Kernelization using structural parameters on sparse graph classes

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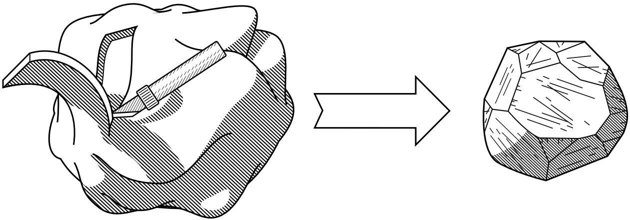
The story so far

Kernelization



- Problem is fixed-parameter tractable iff it has a kernelization algorithm
- Goal: to obtain *polynomial* or even *linear* kernels.

Basic technique of kernelization:
Devise *reduction rules* that preserve equivalence of instances; apply exhaustively, prove kernel size.

Algorithmic meta-results: nail down as many problems as possible

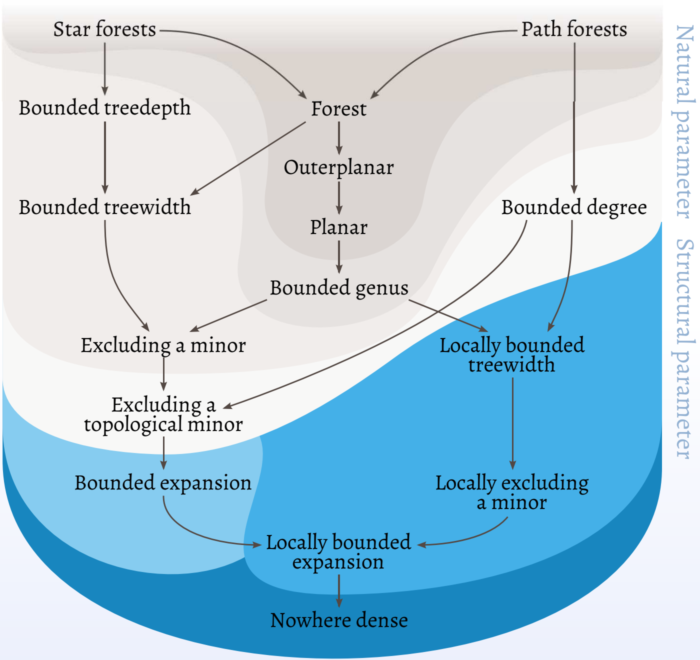
Previous work

- Framework for planar graphs
  Guo and Niedermeier: *Linear problem kernels for NP-hard problems on planar graphs*
- Meta-result for graphs of bounded genus
  Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh and Thilikos: *(Meta) Kernelization*

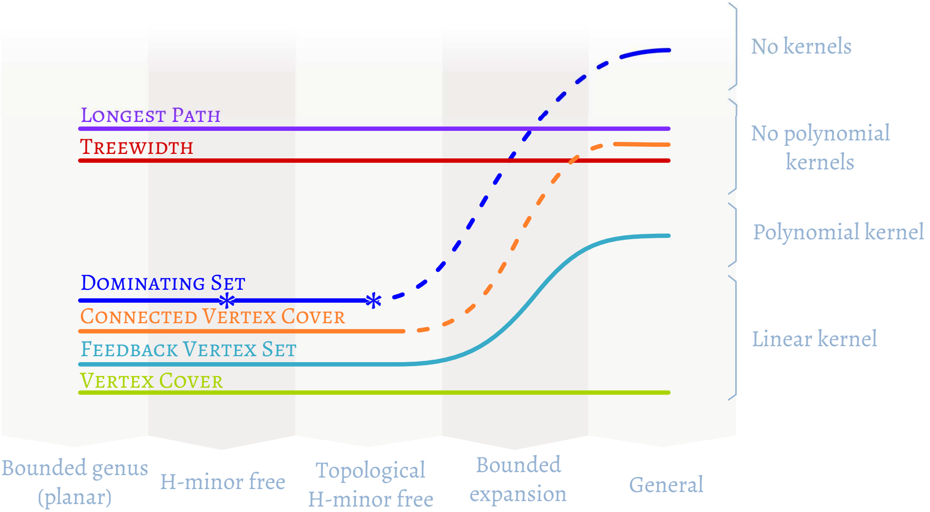
- Meta-result for graphs excluding a fixed graph as a minor
  Fomin, Lokshtanov, Saurabh and Thilikos: *Bidimensionality and kernels*

- Meta-result for graphs excluding a fixed graph as a topological minor
  Kim, Langer, Paul, R., Rossmanith, Sau and Sikdar: *Linear kernels and single-exponential algorithms via protrusion decompositions*
- *Our contribution*: Meta-result for graphs of bounded expansion, local bounded expansion and nowhere-dense graphs using *structural parameterization*

The big picture

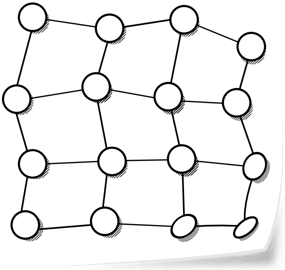


Why we must run into trouble



Bidimensionality does not help

(probably)



Dichotomy: either easy instance or no grid of size $O(k)$

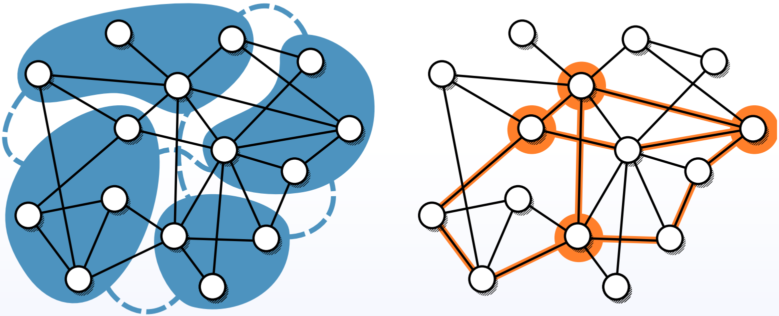
$\Rightarrow$ Bounded treewidth gives enough structure to make reduction rule work (more on that later)

- Need to rely on improvement of the grid minor theorem for graphs beyond H-minor-free
- Known lower bound in general graphs: graphs of treewidth $\Omega(r^2 \log r)$ with no $r \times r$-grid

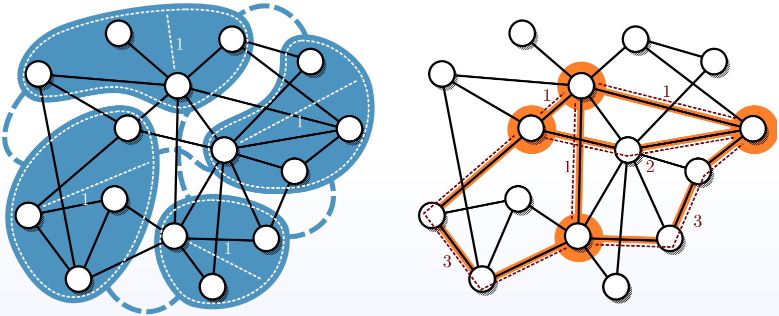
$\Rightarrow$ At least not much hope for linear kernels

Beyond excluded minors

Minors, top-minors



Shallow minors, top-minors



Bounded expansion

For a graph G we denote by $G \triangledown r$ the set of its r-shallow minors.

Definition (Grad, Expansion)

For a graph G, the *greatest reduced average density* is defined as

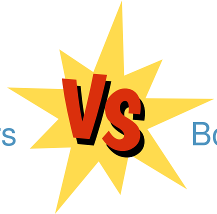
$$\nabla_r(G) = \max_{H \in G \triangledown r} \frac{|E(H)|}{|V(H)|}$$

For a graph class $\mathcal{G}$ the *expansion* of $\mathcal{G}$ is defined as

$$\nabla_r(\mathcal{G}) = \sup_{G \in \mathcal{G}} \nabla_r(G)$$

A graph class $\mathcal{G}$ has *bounded expansion* if there exists a function f such that $\nabla_r(\mathcal{G}) \leq f(r)$ for all $r \in \mathbf{N}$.

Excluded minors



Bounded expansion

d-degenerate (depening on excluded minor)

Linear number of edges

No large cliques

No large clique-minors

Closed under taking minors

Degeneracy of every minor is d

$f(0)$-degenerate (depening on expansion)

Linear number of edges

No large cliques

Can contain large clique minors

"Closed" under taking shallow minors

Degeneracy of minors depends on its "size"

Techniques from result on H-topological-minor-free graphs stop working because they use large (non-shallow) topological minors.

The exemplary obstacle: Treewidth-t-Deletion

The problem

TREEWIDTH-t DELETION

Input: A graph G, an integer k

Problem: Is there a set $X \subseteq V(G)$ of size at most k such that $\mathbf{tw}(G - X) \leq t$?

- TREEWIDTH-1 DELETION = FEEDBACK VERTEX SET
- Model problem for previous results
- $k^{f(t)}$-kernel on general graphs

$\Rightarrow$ Probably none of size $O(f(t)k^c)$ (c independent of t)

Kernel on bounded expansion graphs implies same kernel on general graphs

From general to sparse

1. Treewidth closed under subdivision of edges

$\Rightarrow$ Treewidth-modulator closed under subdivision of edges

$\Rightarrow$ Instances of $\textsc{Treewidth-}t\ \textsc{Deletion}$ closed under subdivision of edges

2. Subdividing each edge of a graph $|G|$ yields a graph of *bounded expansion*

General kernel from sparse kernel:
Reduce (G, k) to $(\tilde{G}, k)$ by subdividing every edge $|G|$ times, output kernel of $(\tilde{G}, k)$.

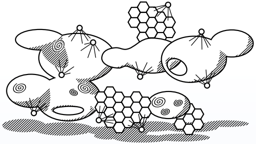
If we want a kernel, we need a parameter that is not closed under edge subdivision

Structural parameterization to the rescue

The natural view

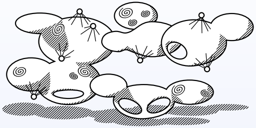
Bounded Expansion — ?

∪



H-Topological-Minor-Free — *Treewidth-bounding*

∪



H-Minor-Free — *Bidimensional* *+separation property*

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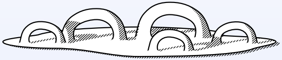

Bounded Genus — *Quasi-compact*

The structural view

Bounded Expansion ?

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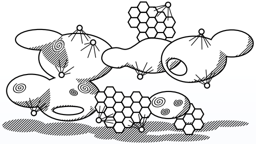

H-Topological-Minor-Free *Treewidth-t Modulator*

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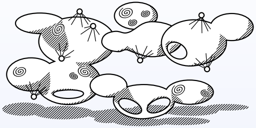

H-Minor-Free *Treewidth-t Modulator* *(implied by Lemma 3.2)*

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Bounded Genus *Treewidth-t Modulator* *(implied by Lemma 9)*

The structural view

Bounded Expansion — *Treedepth-d Modulator*

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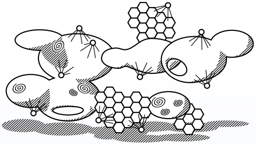

H-Topological-Minor-Free — *Treewidth-t Modulator*

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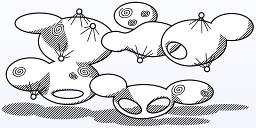

H-Minor-Free — *Treewidth-t Modulator* *(implied by Lemma 3.2)*

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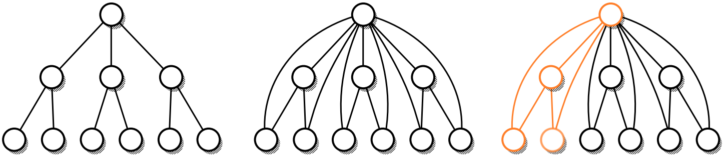
Bounded Genus — *Treewidth-t Modulator* *(implied by Lemma 9)*

Tree*depth*?



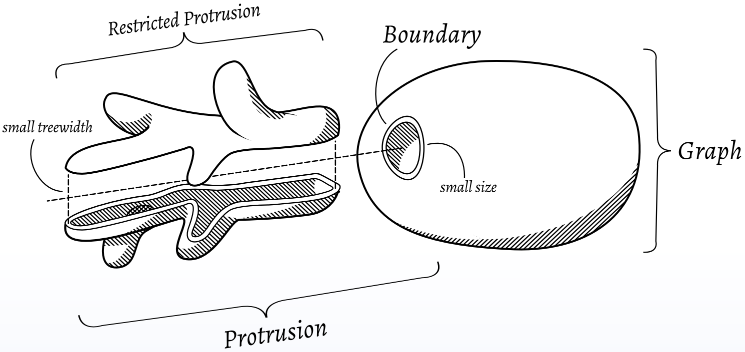
For a graph G with **td**$(G) \leq d$:

- G embeddable in closure of tree (forest) of depth d
- Graph does not contain path of length 2^d
- **tw**$(G) \leq$ **pw**$(G) \leq d - 1$

Not closed under subdivision!

If X is a treedepth-d-modulator, $G - X$ does not contain long paths

Protrusion anatomy

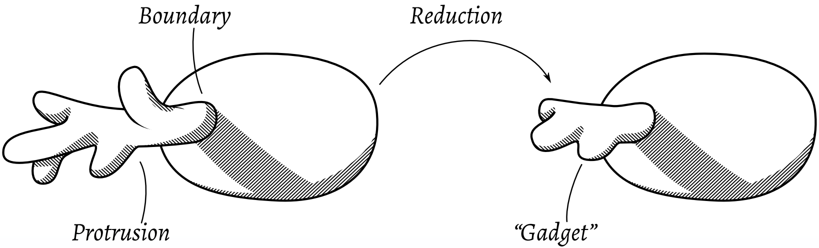


Definition

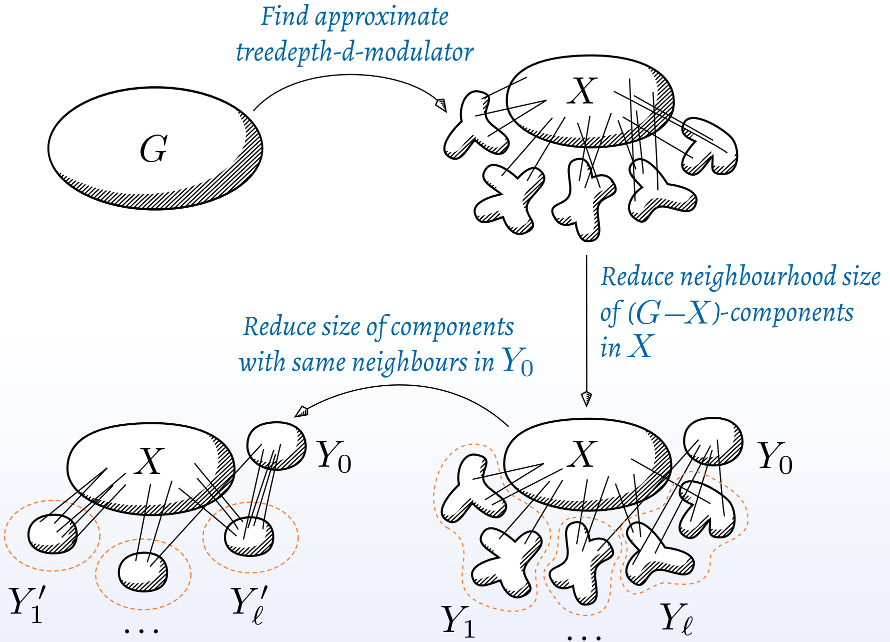
$X \subseteq V(G)$ is a t-*protrusion* if

1. $|\partial(X)| = |N(X) \setminus X| \leq t$ (small boundary)
2. $\textbf{tw}(G[X]) \leq t$ (small treewidth)

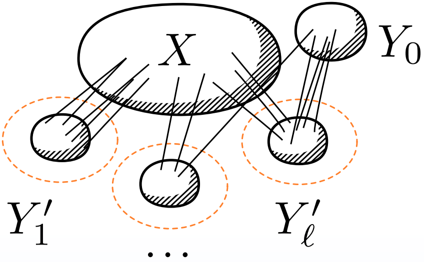
The magic reduction rule



- We want to replace a large protrusion by something smaller
- Possible if problem has *finite integer index*
- Recursive structure of graphs of small treewidth (i.e. protrusion) helps
- Lots of technicalities omitted. . .


Find approximate
treedepth-d-modulator
G
X
Reduce neighbourhood size
of (G − X)-components
in X
Reduce size of components
with same neighbours in Y_0
X
Y_0
Y'_1
...
Y'_ℓ
X
Y_0
Y_1
...
Y_ℓ

Using sparseness



- $Y_i, 1 \leq i \leq \ell$ have constant size after protrusion reduction
- $|Y_0| = O(|X|)$ (follows from degeneracy of 2^d-shallow minors)
- $\ell = O(|Y_0|) = O(|X|)$ (ditto)
- Hidden constants depend on expansion $\nabla_{2^d}(\mathcal{G}) \leq f(2^d)$

The result

Theorem

Any graph-theoretic problem that has finite integer index on graphs of constant treedepth admits linear kernels on graphs of bounded expansion if parameterized by a modulator to constant treedepth.*

- Kernelization possible in linear time

* Structural parameter enables us to relax the FII condition

⇒ Kernels for problems like TREEWIDTH and LONGEST PATH

- Structural parameter helps to include decision problems like 3-COLORABILITY and HAMILTIONIAN PATH
- Quadratic kernels on graphs of locally bounded expansion
- Polynomial kernels on nowhere dense graphs

Consequences

The problems…

Dominating Set, Connected Dominating Set, r-Dominating Set, Efficient Dominating Set, Connected Vertex Cover, Hamiltonian Path/Cycle, 3-Colorability, Independent Set, Feedback Vertex Set, Edge Dominating Set, Induced Matching, Chordal Vertex Deletion, Interval Vertex Deletion, Odd Cycle Transversal, Induced d-Degree Subgraph, Min Leaf Spanning Tree, Max Full Degree Spanning Tree, Longest Path/Cycle, Exact s,t-Path, Exact Cycle, Treewidth, Pathwidth

…parameterized by a treedepth-modulator have …

- …linear kernels on graphs of bounded expansion
- …quadratic kernels on graphs of locally bounded expansion
- …polynomial kernels on nowhere-dense graphs

Conclusion

Our interpretation:

- Underlying reason for previous result is existence of a small treewidth modulator:
  Quasi-compactness and *bidimensionality* are tangible properties which guarantee this on the respective graph classes
- Larger graph classes need stronger parameters
- Treedepth-modulator is a useful parameter (also works well on general graphs as a relaxation of vertex cover)

Open questions:

- Which problems still admit polynomial kernels on these classes using their natural parameter?
- Problem categories: closed under subdivision vs. not closed. Weaker parameterization for latter?
- Linear kernels for graphs with locally bounded treewidth?
- Lower bounds!

Thanks!