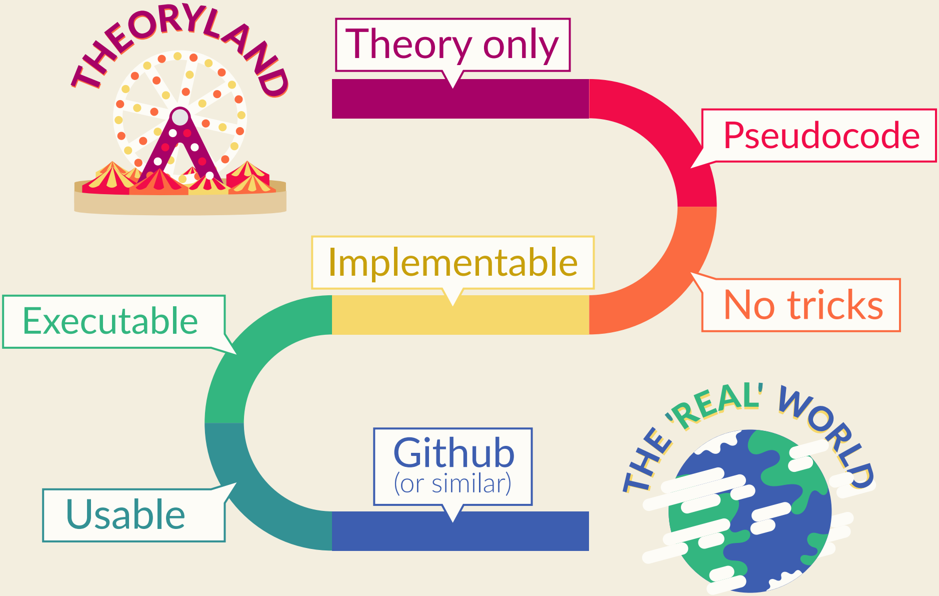
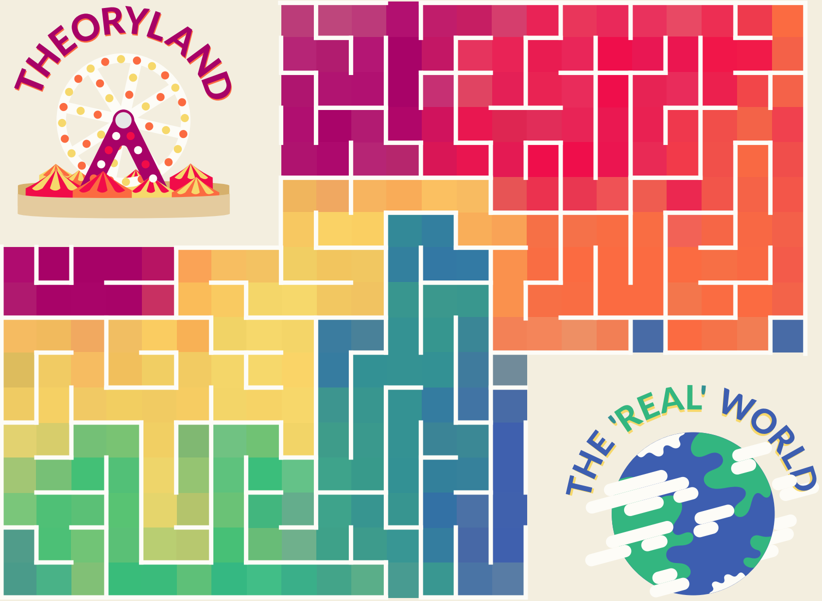
Complex networks & sparsity
Part IV: Implementation

Felix Reidl
Blair D. Sullivan
DOCCOURSE '18

The long and winding road

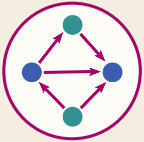


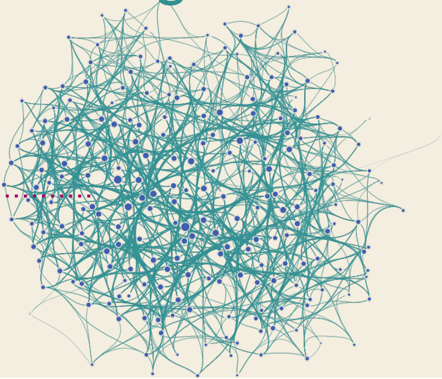
The long and winding ~~road~~



Motif-counting

We want to count the number of times a given **motif graph**





appears in a larger host graph (network).

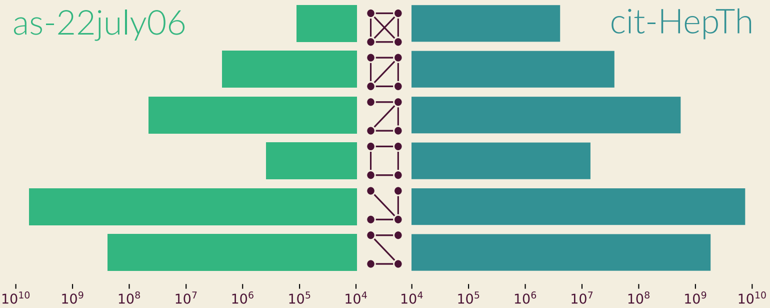
Motifs that appear more often **than expected** *might* play an important **function** in the network.

Milo R, Shen-Orr S, Itzkovitz S, Kashtan N, Chklovskii D, Alon U. **Network motifs: simple building blocks of complex networks**. Science. 2002 Oct 25;298(5594):824-7.

Ribeiro P, Silva F, Kaiser M. **Strategies for network motifs discovery**. InE-Science, 2009. e-Science'09. Fifth IEEE International Conference on 2009 Dec 9 (pp. 80-87). IEEE.

Graphlets

We want to count all (connected) induced subgraphs up to a given size.



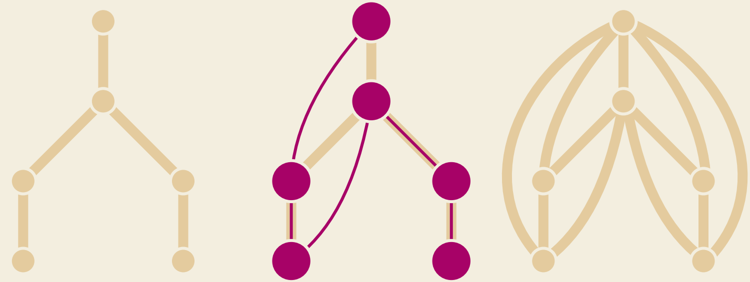
The **graphlet degree distribution** or the **graphlet degree** can be used to compare networks.

Pržulj N, Corneil DG, Jurisica I.
Modeling interactome: scale-free or geometric?.
Bioinformatics. 2004 Jul 29;20(18):3508-15

Pržulj N. B**iological network comparison using graphlet degree distribution.**
Bioinformatics. 2007 Jan 15;23(2):e177-83.

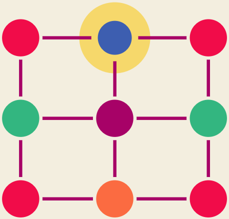
Treedepth

Def. A graph has *treedepth* d if it is the subgraph of the *closure* of a tree of height d.



Treedepth: centered colourings

A vertex colouring is *centered* if every connected subgraph H contains a vertex whose color is unique in H.



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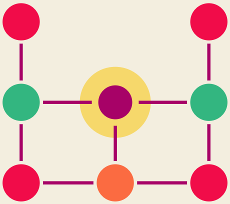

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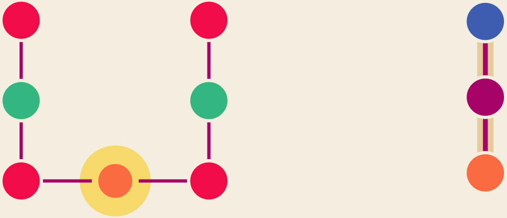

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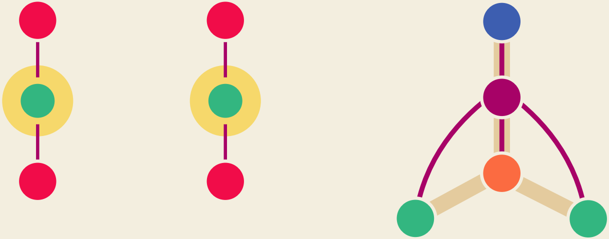

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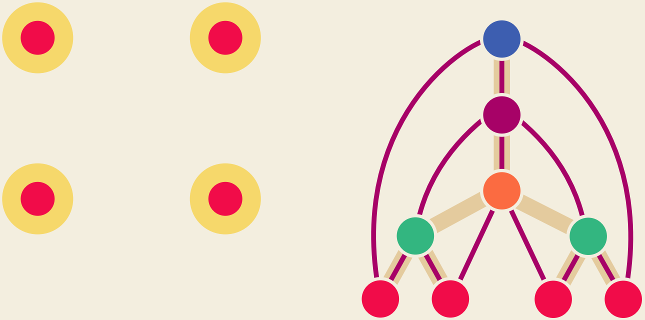

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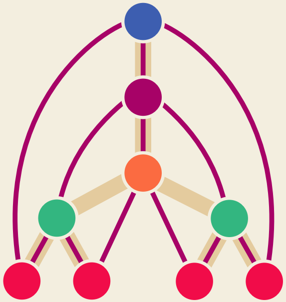
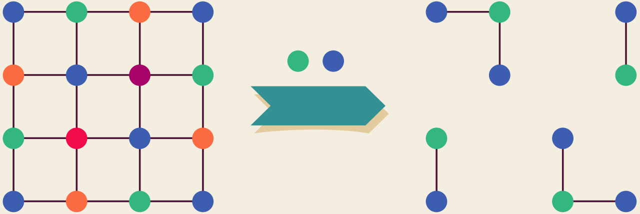


The centered colouring number $\chi_{\mathrm{cen}}(G)$ is equal to the treedepth of G.

Low treedepth colourings

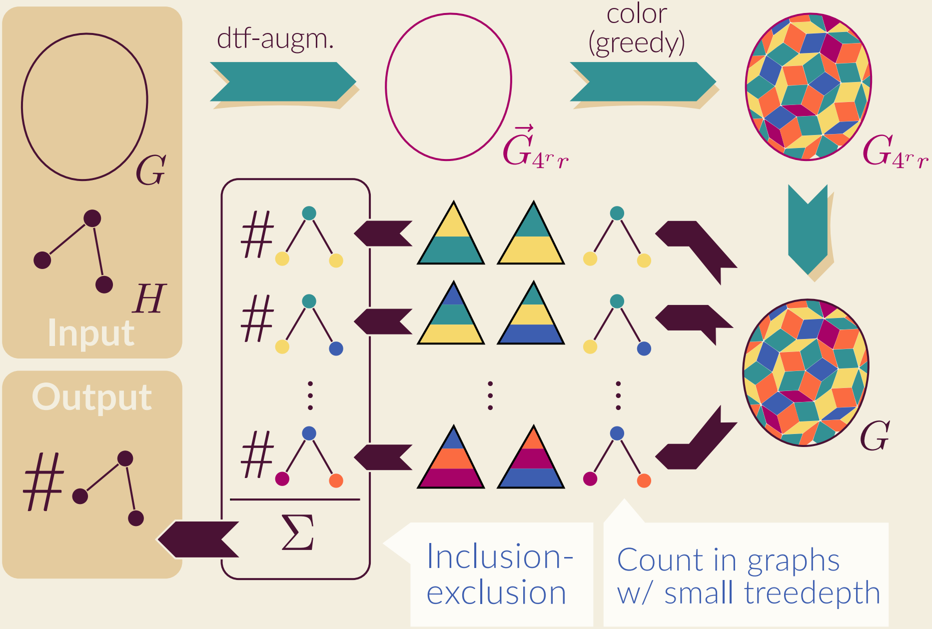
A vertex colouring is a **r-treedepth colouring** if every set of $i < r$ colours induce a subgraph of treedepth i.



$\chi_r(G) :=$ min #colours needed in an r-treedepth colouring of G

A graph class has bounded expansion iff it is χ_r-bounded.

Motif counting using χ_r

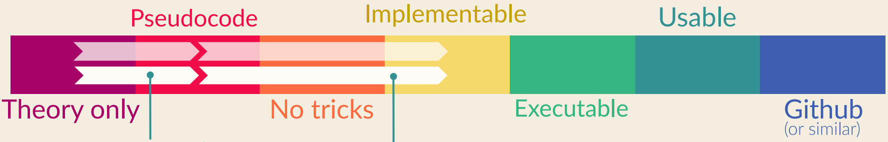


CONCUSS

Engineering motif-counting

$$f(h) \cdot 2^{O(h^2)} n$$

$$f(h) \cdot h^{O(h)} n$$

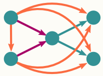


Nešetřil J, De Mendez PO.
Sparsity: graphs, structures, and algorithms.
Springer Science & Business Media; 2012 Apr 24.

Demaine ED, Reidl F, Rossmanith P,
Sánchez Villaamil F, Sikdar S, Sullivan BD.
Structural sparsity of complex networks:
Bounded expansion in random models and real-world graphs.
arXiv preprint arXiv:1406.2587. 2014 Jun 10.

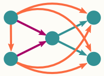
Engineering motif-counting

tf-augmentations



Absolutely impractical

dtf-augmentations



Test colouring after each step

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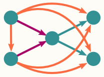

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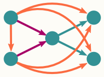

Test colouring after each step

Good engineering & heuristical improvements

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Theory in Practice Group (NCSU)
with great help from students
Clayton G. Hobbs & Brandon Mork
https://github.com/TheoryInPractice/CONCUSS

Is it *practical*?

The current implementation is vastly outperformed by other algorithms (vf2) on practical instances.

Number of colours

Size of motif

$$\binom{f(h)}{h} h^{O(h)} n$$

There are artificial graph classes in which the algorithm performs better.

O'Brien MP, Sullivan BD.
Experimental evaluation of counting subgraph isomorphisms in classes of bounded expansion.
arXiv preprint arXiv:1712.06690. 2017 Dec 18.

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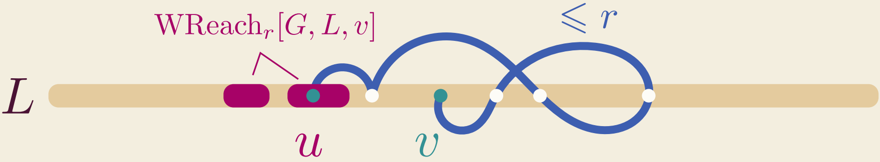
1) Improve colouring algorithm
2) Don't use low-treedepth colourings

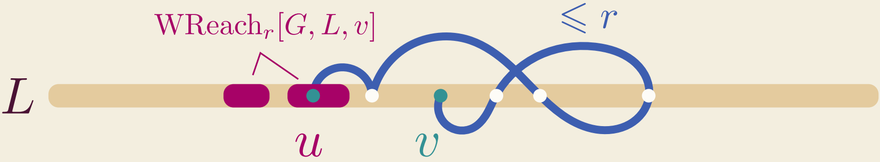
Mandoline

Weak coloring & bounded expansion



u is **weakly r-reachable** from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

Weak coloring & bounded expansion



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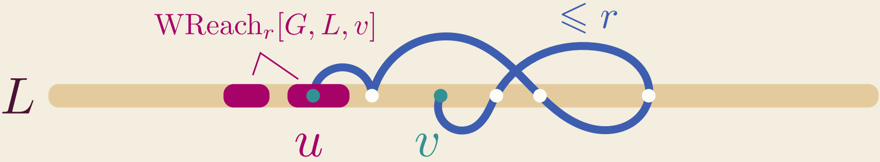
$$\mathrm{wcol}_r(G) := \min_{L \in \Pi(G)} \max_{v \in G} |\,\mathrm{WReach}_r[G, L, v]|$$

Weak coloring & bounded expansion



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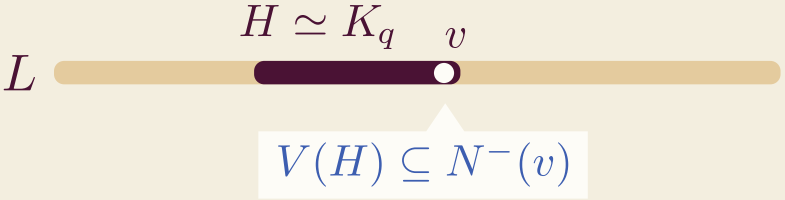
A graph class has bounded expansion iff it is wcol_r-bounded.

Let's start with something easy!

We count cliques in a d-degenerate graph.

Observation: every clique is contained in the left-neighbourhood of its *last* vertex.



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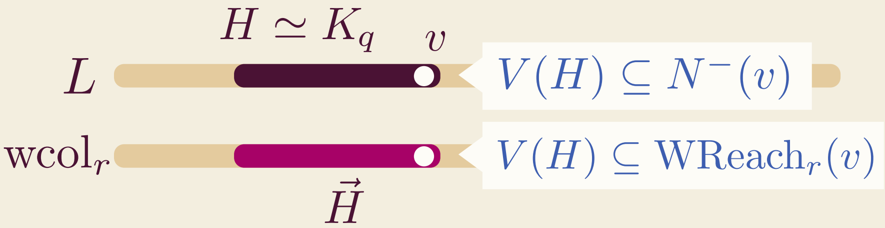
$V(H) \subseteq N^-(v)$

Therefore we can enumerate all cliques by enumerating all cliques in $N^-(v)$ for all $v \in G$!

$O(2^d n)$ time!

Does it blend?

Can we 'lift' this algorithm to wcol?



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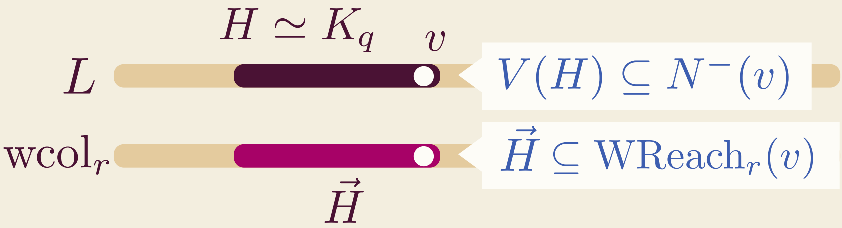

1. What is the 'last' vertex of H?

   Enumerate all orderings $\vec{H}$ of H.

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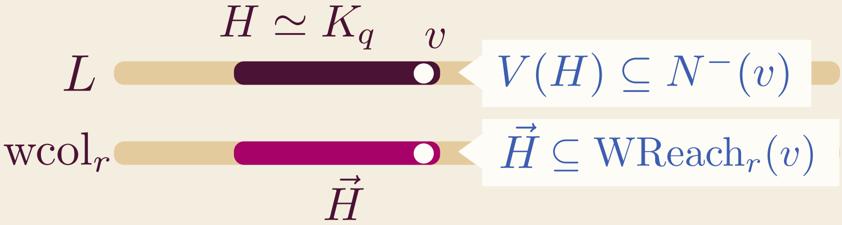


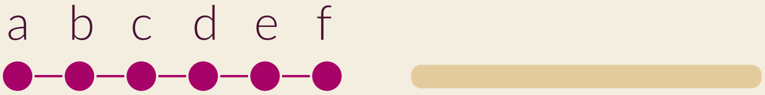
1. What is the 'last' vertex of H?

   Enumerate all orderings $\vec{H}$ of H.

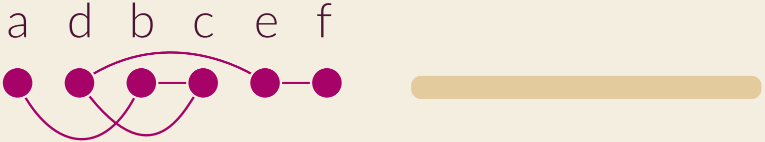
2. Does $\vec{H} \subseteq \mathrm{WReach}_r(v)$ actually hold?

   Only sometimes!

Two ways to order a P_6



wcol_r



wcol_r

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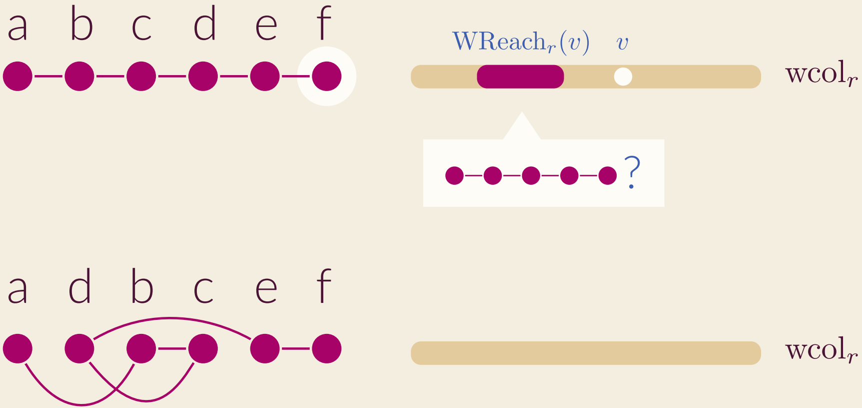
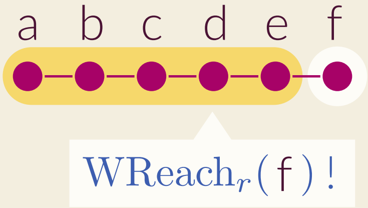
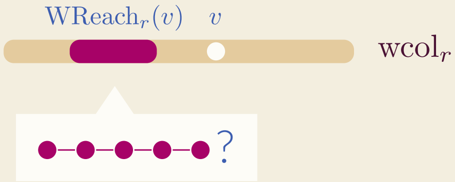
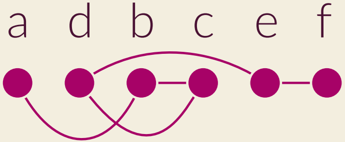
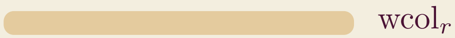


Two ways to order a P_6









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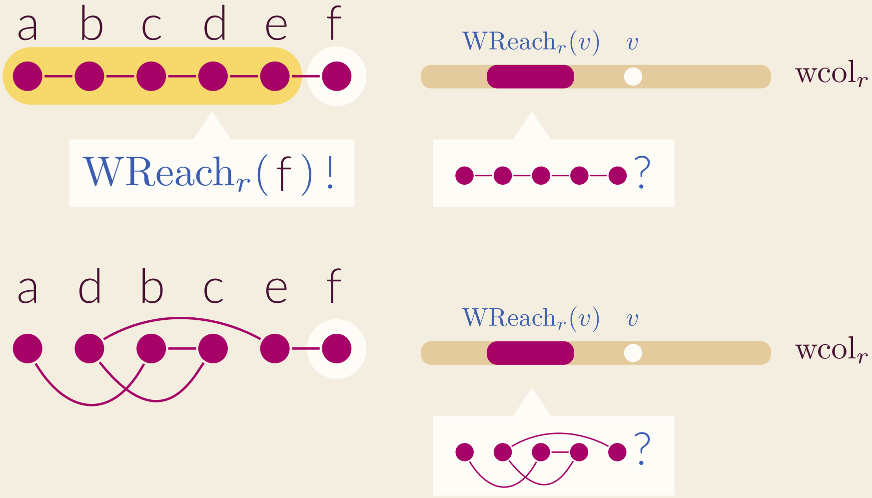

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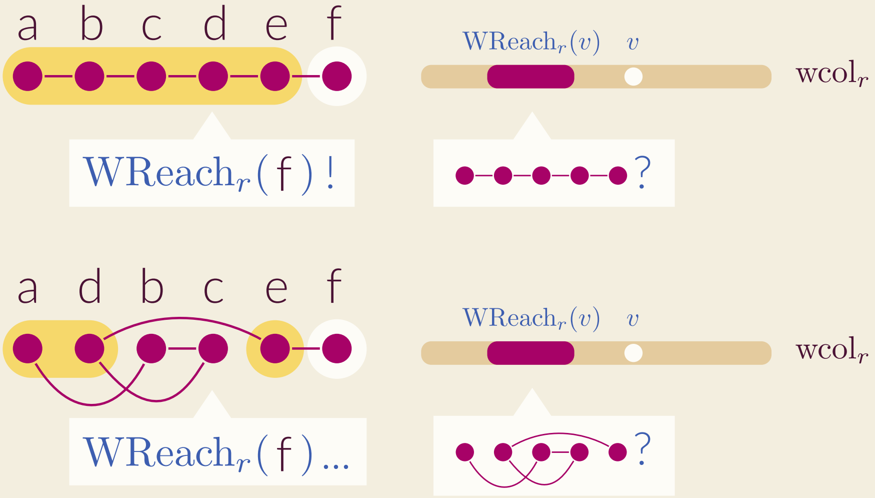

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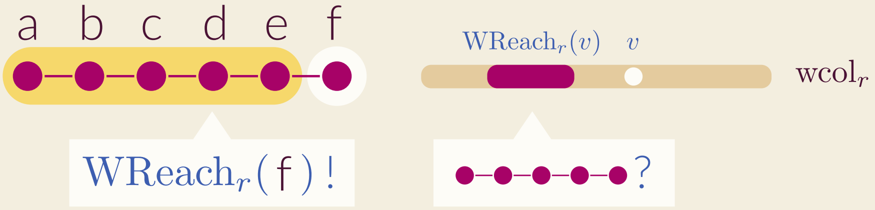

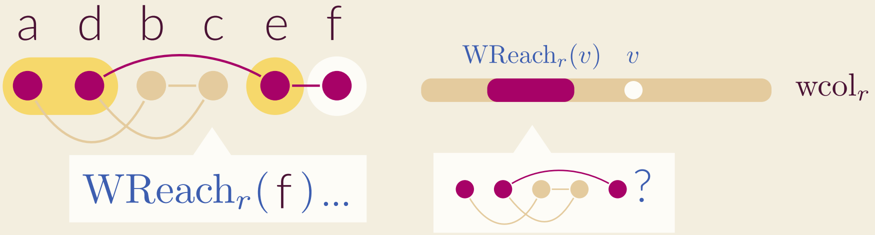

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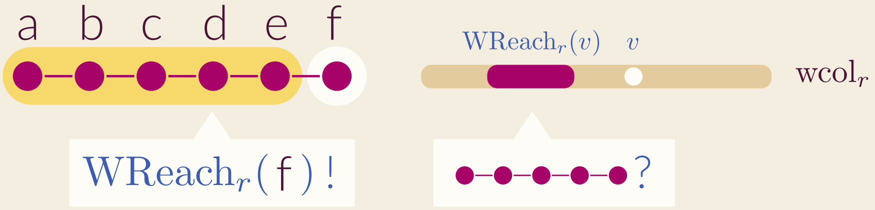
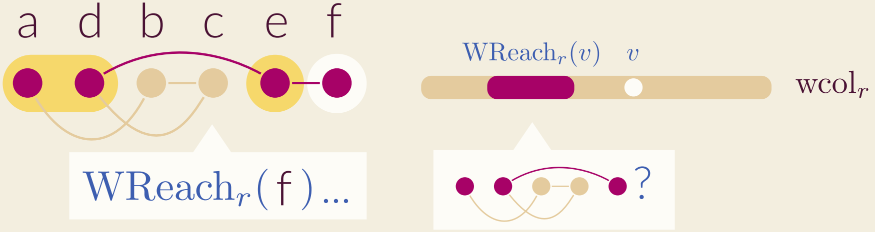




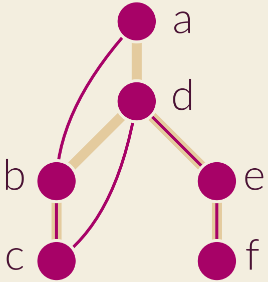
Is there a nice formalization of this property?

Treedepth: elimination orderings

Given an ordering $\prec$ of $V(G)$, we compute a treedepth decomposition as follows:

For every connected component of G, remove the minimum vertex and add it as the current root, then recurse on the resulting components.

adbcef adefbc
adbecf adebfc
adbefc adebcf



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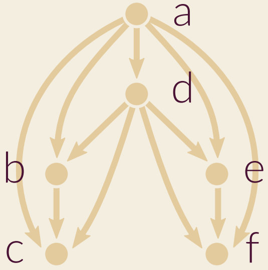
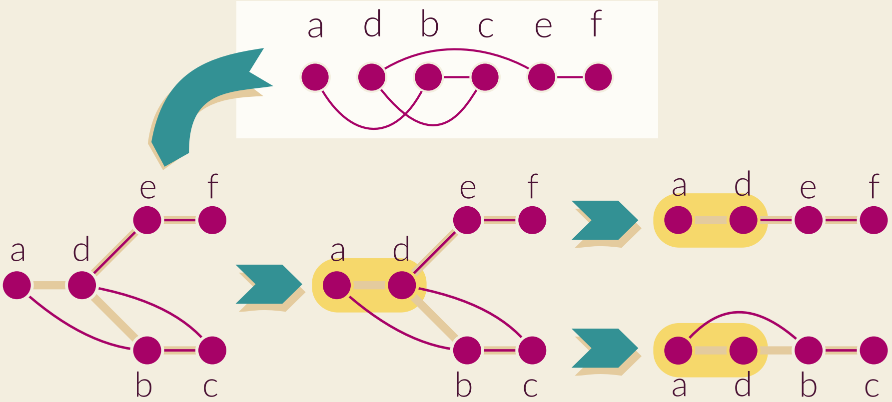


Decomposition!



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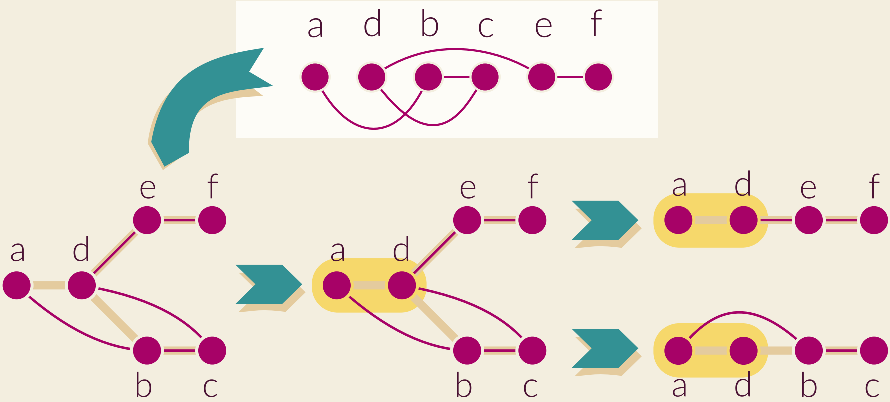
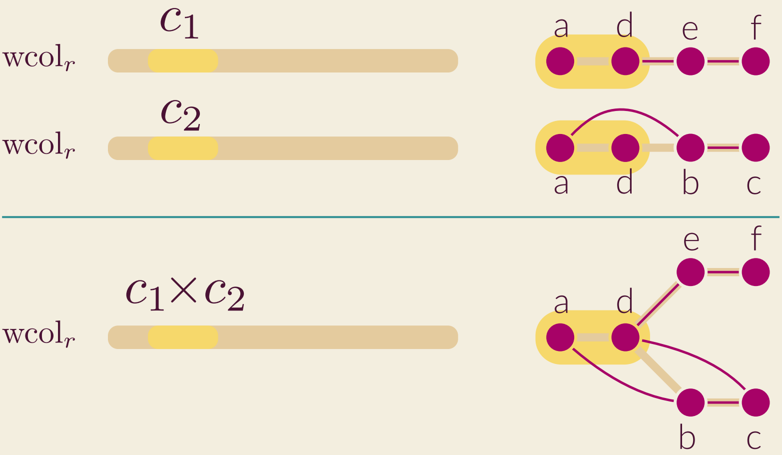


We can count linear pieces!

Progress! These pieces are linear!

Count & combine!



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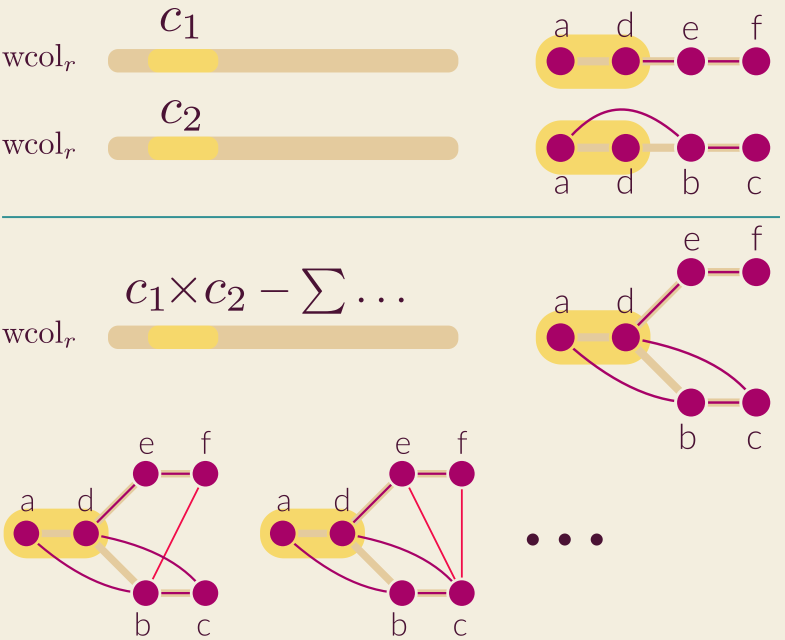

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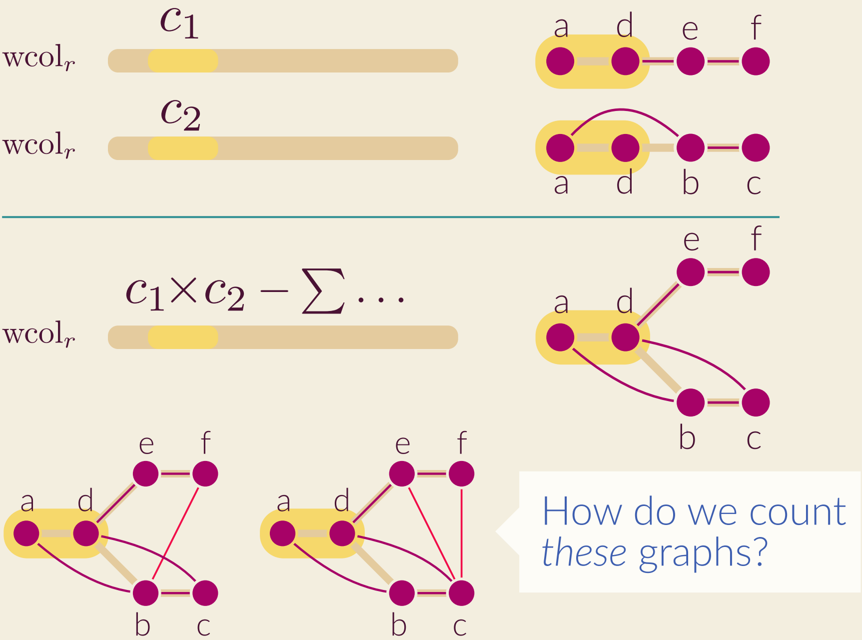

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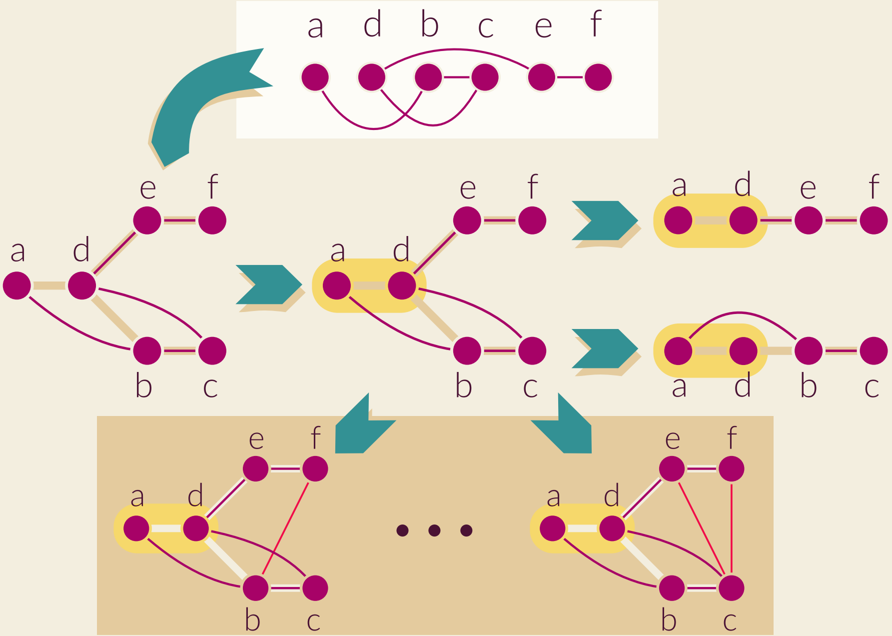

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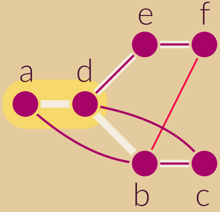

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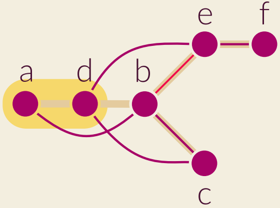
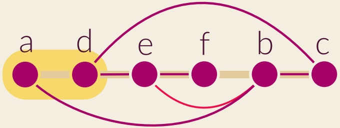


adefbc



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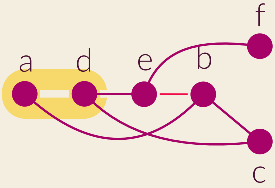

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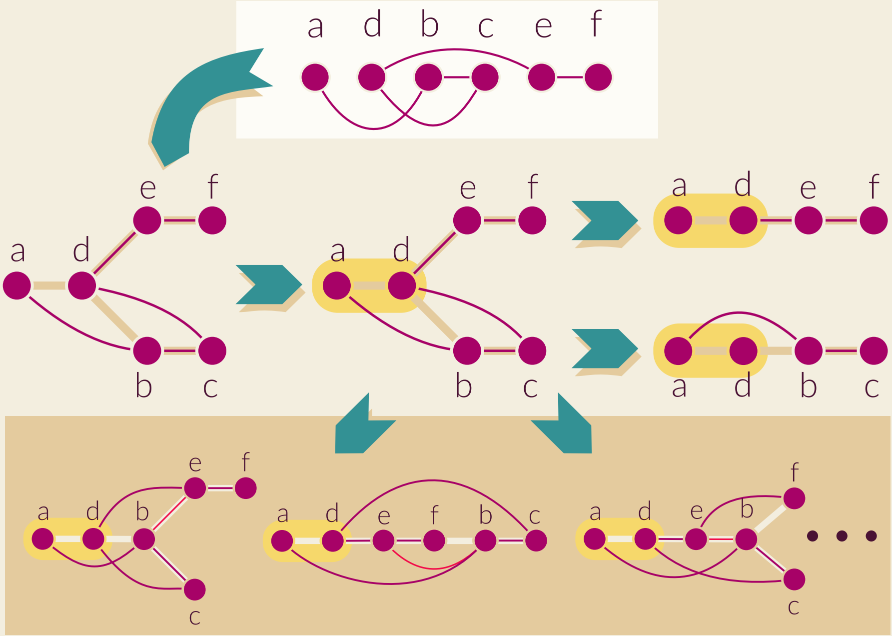

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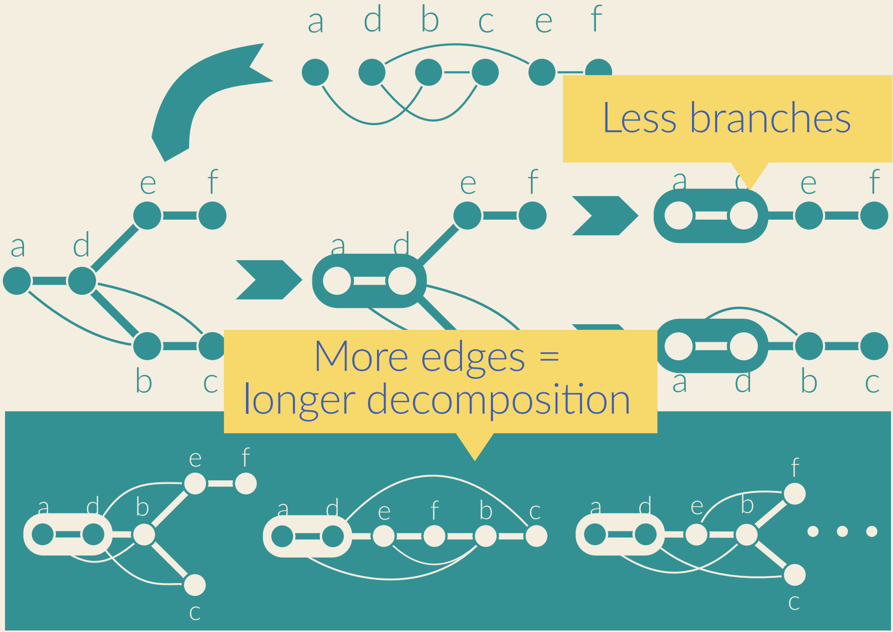
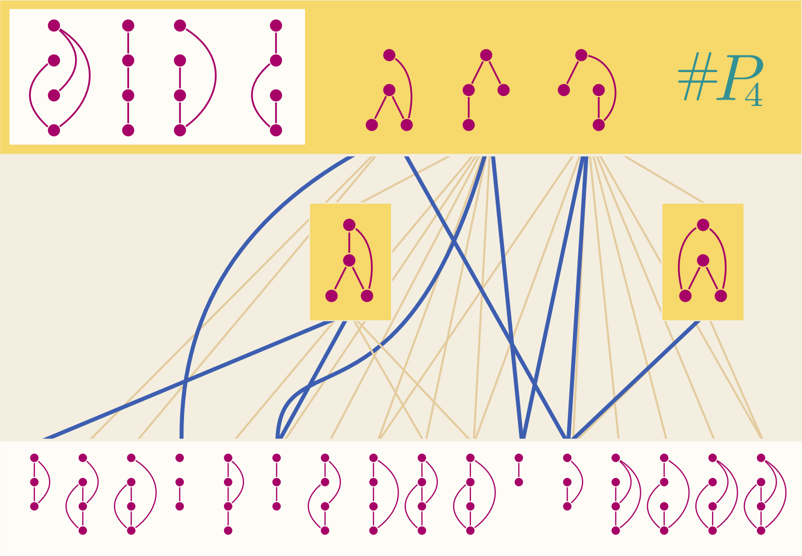
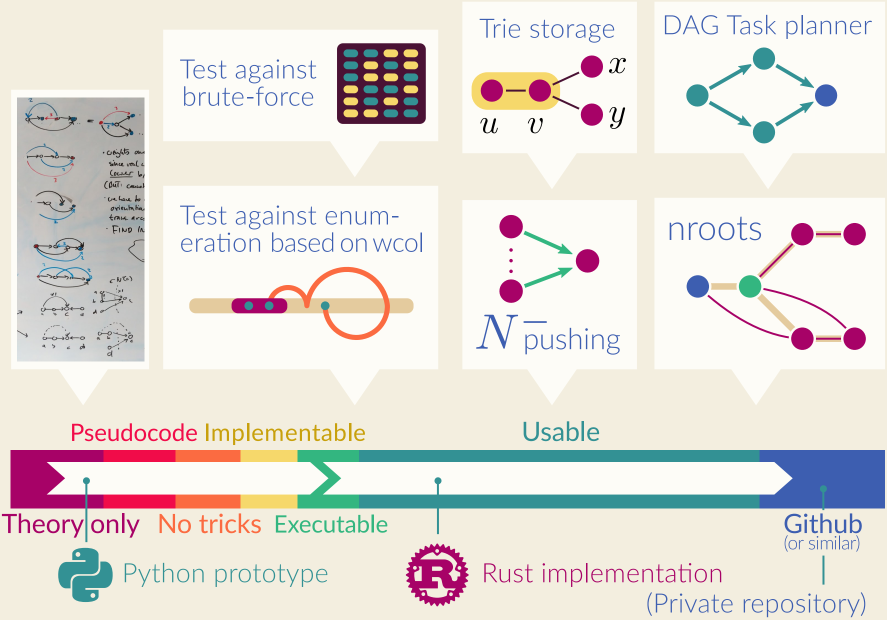


Motif counting using wcol_r

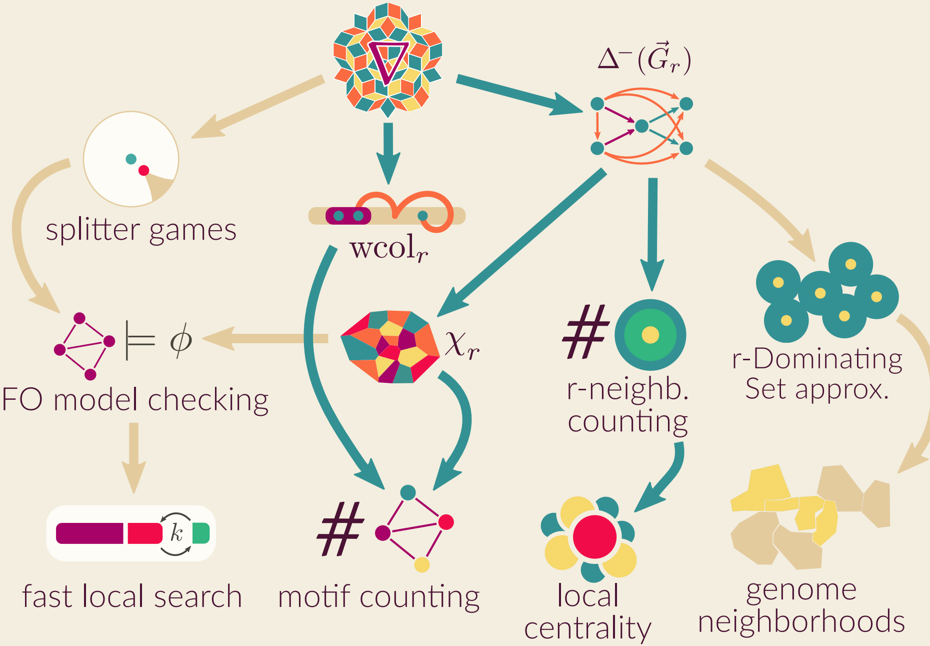


Composition

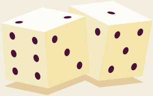
Mandoline progress



Applications & Algorithms



Open questions & future work



Classification of more models!

How do we classify models like preferential attachment?

Can we 'measure' bnd. exp. in practice?

Approximate r-nbhd counting without exponential dependence?

r-nbhd counting in time $O(2^{\omega_r(G)} n)$?

Make existing algorithms *practical*!

wcol/dtf specifically for networks!

Long term: 'sparsity' programming library

Find (interdisciplinary) collaborations

THANKS!

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