Kernels in sparse graph classes

Felix Reidl, Somnath Sikdar

Lehr- und Forschungsgebiet Theoretische Informatik

RWTH AACHEN

Working Seminar on Formal Models, Discrete Structures, and Algorithms 2012

Contents

Fixed parameter tractability and kernels

Sparse graph classes

A useful lemma

Conclusion

Fixed parameter tractability and kernels

Parameterized complexity . . .

. . . deals with decision problems with two components (x, k), where

- x is the input;
- k is the parameter.

Examples:

- VERTEX COVER: given (G, k), does G have a vertex cover of size at most k?
- SUBGRAPH ISOMORPHISM: given (G, H), is $H \subseteq G$?
- LONGEST CYCLE: given (G, l), does G contain a cycle of length at least l?

Fixed-parameter tractability

Running times are measured wrt both x and k.

- $2^k \cdot |x|^{O(1)}$ vs. $|x|^{O(k)}$.
- Only polynomial dependency on $|x|$, but arbitrary for k.

Definition

A parameterized problem is fixed-parameter tractable (fpt) if there is an algorithm with running time $O(f(k) \cdot |x|^c)$, where f is a function of k alone and c is a constant.

A closely related concept: problem kernels.

- in polynomial time strip away easy parts of the input to expose the hard part—the kernel.
- More precise: let $L \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem

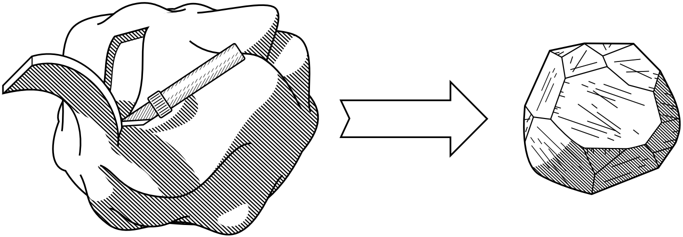
$$(x, k) \xrightarrow{\text{poly time}} (x', k')$$

such that $|x'|, k' \leqslant f(k)$.

and $(x, k) \in L \Leftrightarrow (x', k') \in L$

- f is the kernel size, a kernel is polynomial if $f \in O(n^c)$

Kernelization



- problem is fixed-parameter tractable iff it has a kernelization algorithm
- kernel size usually exponential or worse.
- Goal: to obtain polynomial or even linear kernels.

Basic technique of kernelization:
Devise reduction rules that preserve equivalence of instances; apply exhaustively, prove kernel size.

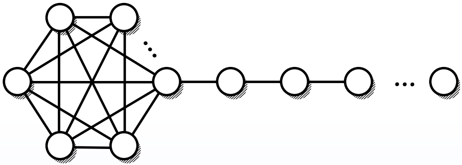
Sparse graph classes

Why sparse classes?

- Many hard problems become fpt on sparse classes of graphs
  - Dominating Set on bounded-genus graphs
  - Independent Set on planar graphs
  - MSO-definable problems on bounded-treewidth graphs
- Meta-results showed that a large class of problems admit linear kernels on certain sparse classes
- No polynomially sized kernels on general graphs for many problems (under certain complexity-theoretic assumptions)
- In particular: “connectivity”-problems (Longest Path, Disjoint Paths, Connected Vertex Cover, Steiner Tree, ...)

What kind of sparseness?

Only requesting a “linear number of edges” not particularly useful.



We need graph classes that are uniformely* sparse.

Definition (d-degenerate)

A graph class $\mathcal{C}$ is d-degenerate if for every $G \in \mathcal{C}$, every subgraph of G contains a vertex of degree $\leqslant d$.

Definition (d-degenerate)

A graph class $\mathcal{C}$ is d-degenerate if for every $G \in \mathcal{C}$, every subgraph of G contains a vertex of degree $\leqslant d$.

Equivalent characterizations:

- G can be erased by succesive deletion of vertices of degree $\leqslant d$
- There exists an ordering of the vertices of G such that every vertex has at most d neighbours to its right
- The edges of G can be oriented such that every vertex has out-degree at most d

Useful properties:

- $|E(G)| \leqslant d|V(G)|$, therefore average degree $\leqslant 2d$
- $\chi(G) \leqslant d + 1$ and $\omega(G) \leqslant d + 1$
- At most $2^d|V(G)|$ cliques
- Hereditary

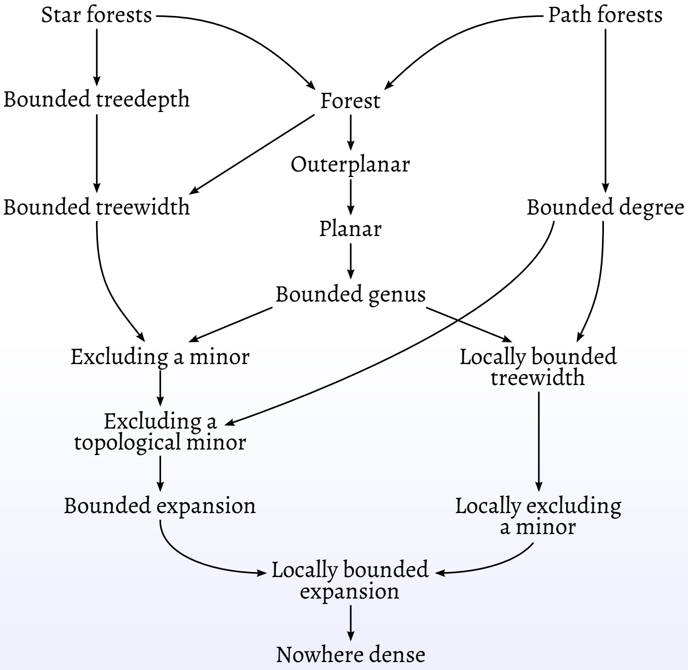
Degeneracy is a good start, but is not strong enough for general results: we can make any graph degenerate by subdividing its edges a lot.

A lot of important problems are invariant under this operation.

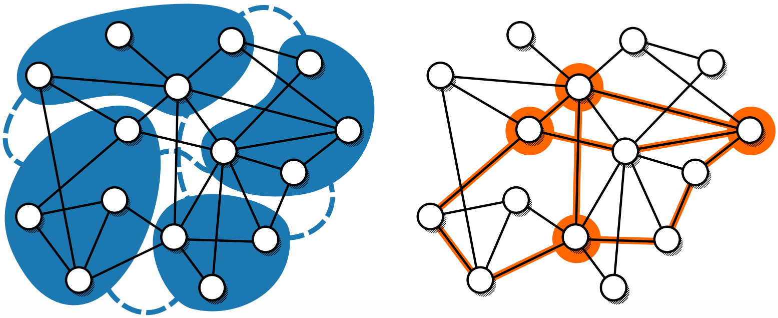
FEEDBACK VERTEX SET, HAMILTONIAN PATH, TREEWIDTH, MINIMUM DEGREE SPANNING TREE, MAXIMUM CUT
(under various parameterizations)

Additionally: DOMINATING SET has no polynomial kernel on d-degenerate graphs

We need structurally* sparse classes.


Star forests
Path forests
Bounded treedepth
Forest
Outerplanar
Bounded treewidth
Bounded degree
Planar
Bounded genus
Excluding a minor
Locally bounded treewidth
Excluding a topological minor
Bounded expansion
Locally excluding a minor
Locally bounded expansion
Nowhere dense

Minors



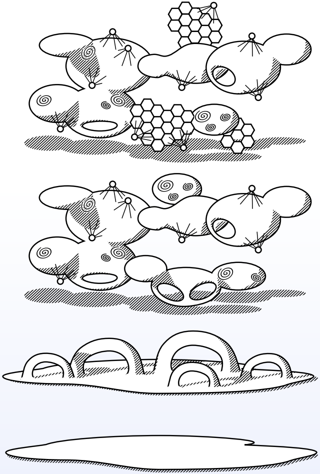
- Minor: take subgraph, contract vertex sets inducing connected subgraphs (branch sets)
- Topological minor: take subgraph, contract vertex-disjoint two-paths between nail vertices
- Characterize graph class by excluding a fixed graph as a (top.) minor

Overview of meta-results

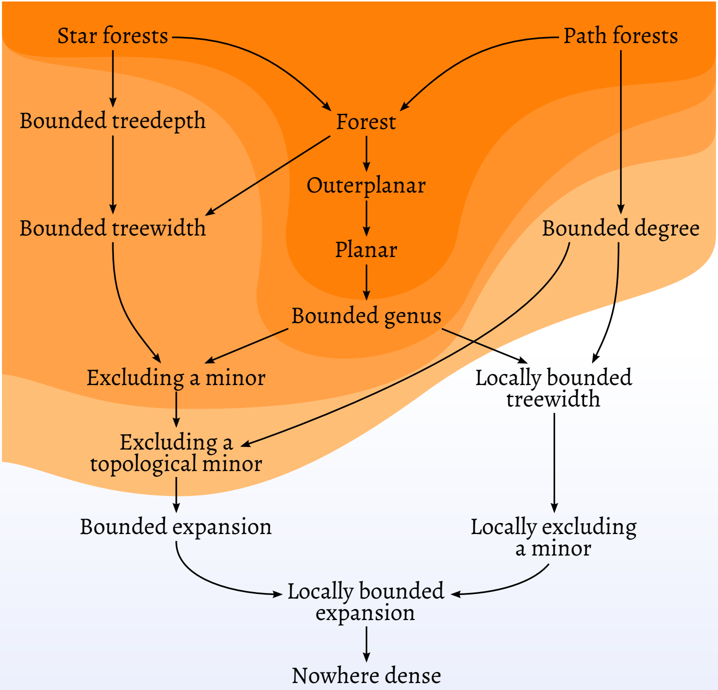
Linear kernels in structurally* sparse classes

- Framework for planar graphs
  Guo and Niedermeier: Linear problem kernels for NP-hard problems on planar graphs
- Meta-result for graphs of bounded genus
  Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh and Thilikos: (Meta) Kernelization
- Meta-result for graphs excluding a fixed graph as a minor
  Fomin, Lokshtanov, Saurabh and Thilikos: Bidimensionality and kernels
- Meta-results for graphs excluding a fixed graph as a topological minor
  Kim, Langer, Paul, R., Rossmanith, Sau, and Sikdar: Linear kernels and single-exponential algorithms via protrusion decompositions

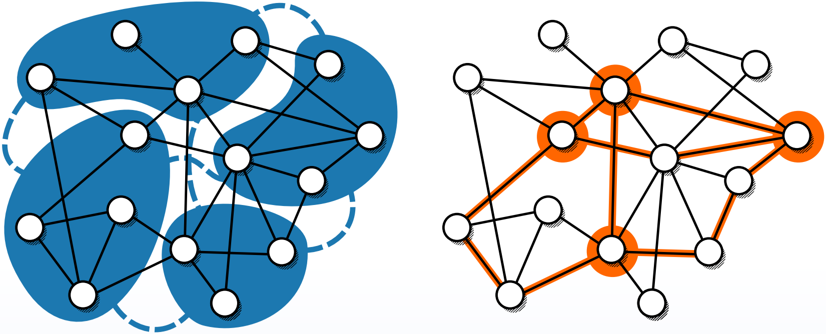
Trade-off: sparseness vs. problem requirements



H-Topological-Minor-Free	*Treewidth-bounding*
∪	
H-Minor-Free	*Bidimensional* *+ separation property*
∪	
Bounded Genus	*Quasi-compact*
∪	
Planar	*“Distance-property”*

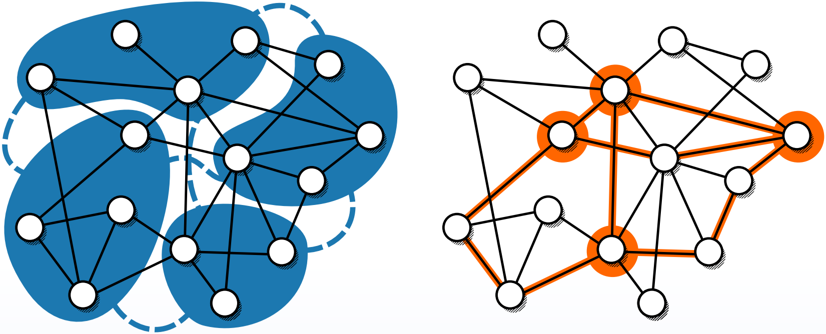

Star forests
Path forests
Bounded treedepth
Forest
Outerplanar
Bounded treewidth
Planar
Bounded degree
Bounded genus
Excluding a minor
Locally bounded treewidth
Excluding a topological minor
Bounded expansion
Locally excluding a minor
Locally bounded expansion
Nowhere dense

More minors



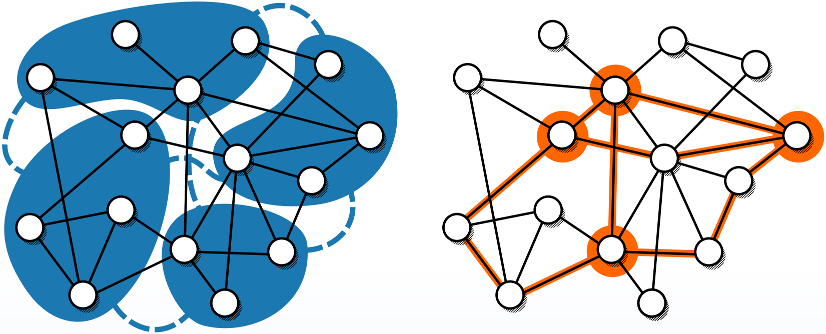
- Shallow minor at depth r: branch-sets have diameter $\leqslant r$
- Shallow top. minor at depth r: paths have length $\leqslant 2r + 1$
- Class of all shallow (top.) minors at depth r of a graph G denoted by $G \triangledown r$ ($G \tilde{\triangledown} r$)

More minors



- Class of all r-depth (top.) minors $G \triangledown r$ ($G \mathbin{\tilde{\triangledown}} r$)
- $G \triangledown 0 = G \mathbin{\tilde{\triangledown}} 0$ contains exactly the subgraphs of G
- $\{G\} \subseteq G \triangledown 0 \subseteq G \triangledown 1 \subseteq \ldots \subseteq G \triangledown \infty$
- $\{G\} \subseteq G \mathbin{\tilde{\triangledown}} 0 \subseteq G \mathbin{\tilde{\triangledown}} 1 \subseteq \ldots \subseteq G \mathbin{\tilde{\triangledown}} \infty$
- $G \mathbin{\tilde{\triangledown}} i \subseteq G \triangledown i$

More minors



- Class of all r-depth (top.) minors $G \triangledown r$ ($G \tilde{\triangledown} r$)
- Natural extension to classes of graphs:

$$\mathcal{C} \triangledown r = \bigcup_{G \in \mathcal{C}} G \triangledown r$$

- $\mathcal{C} \tilde{\triangledown} r$ analogous

Grad, bounded expansion

Introduced by Ossana de Mendez and Nešetřil, encompasses many sparse graph classes. (Most facts and notations taken from Nešetřil, Ossana de Mendez, Wood: Characterisations and examples of graph classes with bounded expansion)

Definition (Greatest reduced average density at depth r)

$$\nabla_r(\mathcal{C}) = \sup_{G \in \mathcal{C} \nabla r} \frac{|E(G)|}{|V(G)|}$$

- Define top-grad $\tilde{\nabla}_r(\mathcal{C})$ analogously via $\tilde{\nabla}$
- Set $\nabla_r(G) := \nabla_r(\{G\})$ and $\tilde{\nabla}_r(G) := \tilde{\nabla}_r(\{G\})$
- $\nabla_0(\mathcal{C}) \leqslant \nabla_1(\mathcal{C}) \leqslant \ldots \leqslant \nabla_\infty(\mathcal{C})$ (same for $\tilde{\nabla}$)

- $\mathcal{C}$ has bounded expansion iff $\nabla_r(\mathcal{C}) < f(r)$ for some function f
- $\mathcal{C}$ excludes a fixed minor iff f is bounded by constant
- $\nabla_i(\mathcal{C}) = \nabla_0(\mathcal{C} \triangledown i)$ (same for $\tilde{\nabla}$)
- $2\nabla_0(G)$ is precisely the degeneracy of G:

$$2\nabla_0(G) = 2 \sup_{H \in G \triangledown 0} \frac{|E(G)|}{|V(G)|} = \max_{H \subseteq G} \frac{2|E(G)|}{|V(G)|}$$

In the following we will look at graph classes $\mathcal{C}$ for which $\nabla_1(\mathcal{C}) < c$ for some constant c.

A useful lemma for graphs of bounded ∇_1

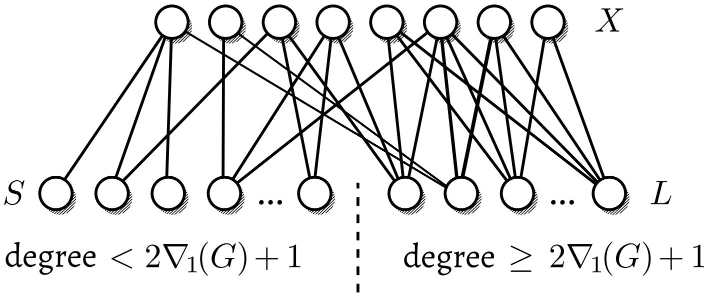
Lemma

Let $G = (X, Y, E)$ *be a bipartite graph. Let* $S = \{v \in Y \mid d(v) < 2\nabla_1(G) + 1\}$ *be the small-degree vertices in* Y *and* $L = Y \setminus S$ *the large-degree vertices in* Y*. Then the following bounds hold:*

- $|L| \leqslant 2\nabla_1(G) \cdot |X|$
- $|\{N(v) \mid v \in S\}| \leqslant (2^{2\nabla_1(G)} + 1)|X|$

Important ingredients for proof:

- A d-degenerate graph has at most $d|V|$ edges and at most $2^d|V|$ cliques
- $2\nabla_0(G)$ is exactly the degeneracy of a graph G
- $\nabla_1(G) = \nabla_0(G \triangledown 1) < c$ (by assumption)
  i.e. the shallow minors at depth 1 are degenerate

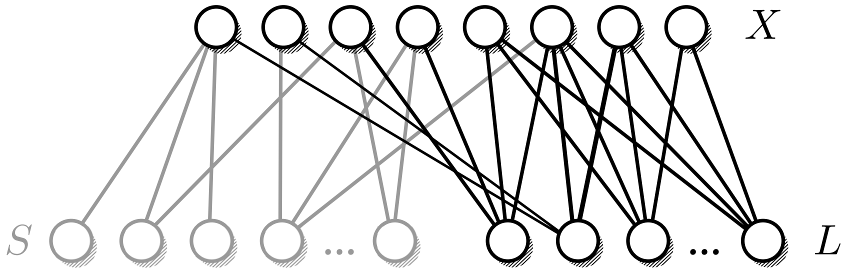


Lemma

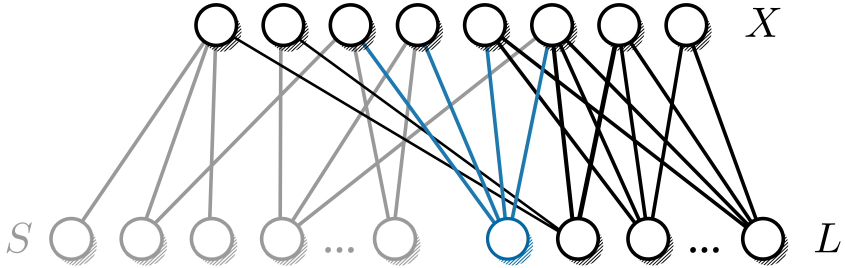
Let $G = (X, Y, E)$ *be a bipartite graph. Let* $S = \{v \in Y \mid d(v) < 2\nabla_1(G) + 1\}$ *be the small-degree vertices in* Y *and* $L = Y \setminus S$ *the large-degree vertices in* Y*. Then the following bounds hold:*

- $|L| \leqslant 2\nabla_1(G) \cdot |X|$
- $|\{N(v) \mid v \in S\}| \leqslant (2^{2\nabla_1(G)} + 1)|X|$

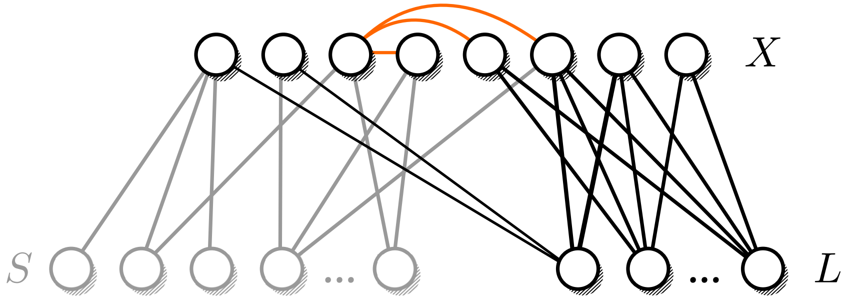
Proof by animation (1)



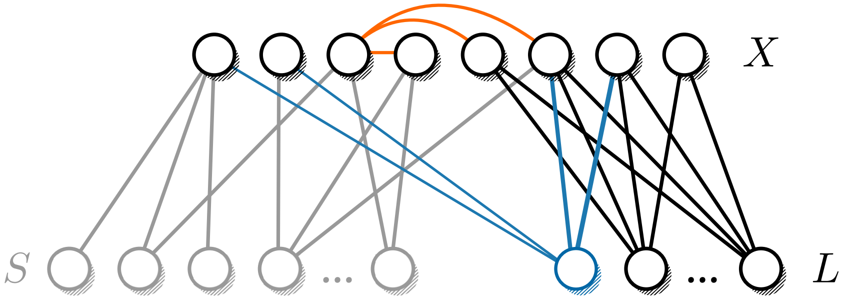
Proof by animation (1)



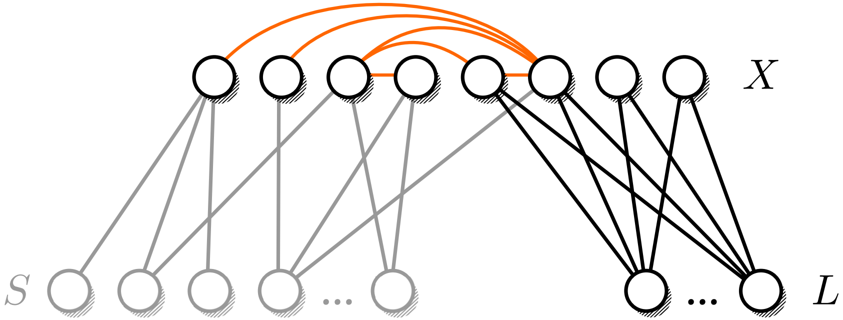
Proof by animation (1)



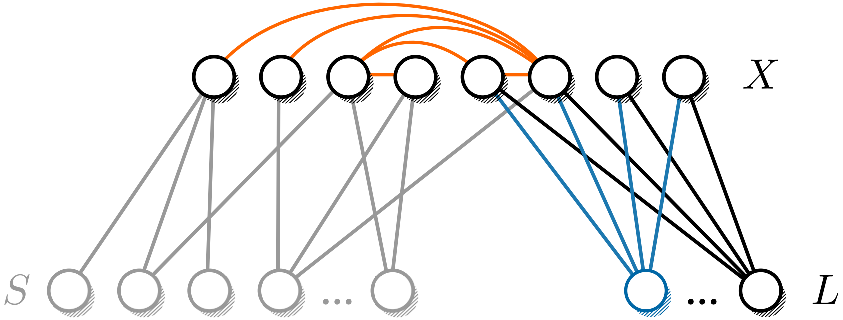
Proof by animation (1)



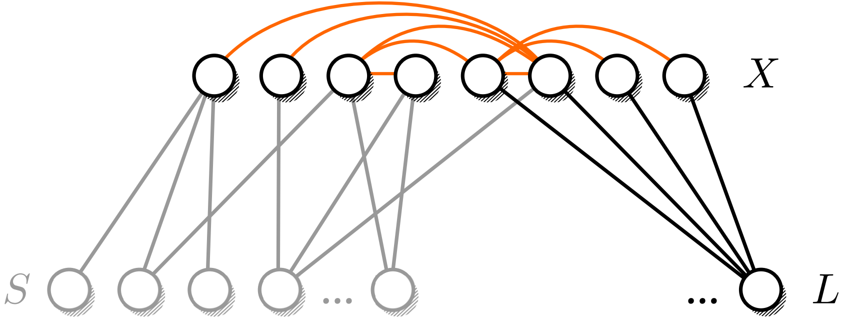
Proof by animation (1)



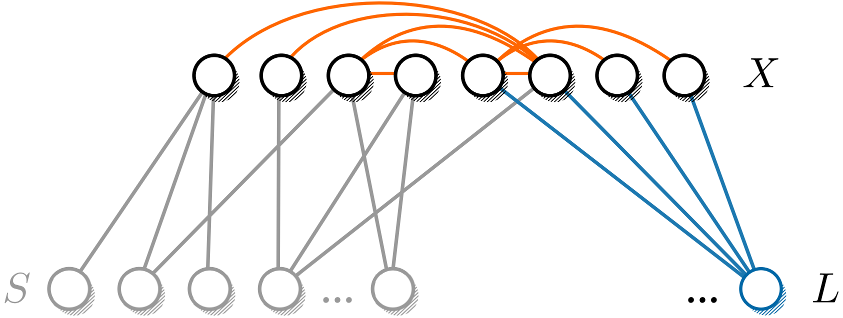
Proof by animation (1)



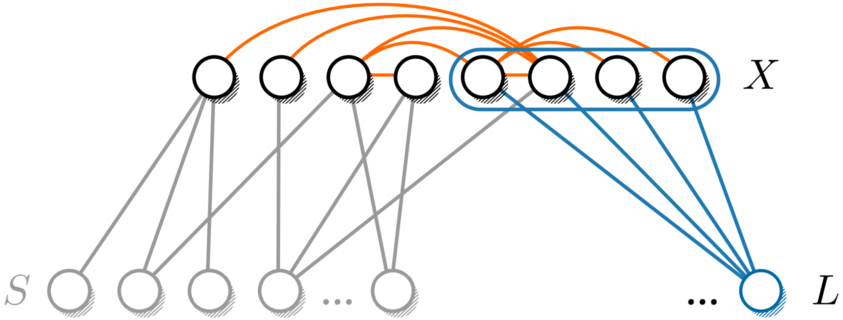
Proof by animation (1)



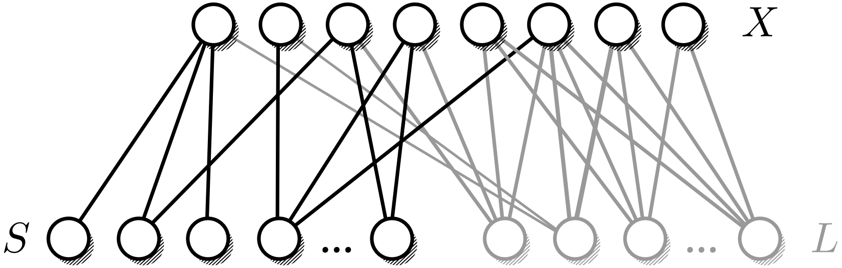
Proof by animation (1)



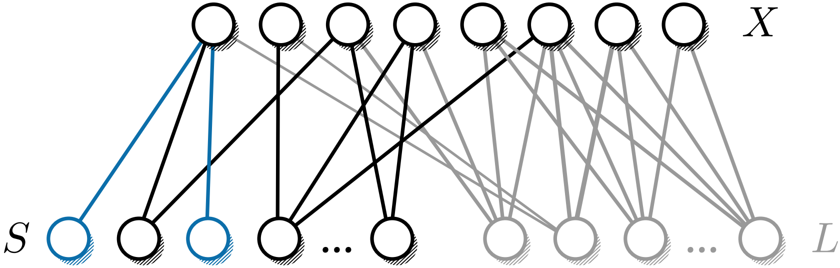
Proof by animation (1)



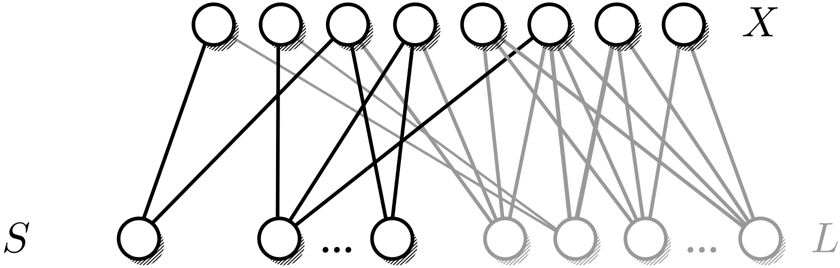
Proof by animation (2)



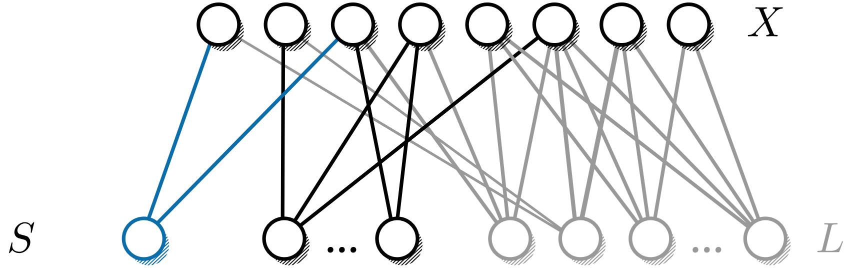
Proof by animation (2)



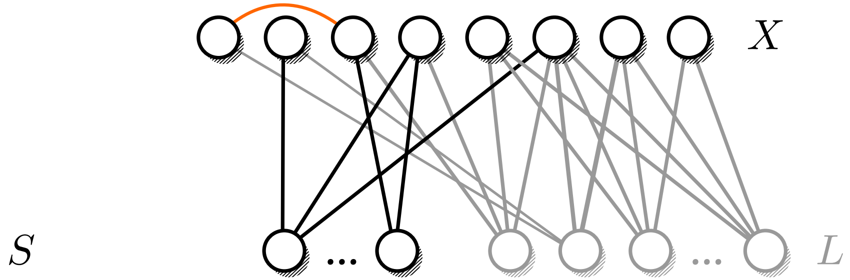
Proof by animation (2)



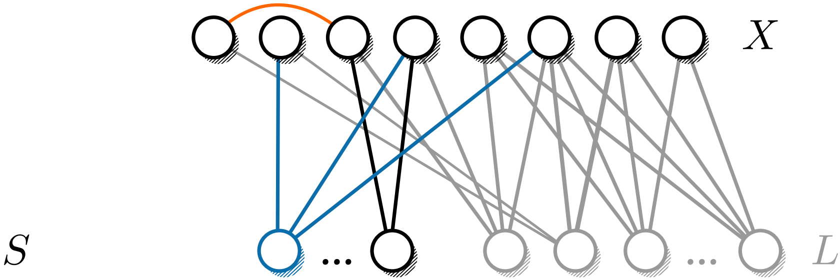
Proof by animation (2)



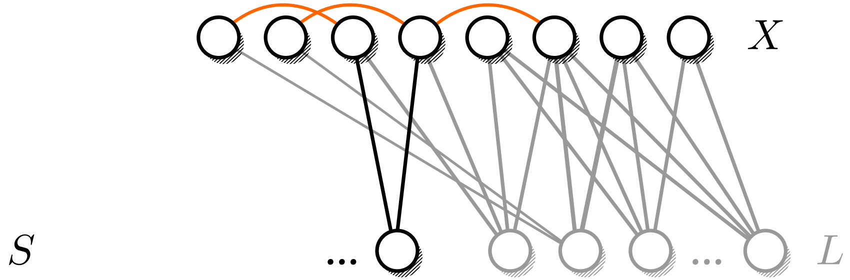
Proof by animation (2)



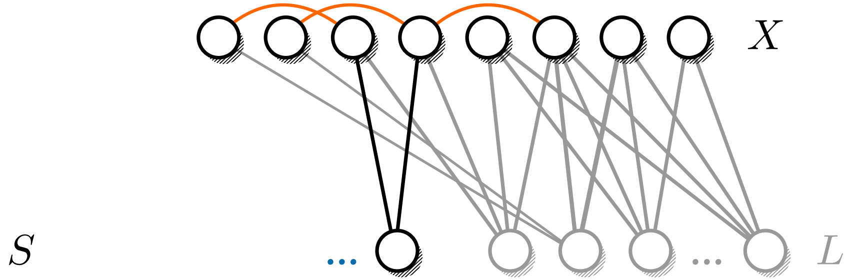
Proof by animation (2)



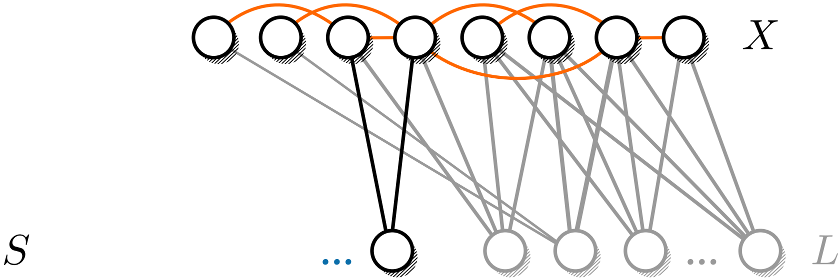
Proof by animation (2)



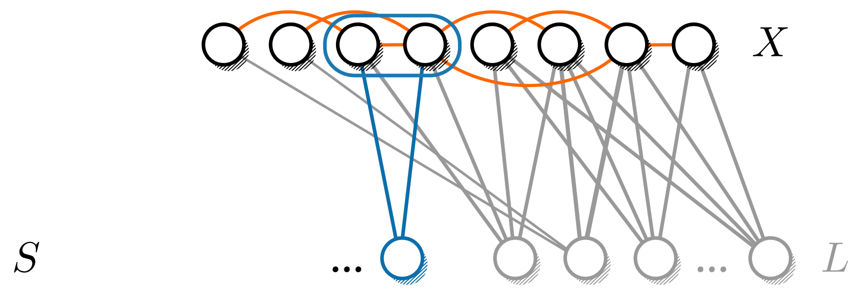
Proof by animation (2)



Proof by animation (2)



Proof by animation (2)



How to apply

Definition (Twin vertices)

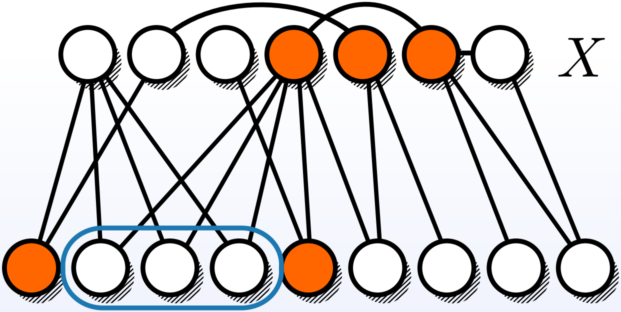
Vertices u, v in a bipartite graph are twins if $N(u) = N(v)$. The equivalence classes under the twin relation are called twin classes.

1. Find bipartition that represents the size of the instance and has bounded ∇_1
2. Make sure one side is small (bounded by parameter)
   - $\Rightarrow$ By lemma: number of large-degree vertices L is small
   - $\Rightarrow$ By lemma: number of twin classes in S is small
3. Find reduction rule to bound the size of twin classes in S

(Toy) Examples

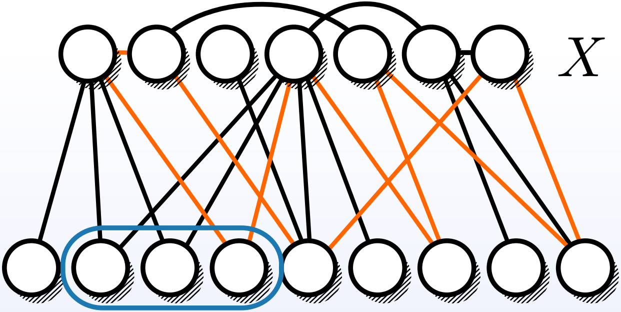
Dominating Set
Param. by Vertex Cover

(works also for connected variant)



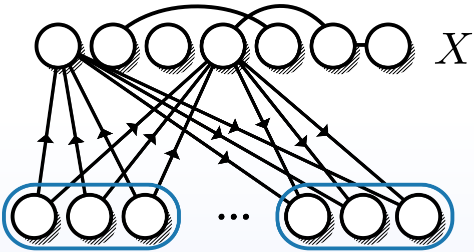
(Toy) Examples

Longest Cycle
Param. by Vertex Cover



(Toy) Examples

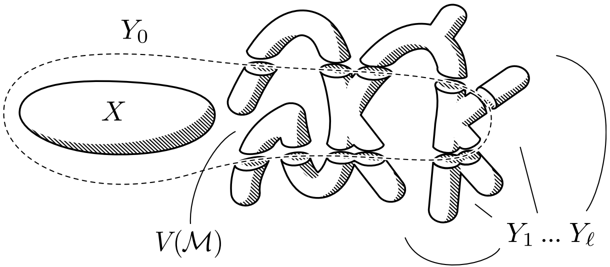
Directed Feedback Vertex Set
Param. by Vertex Cover



$2^{2\nabla_1(G)}$ possible orientation-classes inside each twin class.

Preserve $2\binom{2\nabla_1(G)}{2}$ vertices per orientation-class, remove the rest.

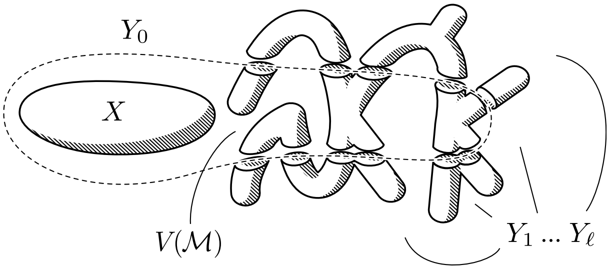
A real example



Protrusion-decomposition of a graph G excluding some fixed graph H as a topological minor.

- X is a treewidth-modulator
- Each bag in $\mathcal{M}$ witnesses a connected subgraph with many neighbours in X
- Each Y_i, $1 \leqslant i \leqslant \ell$ has only constantly many neighbours in Y_0 and has constant size

A real example

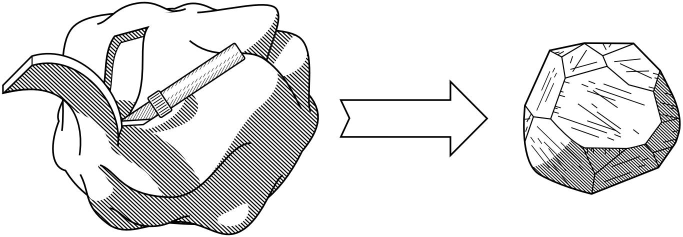


Lemma applied two times:

1. Bipartition X, W where each vertex in W represents a small "witness subgraph"
   - $\Rightarrow$ Bounds size of Y_0 in $O(|X|)$
2. Bipartition Y_0, W where each vertex in W represents a connected component of $G - Y_0$
   - $\Rightarrow$ Bounds size of ℓ in $O(|X|)$

$\Rightarrow$ linear kernel for many problems on H-topological-minor-free graphs

A quick critical reflection



Kernelization algorithm should be feasible in practice

- Linear time algorithm (sparsity should help)
- Ideally, algorithm is agnostic towards graph class
  - Bound dependend on kernel size
  - Running time dependend on kernel size
  - Probabilistic kernel: success probability
- Care for constants: replace heavy weaponry of big results by hand-crafted reduction rules

Conclusion

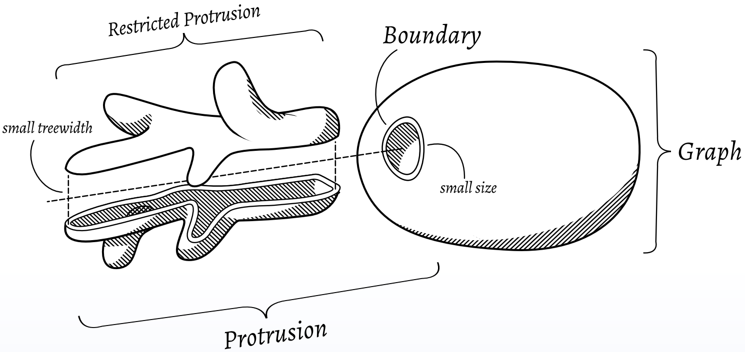
- Important frontier for kernels in sparse graphs are graphs of bounded expansion
- Many tools already available:
    - Low tree-depth coloring
    - Weak k-colorings & co. (Dvořák)
    - p-centered coloring
    - quasi-wideness
- But: must be made applicable for kernelization
- Previous results not generalizable: subdivision-invariant problems as hard as in general graphs

- Structurally* sparse graph classes enable linear kernels even for otherwise hard problems using treewidth-t-modulators
- Bounded ∇_1 yields (somewhat trivial) kernels using vertex covers $=$ treewidth-zero modulator
- Tree-depth a better candidate?

Is there an interesting combination of some notion of sparseness coupled with a parameter weaker than vertex cover that still yields polynomial/linear kernels for a large class of problems?

Thanks!

Appendix: Protrusion anatomy

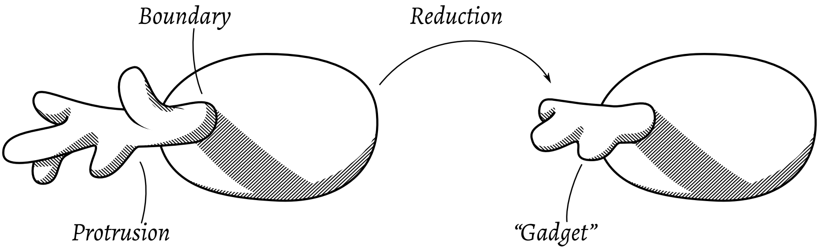


Definition

$X \subseteq V(G)$ is a t-protrusion if

1. $|\partial(X)| = |N(X) \setminus X| \leqslant t$ (small boundary)
2. $\mathbf{tw}(G[X]) \leqslant t$ (small treewidth)

Appendix: Protrusion reduction



We want to replace a large protrusion by a smaller gadget.

1. Requires that the problem has finite integer index
2. The gadgets can always be chosen such that the parameter does not increase
3. This is the only reduction

Caveat: only constantly-sized protrusions can be replaced (if no further restrictions are made), but in a large protrusion such a structure is always present.