\nabla_r(G) = \max_{H \preccurlyeq_{\mathsf{m}}^{r} G} \frac{\|H\|}{|H|}$

$\frac{\|H\|}{|H|}$ + **topological minor depth** r = $\widetilde{\nabla}_r(G) = \max_{H \preccurlyeq_{\mathsf{t}}^{r} G} \frac{\|H\|}{|H|}$

The magic formula

Graph measure + **Depth parametre** = **Notion *of* structural sparseness**

$$\frac{\|H\|}{|H|} \quad + \quad \text{minor depth } r \quad = \quad \nabla_r(G) = \max_{H \preccurlyeq_{\mathrm{m}}^r G} \frac{\|H\|}{|H|}$$

$$\frac{\|H\|}{|H|} \quad + \quad \text{topological minor depth } r \quad = \quad \widetilde{\nabla}_r(G) = \max_{H \preccurlyeq_{\mathrm{t}}^r G} \frac{\|H\|}{|H|}$$

$$\widetilde{\nabla}_r(G) \leqslant \nabla_r(G) \leqslant 4(4\widetilde{\nabla}_r(G))^{(r+1)^2}$$

Nešetřil & Ossona de Mendez

The magic formula

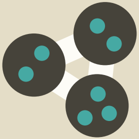
Graph measure	+	Depth parametre	=	Notion of structural sparseness
$\frac{\|H\|}{|H|}$	+	**minor depth** r	=	$\nabla_r(G) = \max_{H \preccurlyeq_m^r G} \frac{\|H\|}{|H|}$
$\frac{\|H\|}{|H|}$	+	**topological minor depth** r	=	$\widetilde{\nabla}_r(G) = \max_{H \preccurlyeq_t^r G} \frac{\|H\|}{|H|}$

$$\widetilde{\nabla}_r(G) \asymp \nabla_r(G)$$

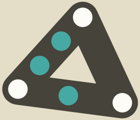
$\log(\widetilde{\nabla}_r(G)) = \Theta(\log(\nabla_r(G)))$

Nešetřil & Ossona de Mendez

Bounded expansion



A graph class $\mathcal{G}$ has *bounded expansion* iff it is ∇_r-bounded.

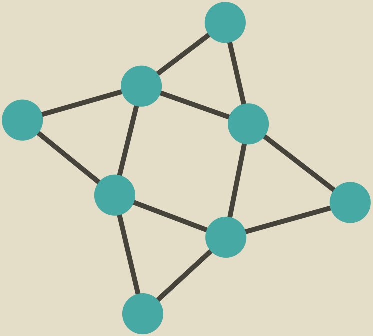


A graph class $\mathcal{G}$ has *bounded expansion* iff it is $\widetilde{\nabla}_r$-bounded.

“We allow dense minors in our graph, but only on a large scale”

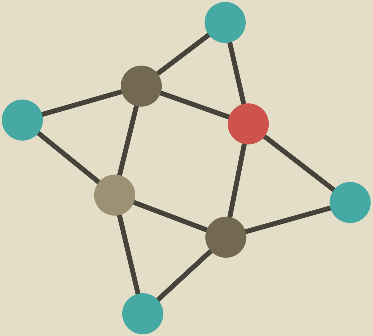
Centred colourings

A vertex colouring is *centred* if every connected subgraph contains a unique colour, a *centre*.



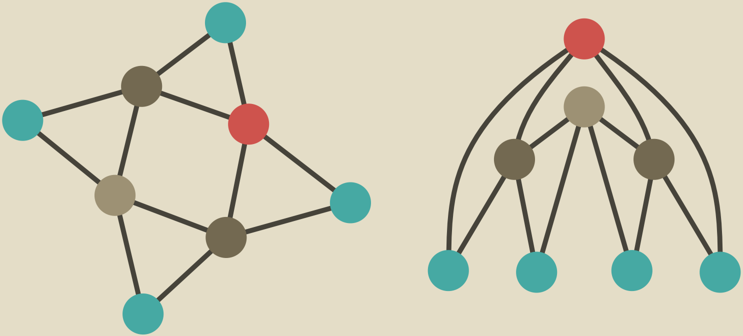
Centred colourings

A vertex colouring is *centred* if every connected subgraph contains a unique colour, a *centre*.



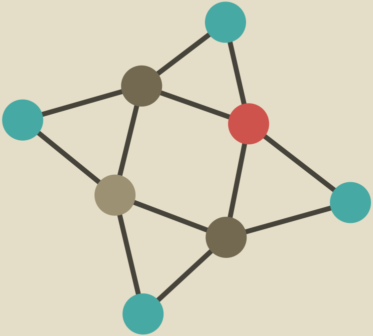
Centred colourings

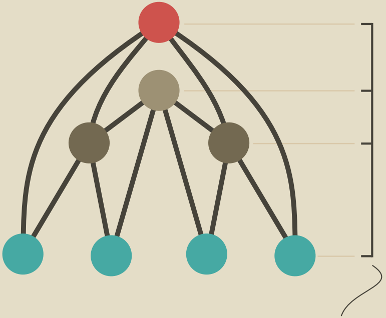
A vertex colouring is *centred* if every connected subgraph contains a unique colour, a *centre*.



Centred colourings

A vertex colouring is *centred* if every connected subgraph contains a unique colour, a *centre*.





$$\mathrm{tw}(G) + 1 \leqslant \mathrm{pw}(G) + 1 \leqslant \mathrm{td}(G)$$

The magic formula

Graph measure	+	Depth parametre	=	Notion *of* Structural sparseness
$\mathrm{td}(H)$	+	?	=	?

The magic formula

Graph measure + **Depth parametre** = **Notion *of* Structural Sparseness**

$\text{td}(H)$ + colour subsets of size r = ?

A vertex colouring is *r-centred* if every connected subgraph either
- contains a centre, or
- contains r or more colours.

The magic formula

Graph measure	+	Depth parametre	=	Notion of structural sparseness
$\text{td}(H)$	+	colour subsets of size r	=	$\chi_r(G)$

Minimum number of colours needed for an r-centred colouring of G

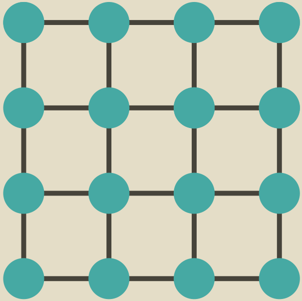
A vertex colouring is *r-centred* if every connected subgraph either

- contains a centre, or
- contains r or more colours.

r-Centred colourings

A vertex colouring is *r-centred* if every connected subgraph either

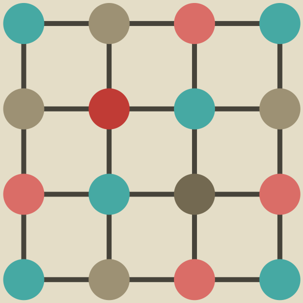
- contains a centre, or
- contains r or more colours.



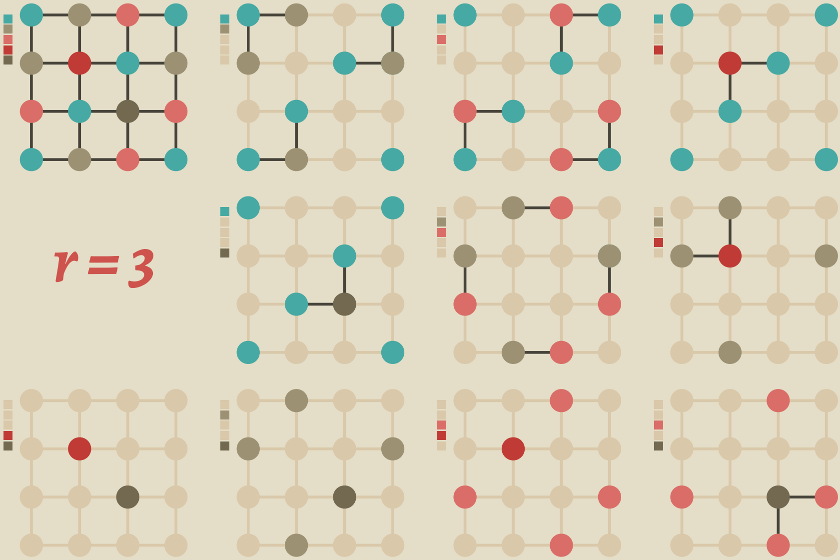
r-Centred colourings

A vertex colouring is *r-centred* if every connected subgraph either

- contains a centre, or
- contains r or more colours.



r-Centred colourings



Bounded expansion (again)

$$\chi_r(G) \asymp \widetilde{\nabla}_r(G)$$

Nešetřil & Ossona de Mendez

A graph class $\mathcal{G}$ has bounded expansion iff it is χ_r-bounded.

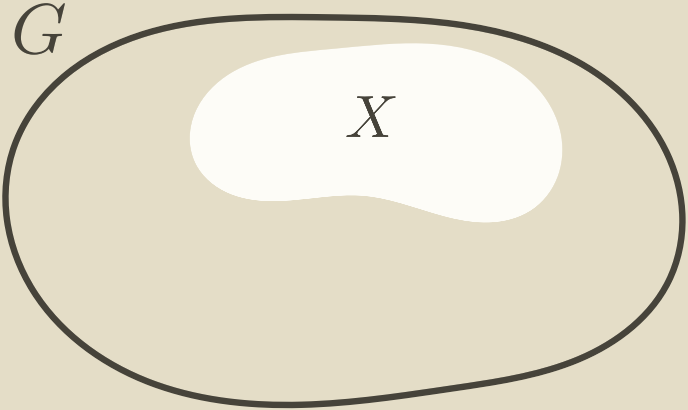
“Structurally sparse graphs can be decomposed locally into parts of bounded width”

Interlude

~~CUT~~ COMPLEXITY

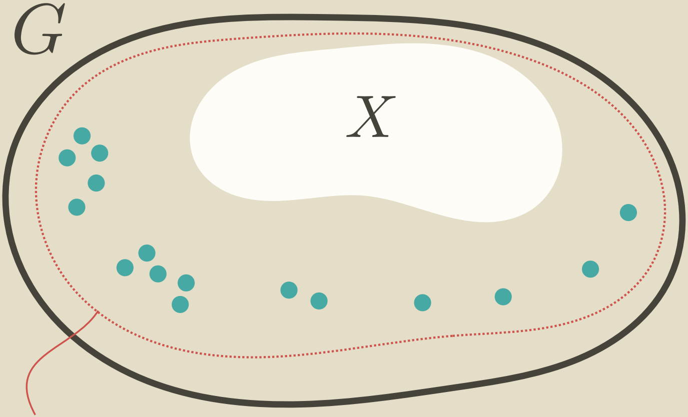
Bounded cut complexity

(Works in many sparse classes)



Bounded cut complexity

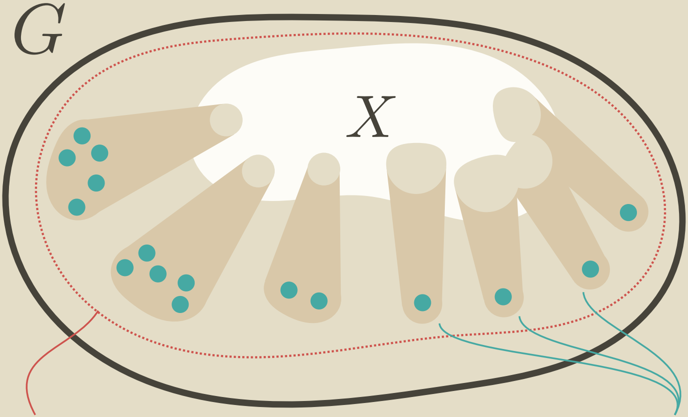
(Works in many sparse classes)



$N(X)$ **is large...**

Bounded cut complexity

(Works in many sparse classes)



$N(X)$ **is large...** **...but** $N(X)/N_X$ **has size** $O(|X|)$

Cut complexity as a measure

Alternatively $N[v]$

Alternatively $v \in G \setminus X$

$$\xi(G) = \max_{X \subseteq V(G)} \frac{|\{N(v) \cap X\}_{v \in G}|}{|X|}$$

Normalise to make subsets comparable

Cut complexity as a measure

Alternatively $N[v]$

Alternatively $v \in G \setminus X$

$$\xi(G) = \max_{X \subseteq V(G)} \frac{|\{N(v) \cap X\}_{v \in G}|}{|X|}$$

Normalise to make subsets comparable

Graph measure + **Depth parametre** = etc. etc.

From cuts to neighbourhoods

$$\xi(G) = \max_{X \subseteq V(G)} \frac{|\{N(v) \cap X\}_{v \in G}|}{|X|}$$

From cuts to neighbourhoods

$$\xi(G) = \max_{X \subseteq V(G)} \frac{|\{N(v) \cap X\}_{v \in G}|}{|X|}$$

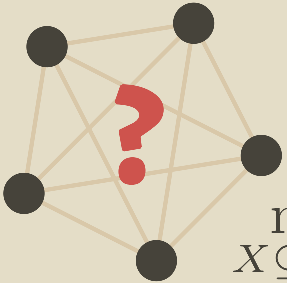
DEPTH

$$\max_{X \subseteq V(G)} \frac{|\{N^r[v] \cap X\}_{v \in G}|}{|X|}$$

From cuts to neighbourhoods

$$\xi(G) = \max_{X \subseteq V(G)} \frac{|\{N(v) \cap X\}_{v \in G}|}{|X|}$$

DEPTH



$$\max_{X \subseteq V(G)} \frac{|\{N^r[v] \cap X\}_{v \in G}|}{|X|}$$

From cuts to neighbourhoods

$$\xi(G) = \max_{X \subseteq V(G)} \frac{|\{N(v) \cap X\}_{v \in G}|}{|X|}$$

DEPTH

$$\max_{X \subseteq V(G)} \frac{|\{N^r[v] \cap X\}_{v \in G}|}{|X|}$$

Dense graphs

From cuts to neighbourhoods

$$\xi(G) = \max_{X \subseteq V(G)} \frac{|\{N(v) \cap X\}_{v \in G}|}{|X|}$$

Consider all subgraphs

Depth

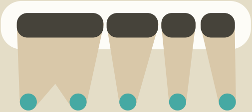
$$\nu_r(G) = \max_{H \subseteq G, X \subseteq V(H)} \frac{|\{N^r[v] \cap X\}_{v \in H}|}{|X|}$$

From cuts to neighbourhoods

$$\xi(G) = \max_{X \subseteq V(G)} \frac{|\{N(v) \cap X\}_{v \in G}|}{|X|}$$

$$\nu_r(G) = \max_{H \subseteq G, X \subseteq V(H)} \frac{|\{N^r[v] \cap X\}_{v \in H}|}{|X|}$$

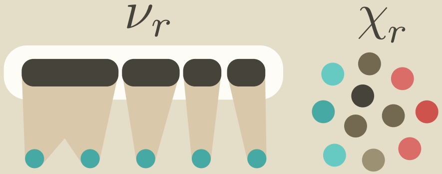
"Neighbourhood Complexity"



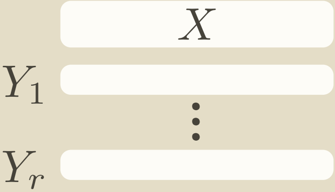
Characterising Structural Sparseness by Neighbourhood Complexity

Part I

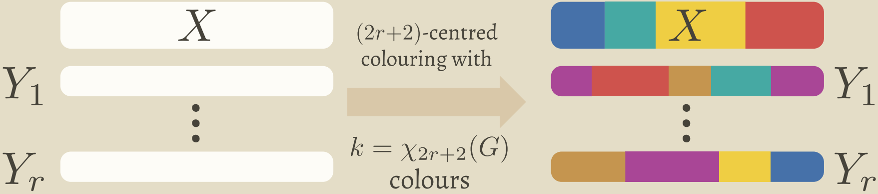
The story of $\nu_r(G) \leqslant f(\chi_{2r+2}(G))$



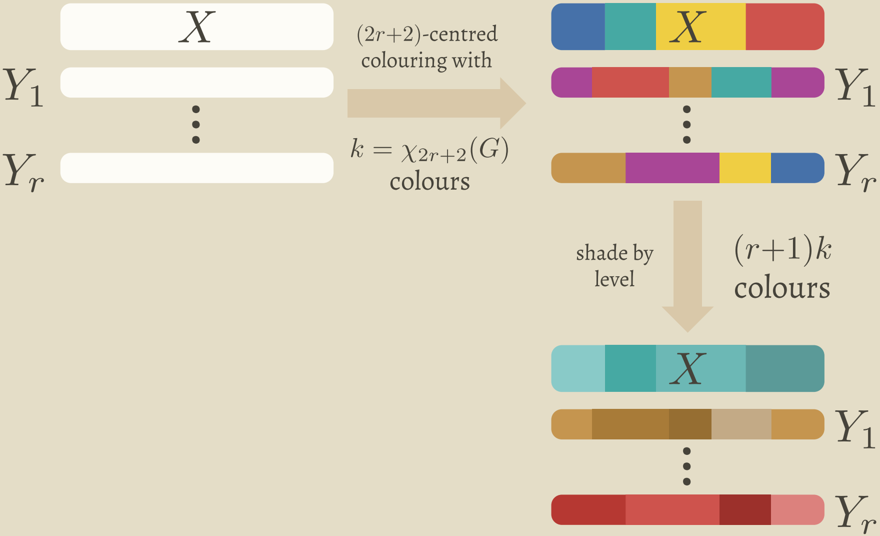
Bounding N.C. by centred cols.



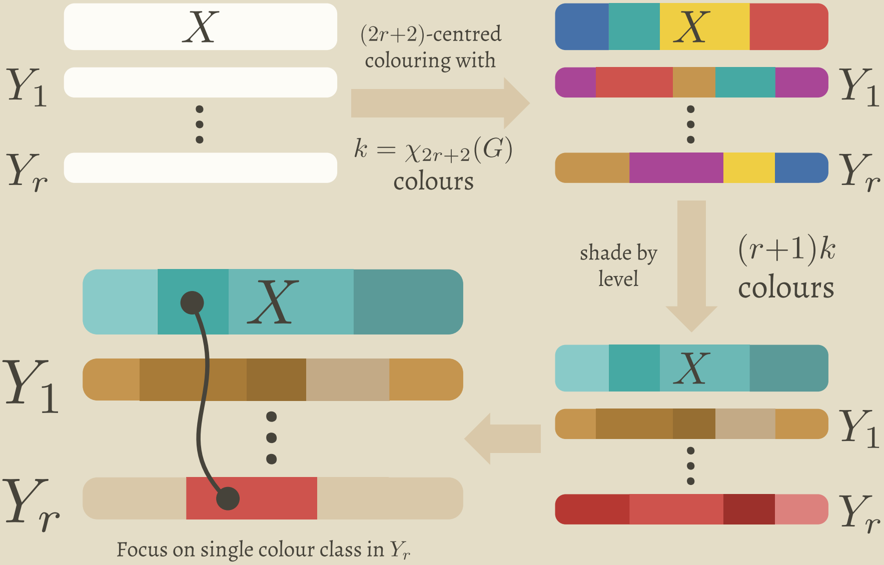
Bounding N.C. by centred cols.



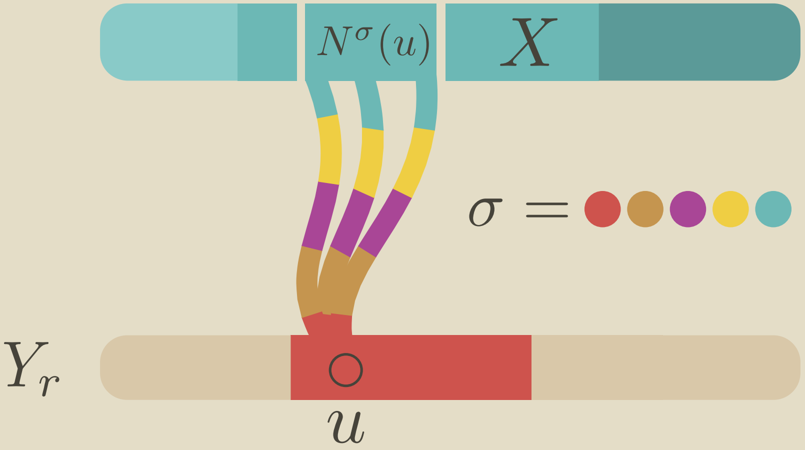
Bounding N.C. by centred cols.



Bounding N.C. by centred cols.

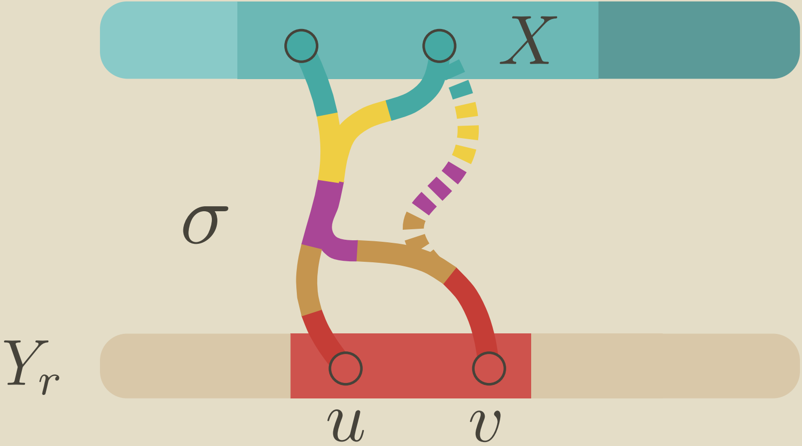


Signatures & Neighbourhoods



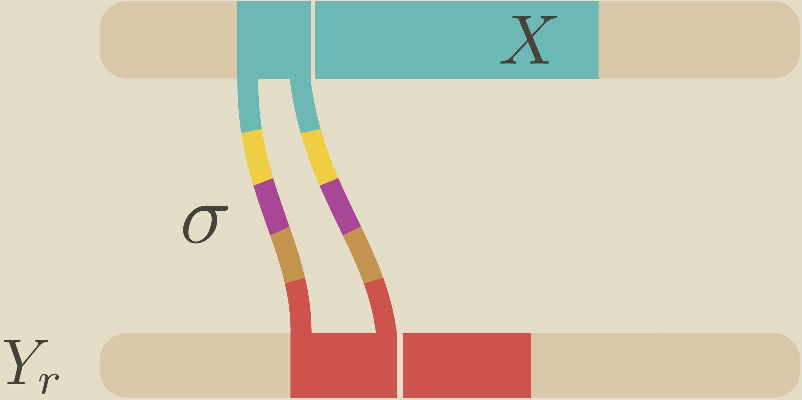
$$N^r(u) \cap X = \bigcup_\sigma N^\sigma(u)$$

Signatures: observation I



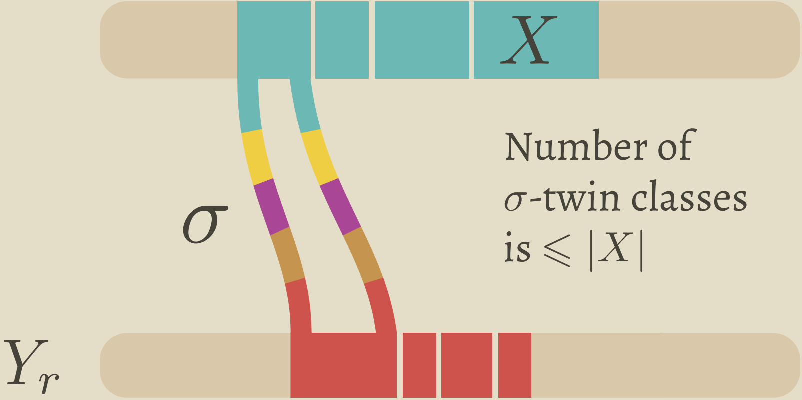
For a fixed signature σ and $u, v \in Y_r$, either $N^\sigma(u) = N^\sigma(v)$ or $N^\sigma(u) \cap N^\sigma(v) = \emptyset$.

Signatures: observation I



For a fixed signature σ and $u, v \in Y_r$, either $N^\sigma(u) = N^\sigma(v)$ or $N^\sigma(u) \cap N^\sigma(v) = \emptyset$.

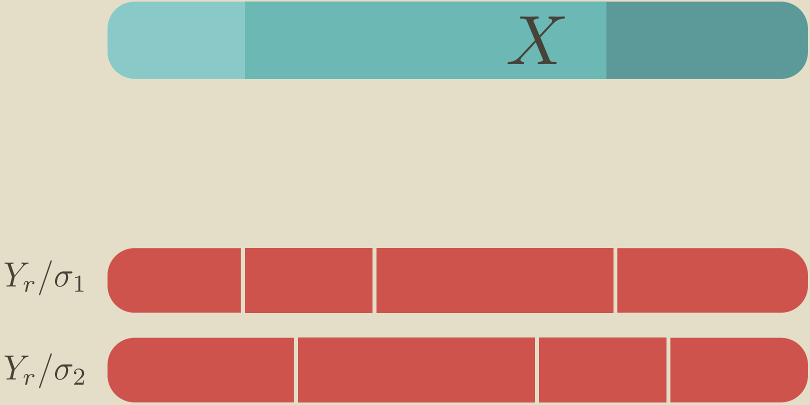
Signatures: observation I



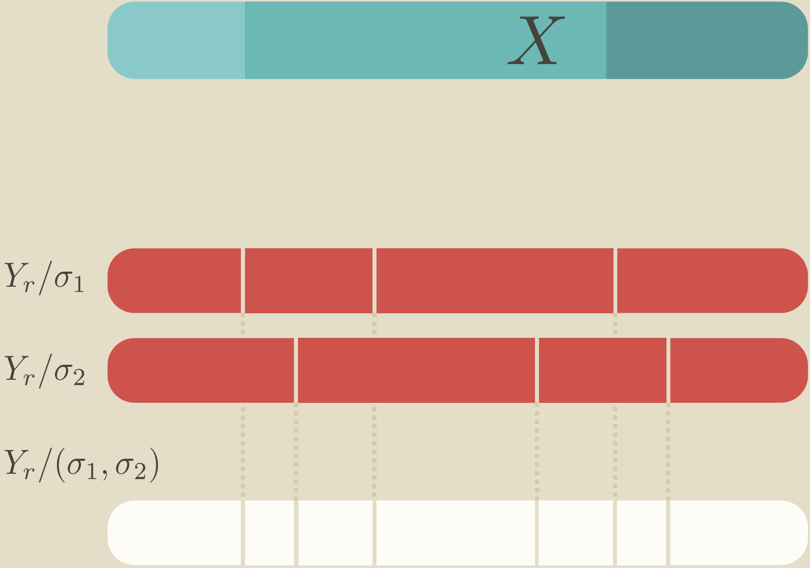
Number of σ-twin classes is $\leqslant |X|$

For a fixed signature σ and $u, v \in Y_r$, either $N^\sigma(u) = N^\sigma(v)$ or $N^\sigma(u) \cap N^\sigma(v) = \emptyset$.

Signatures: observation II

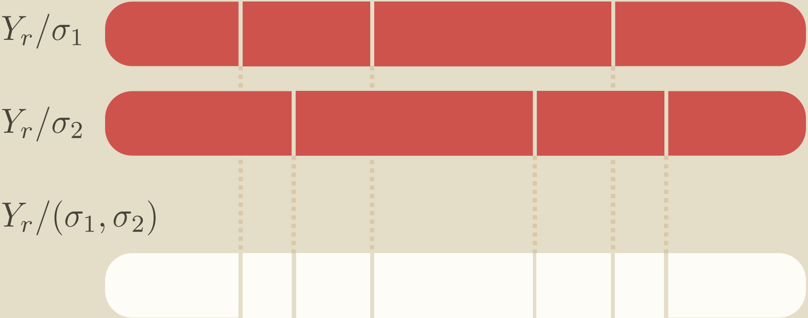


Signatures: observation II



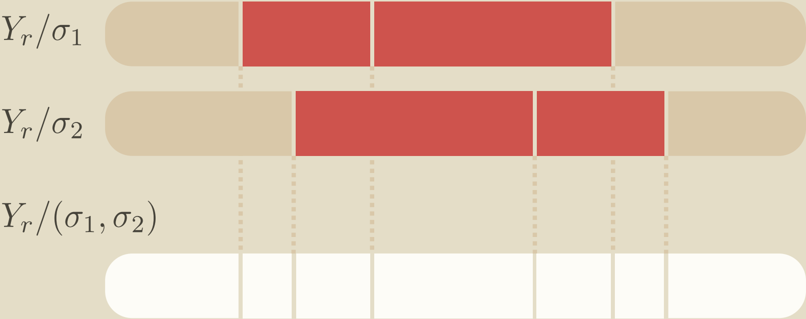
Signatures: observation II

This would be bad:
$\approx 2^{|X|}$ neighbourhood classes

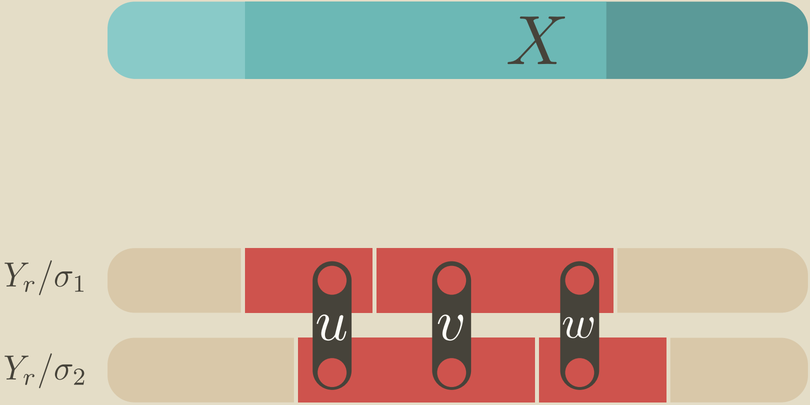


Signatures: observation II

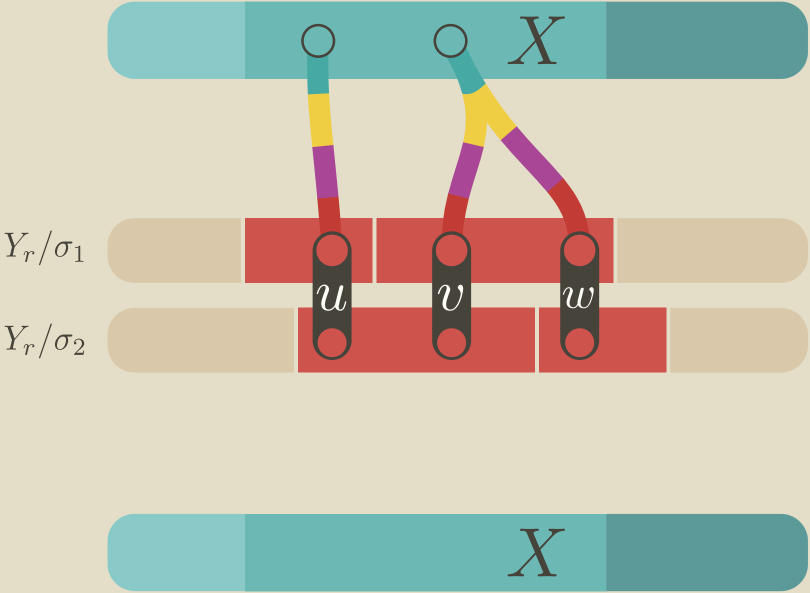
This would be bad:
$\approx 2^{|X|}$ neighbourhood classes



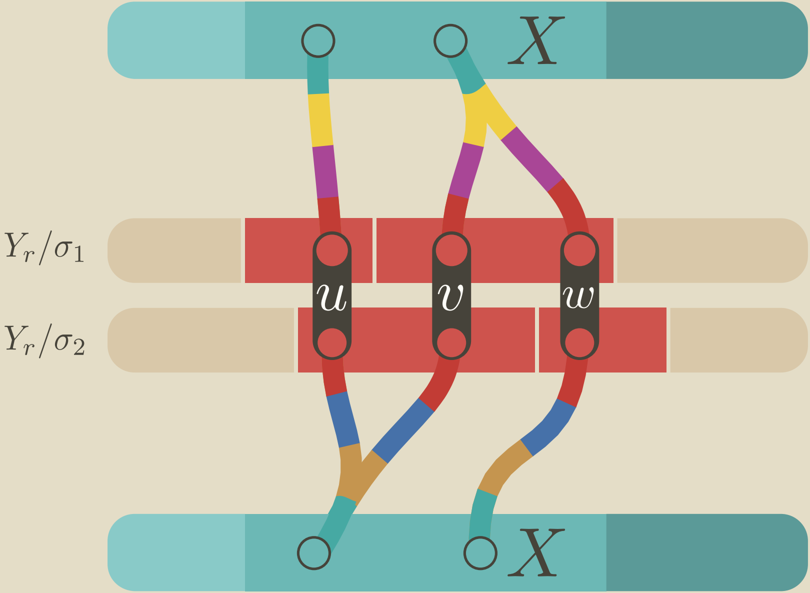
Signatures: observation II



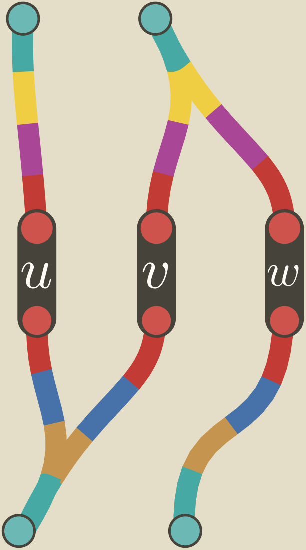
Signatures: observation II



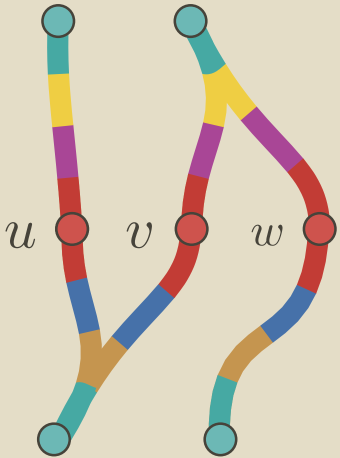
Signatures: observation II



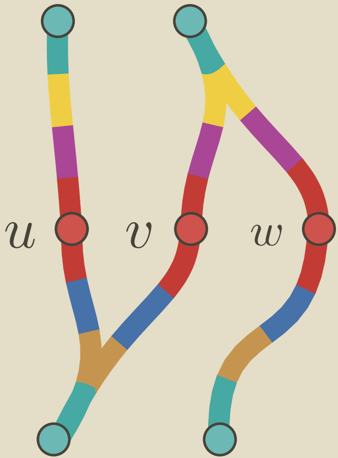
Signatures: observation II



Signatures: observation II

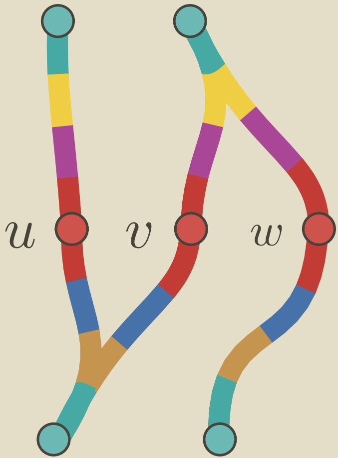


Signatures: observation II



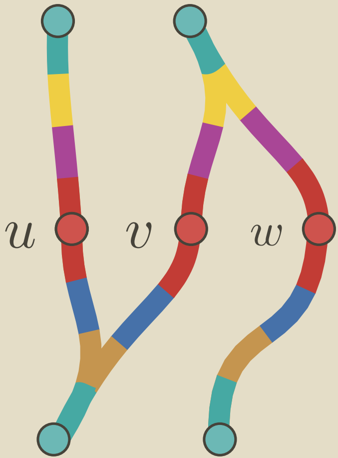
- **Connected subgraph**

Signatures: observation II



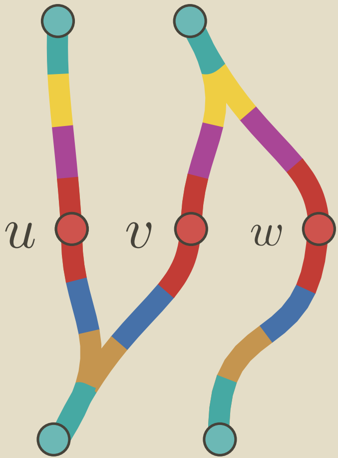
- **Connected subgraph**
- **At most $2r + 1$ colours**

Signatures: observation II



- **Connected** subgraph
- At most $2r + 1$ colours
- Every colour appears at least **twice**

Signatures: observation II



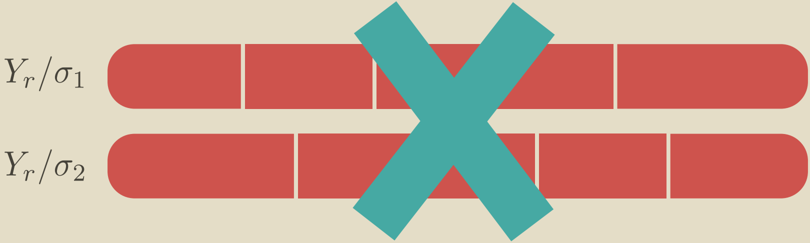
- Connected subgraph
- At most $2r + 1$ colours
- Every colour appears at least twice

This subgraph has few colours and no centre!

Contradiction!

Signatures: observation II

For every σ_1-twin class $C_1 \subseteq Y_r$ and σ_2-twin class $C_2 \subseteq Y_r$ we have that either $C_1 \subseteq C_2, C_2 \subseteq C_1$ or $C_1 \cap C_2 = \emptyset$.



Signatures: observation II

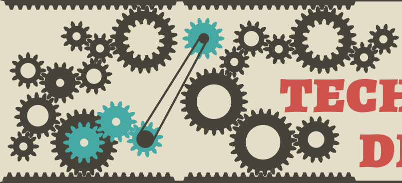
For every σ_1-twin class $C_1 \subseteq Y_r$ and σ_2-twin class $C_2 \subseteq Y_r$ we have that either $C_1 \subseteq C_2$, $C_2 \subseteq C_1$ or $C_1 \cap C_2 = \emptyset$.

The number of twin-classes in Y_r is at most $f(\chi_{2r+2}(G))|X|$.

Signatures: observation II

For every σ_1-twin class $C_1 \subseteq Y_r$ and σ_2-twin class $C_2 \subseteq Y_r$ we have that either $C_1 \subseteq C_2, C_2 \subseteq C_1$ or $C_1 \cap C_2 = \emptyset$.

The number of twin-classes in Y_r is at most $f(\chi_{2r+2}(G))|X|$.



Many technical details

$$\nu_r(G) \leqslant f(\chi_{2r+2}(G))$$

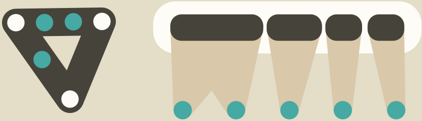
Characterising Structural Sparseness by Neighbourhood Complexity

Part II

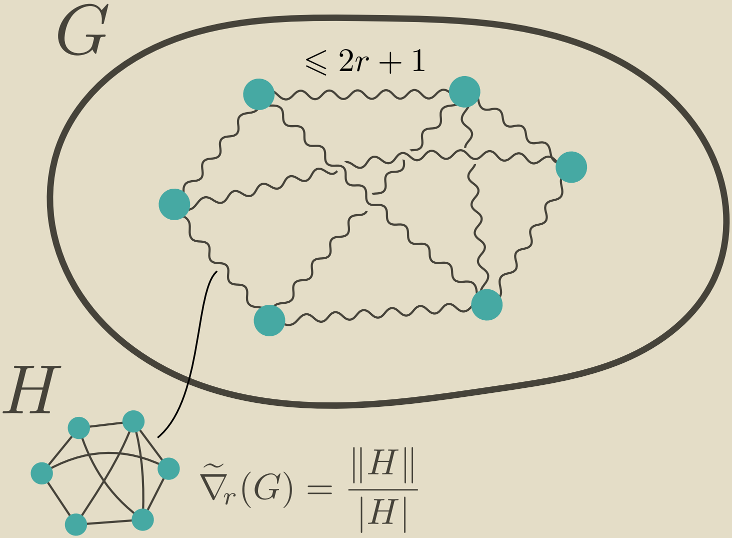
The (shorter) story of

$$\widetilde{\nabla}_{r-1}(G) \leqslant f(\nu_{\leqslant r}(G))$$

$\widetilde{\nabla}_r$ ν_r

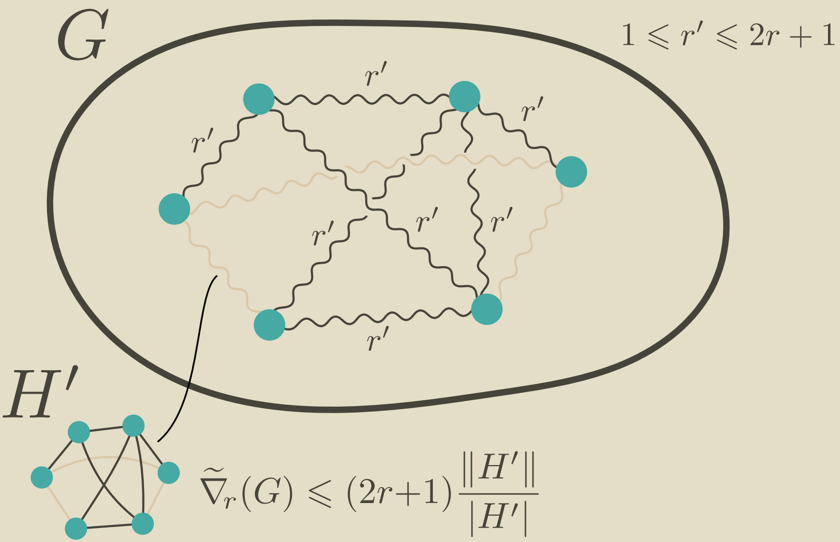


Dense top. minors = high complexity



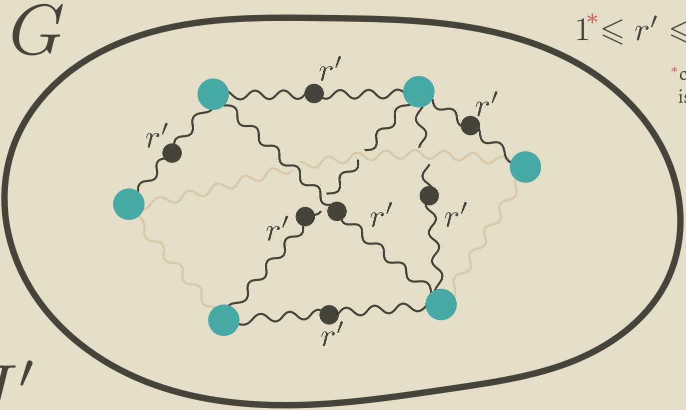
$$\widetilde{\nabla}_r(G) = \frac{\|H\|}{|H|}$$

Dense top. minors = high complexity



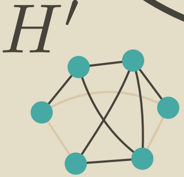
$$\widetilde{\nabla}_r(G) \leqslant (2r+1)\frac{\|H'\|}{|H'|}$$

Dense top. minors = high complexity



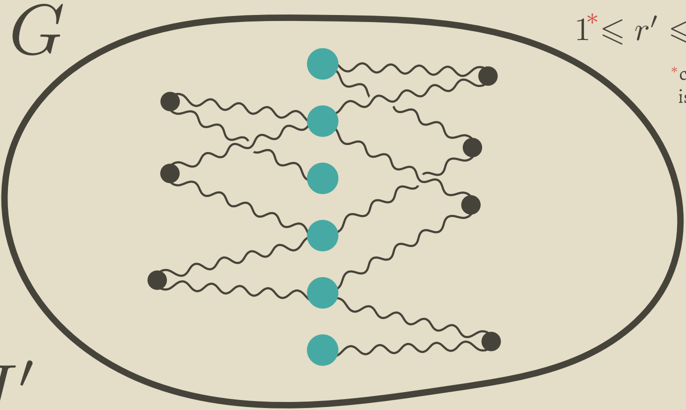
$1^{*} \leqslant r' \leqslant 2r+1$

*case $r' = 1$ is more involved



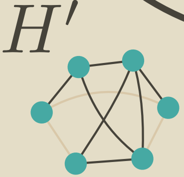
$$\widetilde{\nabla}_r(G) \leqslant (2r{+}1)\frac{\|H'\|}{|H'|}$$

Dense top. minors = high complexity



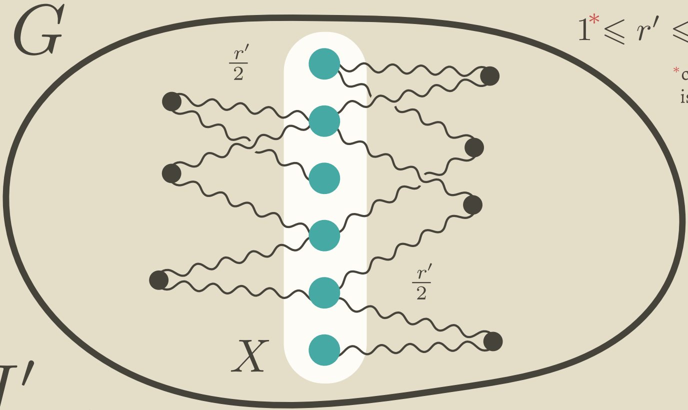
$1^{*} \leqslant r' \leqslant 2r+1$

* case $r' = 1$ is more involved



$$\widetilde{\nabla}_r(G) \leqslant (2r+1)\frac{\|H'\|}{|H'|}$$

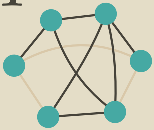
Dense top. minors = high complexity



$1^{*} \leqslant r' \leqslant 2r + 1$

*case $r' = 1$ is more involved

H'



$$\widetilde{\nabla}_r(G) \leqslant (2r+1)\frac{\|H'\|}{|H'|} \leqslant (2r+1)\nu_{\leqslant r+1}(G)$$

Bounded expansion (again)

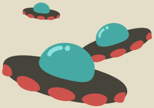
$$f_1(\widetilde{\nabla}_{r-1}(G)) \leqslant \nu_{\leqslant r}(G) \leqslant f_2(\chi_{2r+2}(G))$$

R., Sánchez Villaamil, Stavropulos

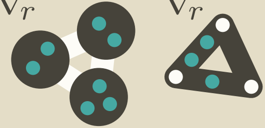
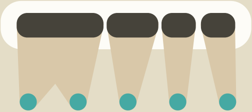
A graph class $\mathcal{G}$ has bounded expansion iff it is ν_r-bounded.

“Structurally sparse graphs have simple neighbourhood structures”

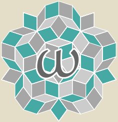
Summary
& Future work



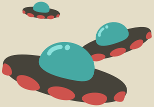
Summary

Graph measure	+	Depth parametre	=	Notion of structural sparseness
$\frac{\lVert H \rVert}{\lvert H \rvert}$	+	topological minor depth r	=	∇_r $\widetilde{\nabla}_r$ 
$\mathrm{td}(H)$	+	colour subsets of size r	=	χ_r 
cut complexity	+	neighbourhood radius r	=	NEW ν_r 

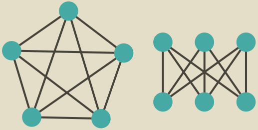
Future work



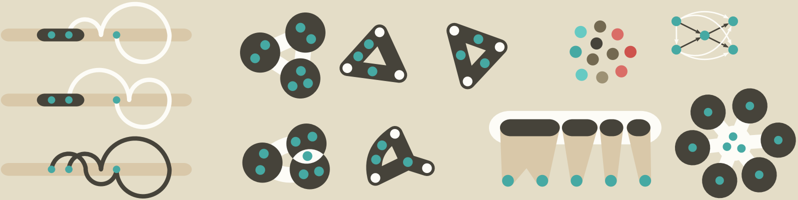
Characterise nowhere dense classes by neighbourhood complexity? Need polynomial bound!



What about dense classes? Neighbourhood complexity without taking subgraphs...



Find more characterisations!



THANKS!

Questions?

References

Nešetřil, J., & de Mendez, P. O. (2012).
Sparsity: graphs, structures, and algorithms (Vol. 28).
Springer Science & Business Media.

FR. (2015). **Structural sparseness and complex networks.**

FR, Sánchez Villaamil, F., & Stavropoulos, K. (2016).
Characterising Bounded Expansion by Neighbourhood Complexity.
arXiv preprint arXiv:1603.09532.