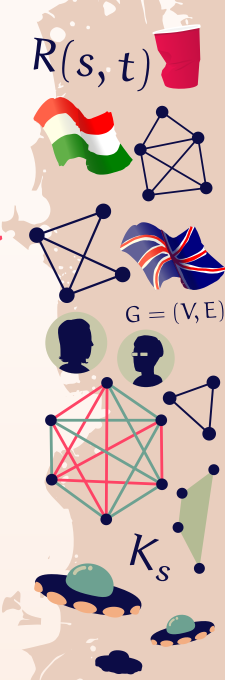
# How Ramsey ruined parties forever.

Felix Reidl
SUM Series 2016

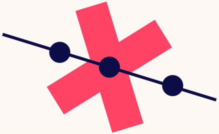
# Things we will cover

- Why parties usually either have too many people **all knowing each other**, or too many people that do **not know each other at all**
- What **graphs** are
- **Ramsey numbers!**
- Why graph theoreticians are usually not invited to formal dinners



# The lamest party trick ever

Draw **five points** on a paper such that **no three points lie on a line**



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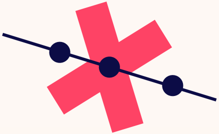

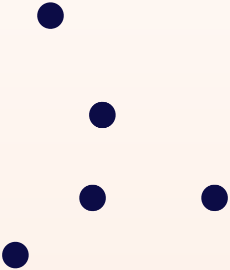

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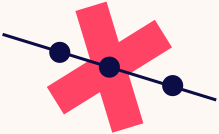

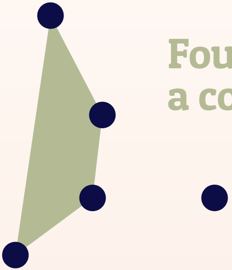

Four of these poins will form a convex quadrilateral

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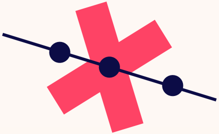
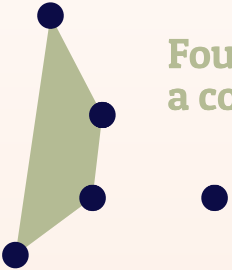
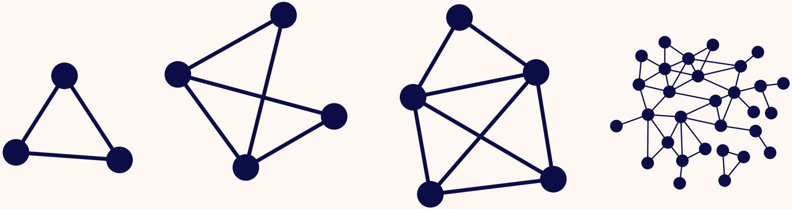




Four of these poins will form a convex quadrilateral

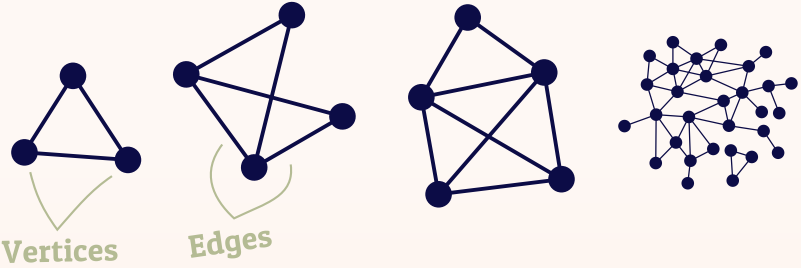
**Now try to avoid forming a convex quadrilateral.**

# Graphs



Ramsey numbers

# Graphs



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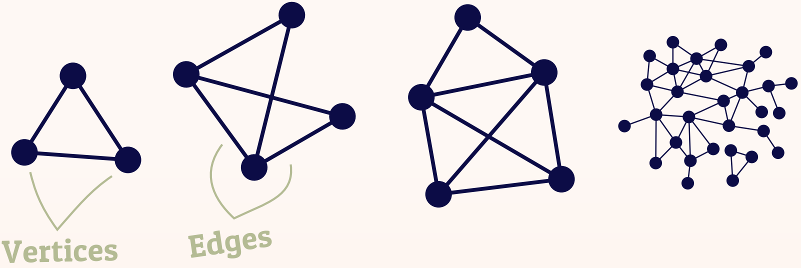
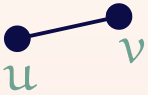


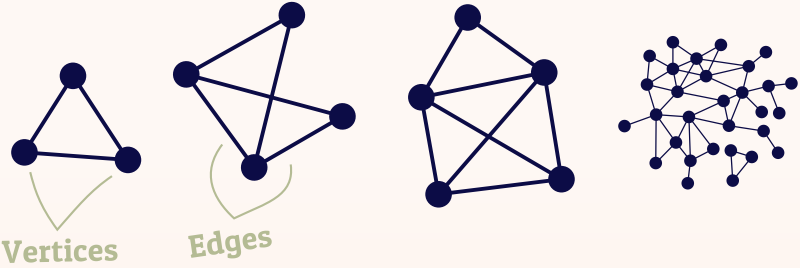

$$G = (V, E)$$

$$E \subseteq \binom{V}{2}$$




$$\iff \{u, v\} \in E$$

# Graphs

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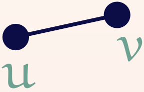
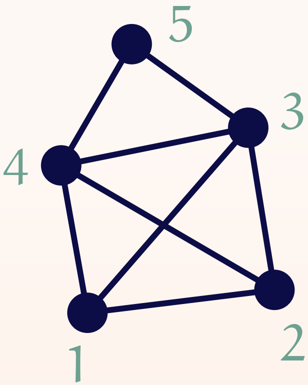


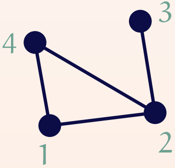

$$\iff \{u, v\} \in E$$

**That's all!**

# Graph vocab




- $4$ and $5$ are **adjacent**
- $3$ is **incident** to the edge $\{3, 4\}$
- $2, 3, 4$ are the **neighbors** of $1$
- the **degree** of $3$ is four
- the graph has $n = 5$ vertices and $m = 8$ edges
- this smaller graph below is a **subgraph** of the above graph



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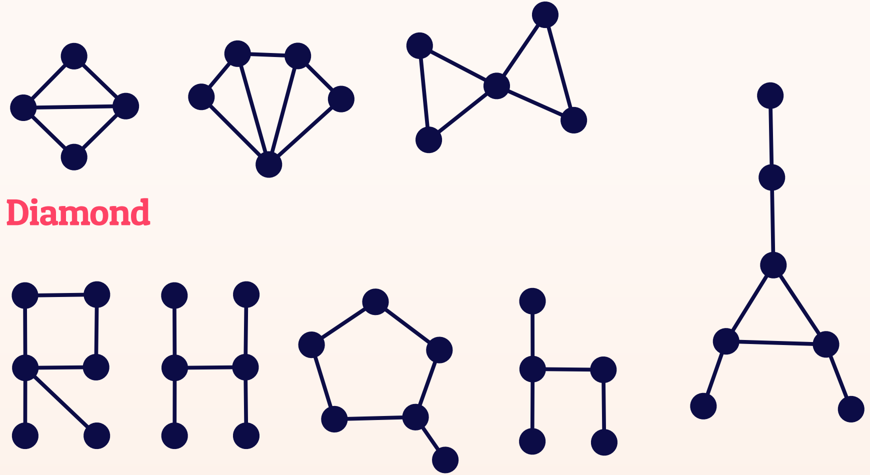

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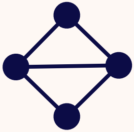
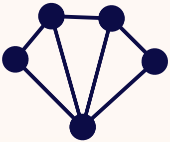
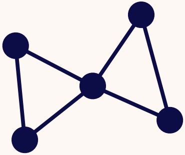
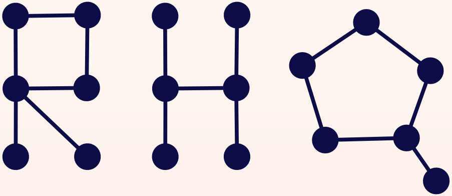


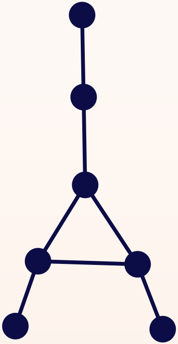
Diamond



Gem







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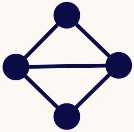

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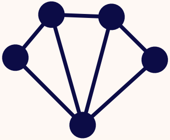

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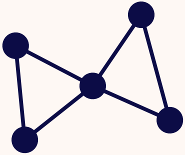
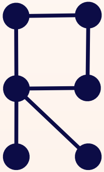
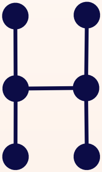
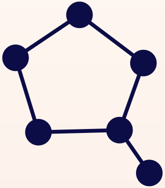
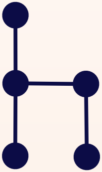


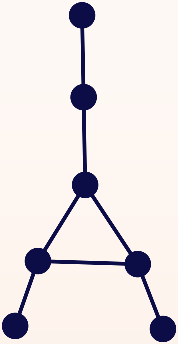
Butterfly
= Hourglass











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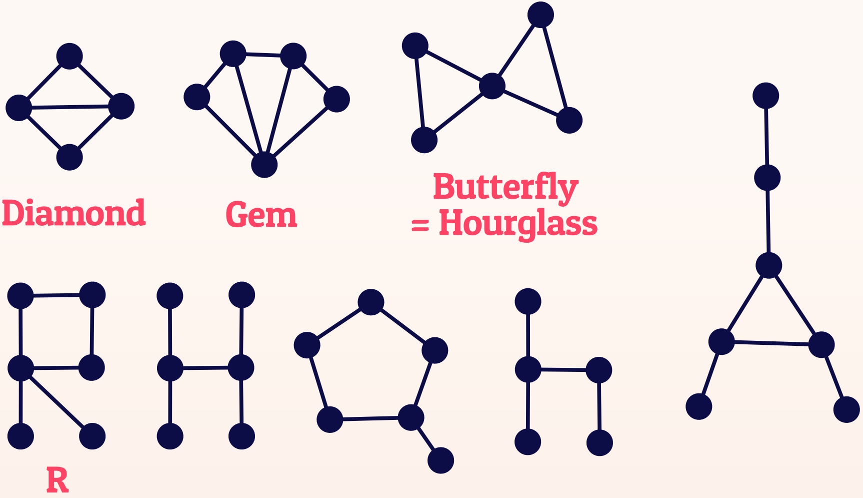

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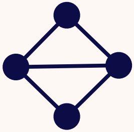
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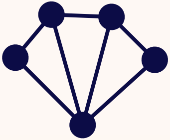
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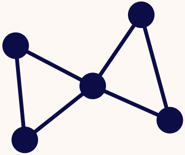
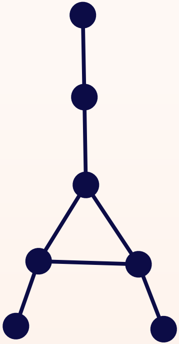
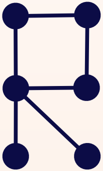
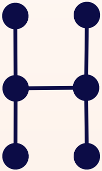
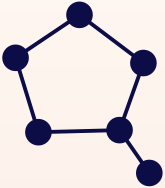
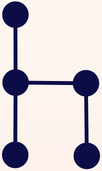

Butterfly
= Hourglass




R


H





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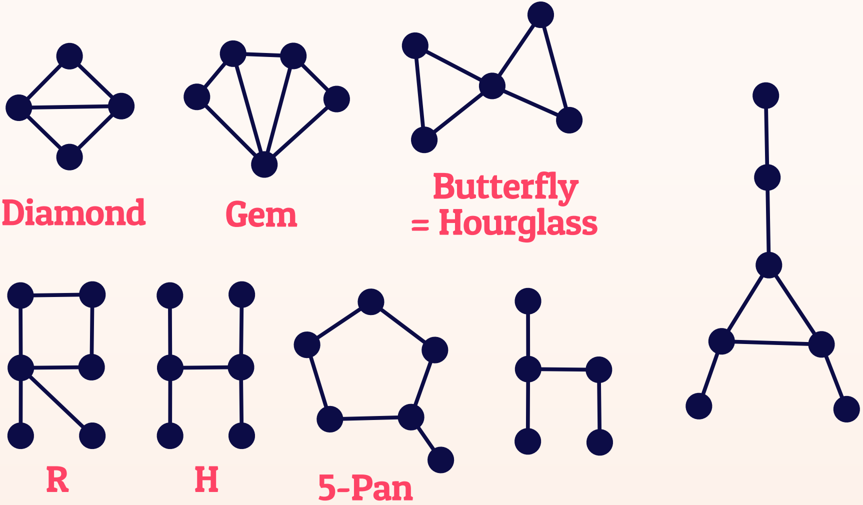

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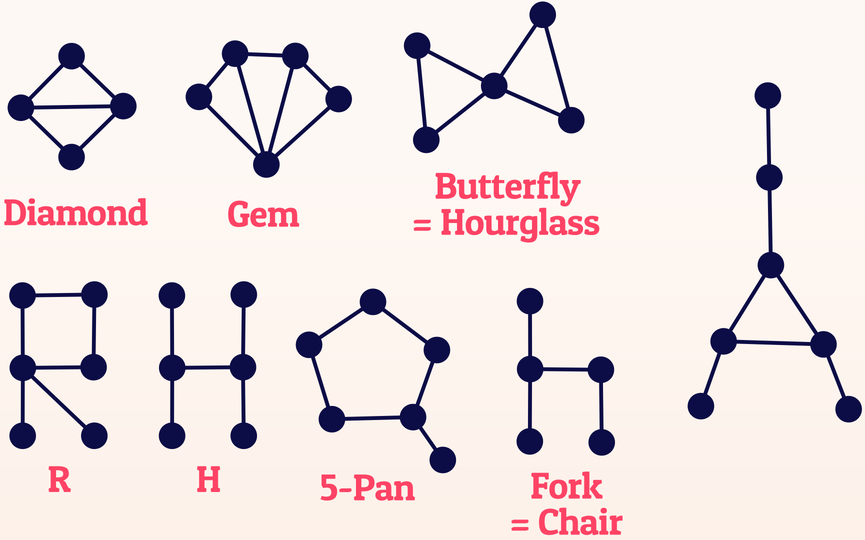

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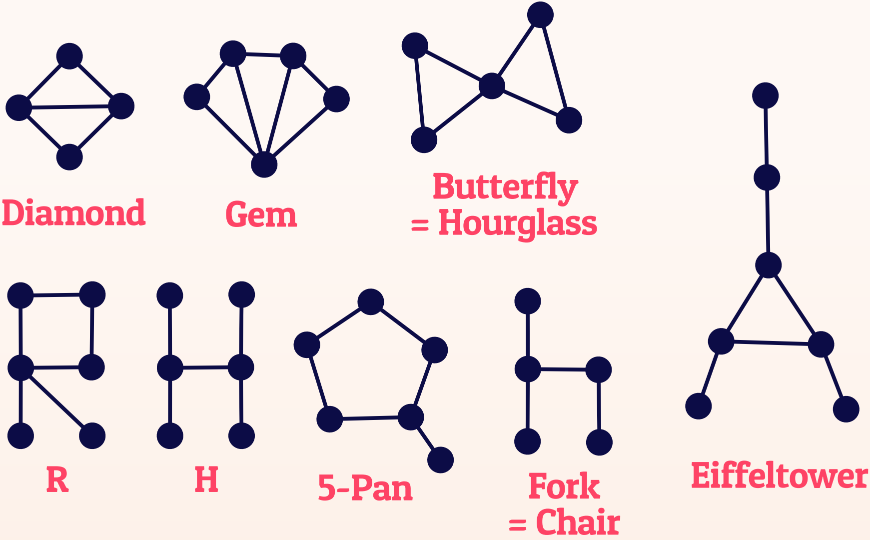
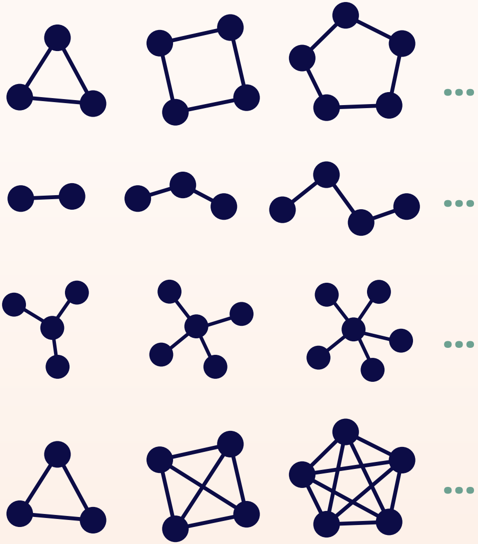

Diamond
Gem
Butterfly
= Hourglass
R
H
5-Pan
Fork
= Chair
Eiffeltower

# Classy graphs



**Cycles** $C_n$

**Paths** $P_n$

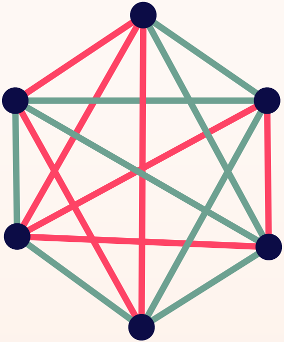
**Stars** $S_n$

**Complete graphs** **(Cliques)** $K_n$

**Ramsey numbers**

# A slightly less lame party trick

(please do not try this at an actual party)



**Color the edges of $K_6$ with two colors such that no monochromatic triangle exists!**

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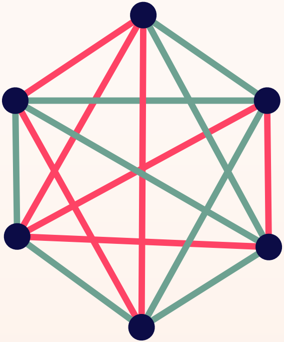
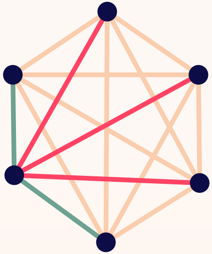


Color the edges of $K_6$ with two colors such that **no monochromatic triangle** exists!

**Seems unavoidable!**

Can we prove this?

# Proof



- Consider the five edges incident to an arbitrary vertex

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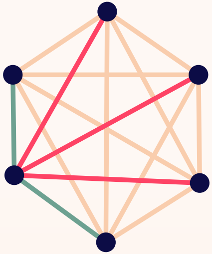

- Consider the five edges incident to an arbitrary vertex
- At least three of these edges will have the same color (say red)

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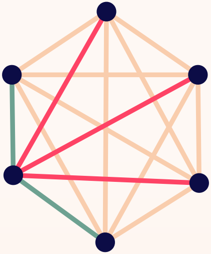

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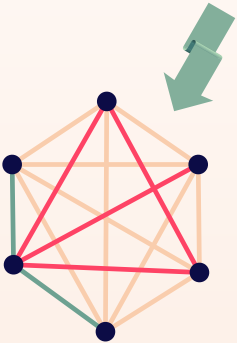

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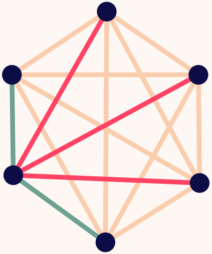

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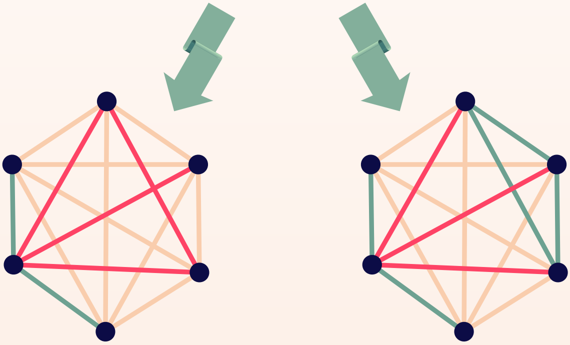

# So many questions

Different class?

Larger cliques?

More colors?

"Every two-coloring of the edges of $K_6$ will contain a monochromatic triangle."

Things other than graphs?

Other graphs?

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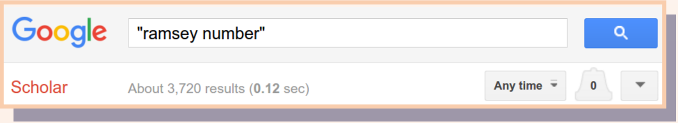

# Ramsey?

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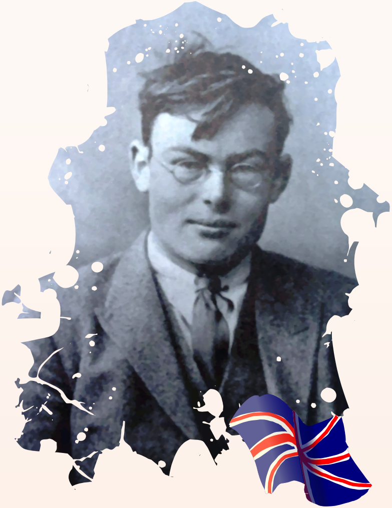
Ramsey numbers

# Frank P. Ramsey

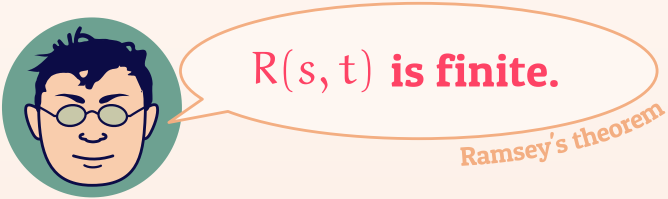


## Ramsey facts:

- P. stands for "Plumpton"
- Philosopher, economist and mathematician
- Translated Wittgenstein's Tractatus from German
- Died at age 26

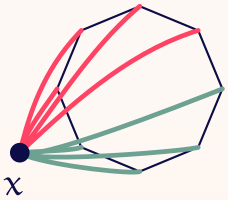
# Ramsey numbers

Let $R(s,t)$ denote the smallest number such that every red/green-coloring of the edges of $K_{R(s,t)}$ either contain a **red** $K_s$ or a **green** $K_t$ .



# Proof by unravelling

1 Pick a vertex $x$



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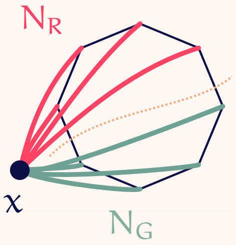


1 Pick a vertex $x$

2 Partition neighbors into sets $N_R$ and $N_G$

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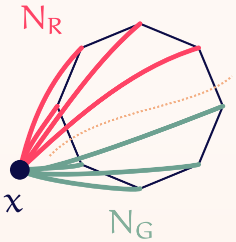


1 Pick a vertex $x$

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3 If $N_R$ is larger, color $x$ **red** otherwise color it **green**

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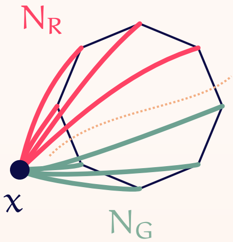


1. Pick a vertex $x$
2. Partition neighbors into sets $N_R$ and $N_G$
3. If $N_R$ is larger, color $x$ **red** otherwise color it **green**
4. Repeat inside larger set

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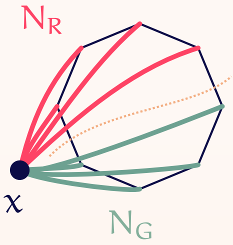


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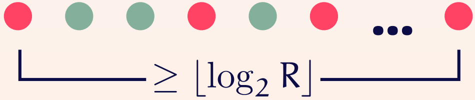
**Result:**



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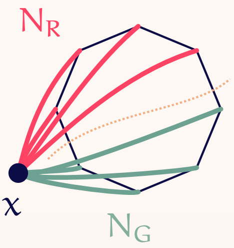


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**Result:**


One color set has size $\geq \lfloor \log_2 R \rfloor / 2$

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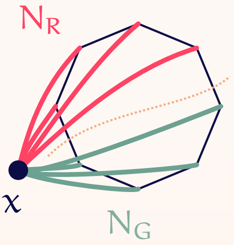


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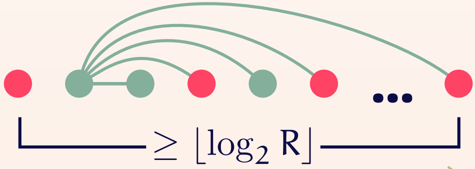
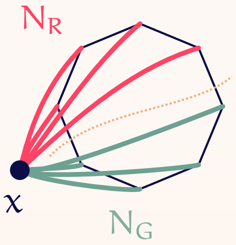


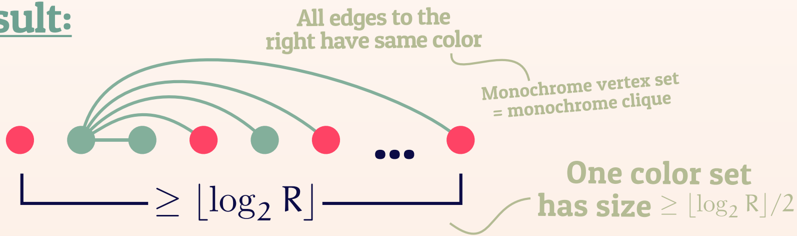

**One color set has size** $\geq \lfloor \log_2 R \rfloor / 2$

# Proof by unravelling




1. Pick a vertex $x$
2. Partition neighbors into sets $N_R$ and $N_G$
3. If $N_R$ is larger, color $x$ **red** otherwise color it **green**
4. Repeat inside larger set

## Result:



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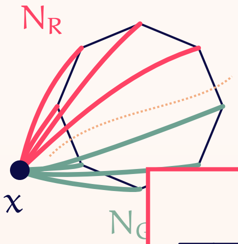



1 Pick a vertex $x$

2 Partition neighbors into sets $N_R$ and $N_G$

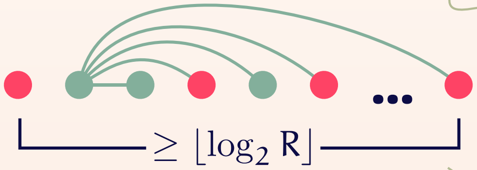
3 If $N_R$ is larger, color $x$ **red**

$$\implies R(s,s) \le 2^{2s+1}$$

**Result:**

All edges to the right have same color

Monochrome vertex set = monochrome clique




One color set has size $\ge \lfloor \log_2 R \rfloor / 2$

Ramsey numbers

# Some Ramsey numbers

| s \ t | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 3 | 6 | 9 | 14 | 18 |
| 4 | 1 | 4 | 9 | 18 | 25 | 35–41 |
| 5 | 1 | 5 | 14 | 25 | 43–49 | 58–87 |
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- **Symmetry**
  $R(s, t) = R(t, s)$

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  $R(s,t) \leq R(s-1,t) + R(s,t-1)$

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- **Symmetry**
  $R(s, t) = R(t, s)$
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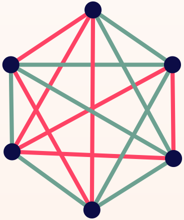
  $R(s, t) \leq R(s-1, t) + R(s, t-1)$
- **Lower bounds?**

# Lower bounds by brute force

- Try all colorings for $K_n$ and check for monochromatic $K_s / K_t$



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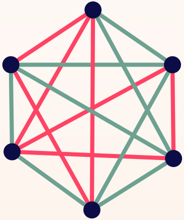

$2^{n(n-1)/2}$ **colorings**

$\times n^{s+t}$ **time to check**

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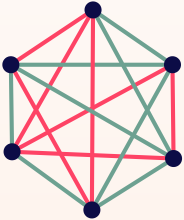

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---

**Ain't nobody got time for that.**

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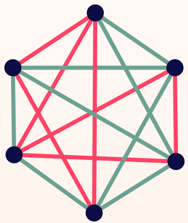


$$2^{n(n-1)/2}$$ colorings
$$\times n^{s+t}$$ time to check

---

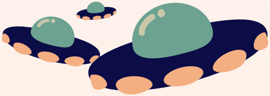
**Ain't nobody got time for that.**

- To check whether $R(5,5) = 43$ it would take around $10^{272}$ cpu years

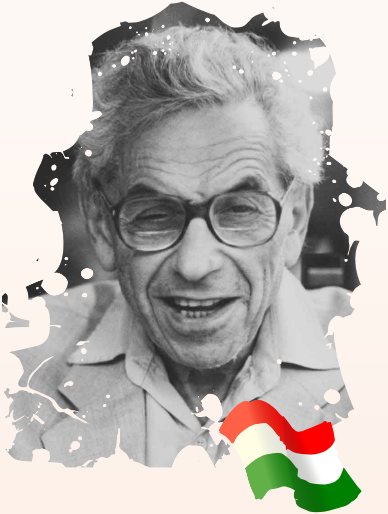
# Erdős weighs in

Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the **Ramsey number for red five and blue five**.

We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the **Ramsey number for red six and blue six**, however, we would have no choice but to launch a preemptive attack.



# Paul Erdős



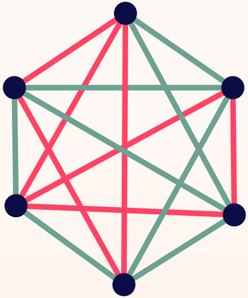
## Erdős facts:

- Most important mathematician of 20th century
- Called children "epsilons"
- Erdős number: how many publications away?
- About 1500 publications

# Erdős' probabilistic lower bound

**Color the edges of $K_{2^{s/2}}$ randomly with red and green.**

with equal probability



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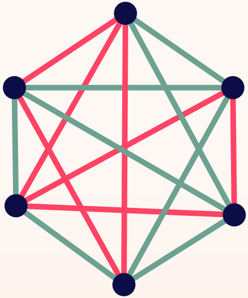
Probability that a monochromatic $K_s$ exists:



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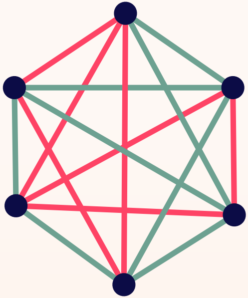
$$\binom{2^{s/2}}{s} \frac{2}{2^{s(s-1)/2}} < 1$$



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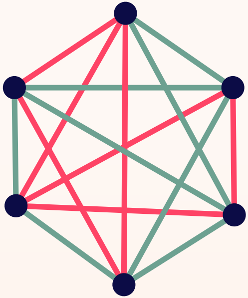

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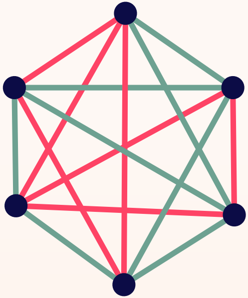

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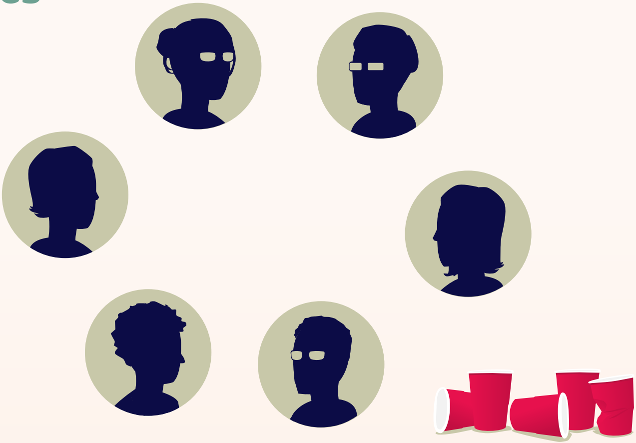
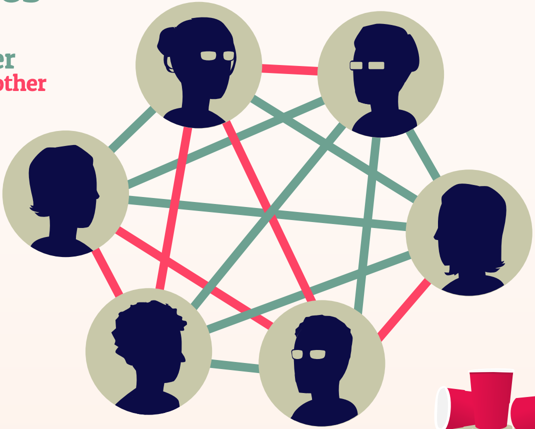
There is at least one coloring without a monochromatic $K_s$

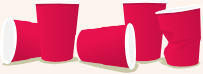
# Back to parties



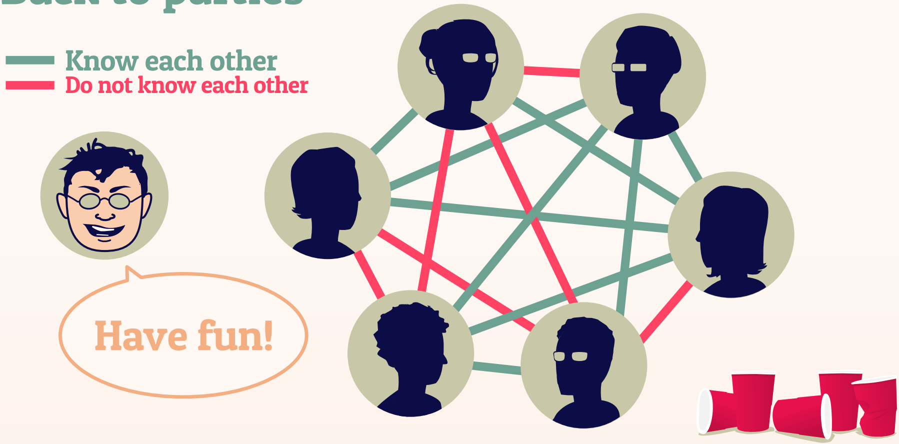
# Back to parties

Know each other
Do not know each other



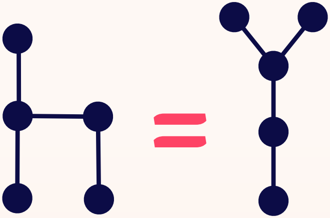


# Back to parties




**This is why (large) parties either have too many people all knowing each other, or too many people that do not know each other at all.**

## Done!



Chair    Fork

# Thanks!