Solving a Family of Multivariate Optimization and Decision Problems on Classes of Bounded Expansion

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Joint work with Peter Rossmanith

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Theoretical Computer Science | RWTH AACHEN UNIVERSITY

Algorithms For...

Max Partial Vertex Cover

Problem: How many edges can be covered by set of k vertices?

Red Blue Partial Dominating Set

Problem: Are there k vertices dominating $\geq t_{\text{red}}$ red and $\geq t_{\text{blue}}$ blue vertices?

Fair Dominating Matching

Problem: Is there a matching of size k that dominates twice as many red as blue vertices?

Half Triangle Deletion

Problem: Can we destroy half of the triangles and squares by deleting k vertices?

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On graph classes

- Of bounded treewidth
- Of bounded degree
- Planar Graphs

We give one algorithm for that!

We give one meta-algorithm for that!

Algorithmic Meta-Theorems

"Every problem expressible in logic L
can be solved efficiently on graph class $\mathcal{C}$."

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MSO on treewidth

FO on nowhere dense

Model-Checking

$MC(L, \mathcal{C})$

Input:	A graph $G \in \mathcal{C}$ and a sentence $\varphi \in L$
Problem:	Is φ true in G? ($G \models \varphi$?)
Parameter:	$\lvert\varphi\rvert$

$MC(FO, \mathcal{G})$ is hard (PSPACE-hard and AW[*]-hard)

Goal: Find classes $\mathcal{C}$ and logic L where $MC(L, \mathcal{C})$ is FPT (time complexity of $f_{\mathcal{C}}(\lvert\varphi\rvert)n^d$)

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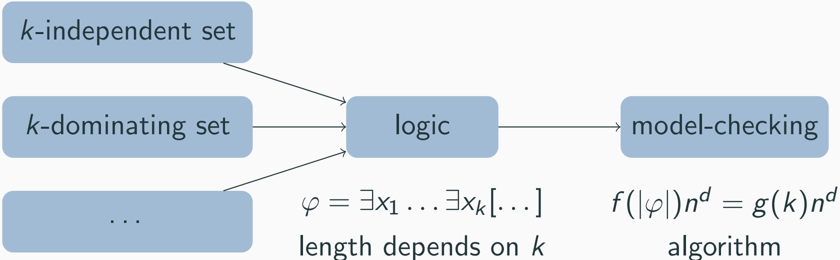
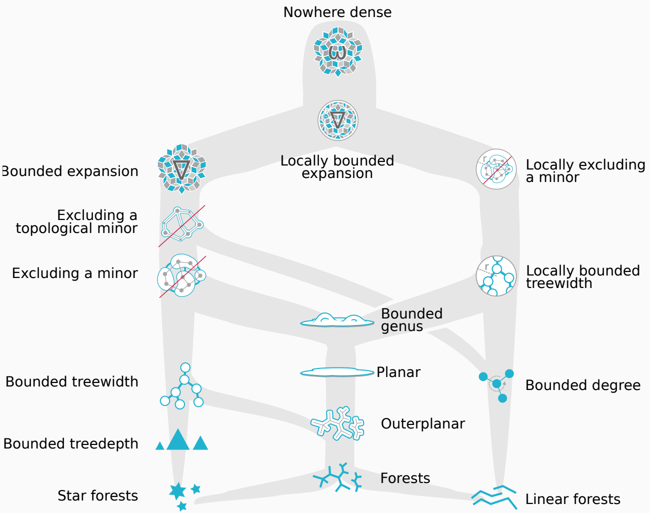
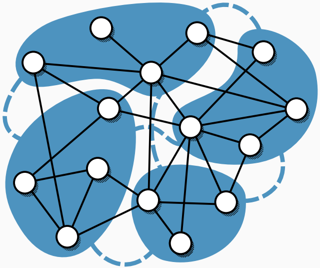


Sparse Graph Classes



By Felix Reidl

Bounded Expansion



Class has bounded expansion if:

there is a function f s.t.

$$\frac{|E|}{|V|} \leq f(r)$$

for every r-shallow minor

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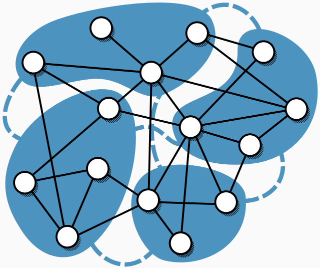

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Many other characterizations:

copwidth game, weak coloring numbers,

neighborhood complexity, treedepth

colorings, neighborhood covers, ...

First-Order with Some Counting: FO($\{>0\}$)

Definition of FO($\{>0\}$) (Dreier, Rossmanith, '21)

Built recursively using

- the rules of FO
- $\#y\, \varphi(y, x_1, \ldots, x_k) \geq m$ for every $m \in \mathbb{N}$ and FO($\{>0\}$) formula φ

Fragment of FO($\mathbb{P}$) and FOC($\mathbb{P}$) (Kuske, Schweikardt '17)

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PartDomSet:

$$\exists x_1 \ldots \exists x_k \#y \left(\bigvee_{1 \leq i \leq k} E(y, x_i) \vee y = x_i \right) \geq t$$

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h-index:

$$\#\text{mypaper}\left(\#\text{otherpaper}\,\text{cite}(\text{mypaper}, \text{otherpaper}) \geq h\right) \geq h$$

Known Results for $\mathrm{FO}(\{>0\})$

Model-checking of $\mathrm{FO}(\{>0\})$ is *hard* on forests of depth 4

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Theorem (Dreier, Rossmanith '21)

On classes of bounded expansion, in linear FPT time

1. $(1+\varepsilon)$-*approximation of* FO($\{>0\}$),
2. *Exact evaluation of formulas* $\exists x_1 \dots \exists x_k \# y \underbrace{\varphi(y, x_1, \dots, x_k)}_{\text{FO w/o \#}} \geq m$

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Theorem (Dreier, M., Rossmanith '23)

On nowhere dense classes, in almost linear FPT time:

Exact evaluation of formulas $\exists x_1 \ldots \exists x_k \# y \underbrace{\varphi(y, x_1, \ldots, x_k)}_{\text{quantifier-free}} \geq m$

$\Rightarrow$ PartDomSet in (almost) linear FPT time on bounded expansion & nowhere dense

Other variants

Red Blue Partial Dominating Set

Input: A graph G and $k, t_{\text{red}}, t_{\text{blue}} \in \mathbb{N}$

Problem: Are there k vertices dominating $\geq t_{\text{red}}$ red and $\geq t_{\text{blue}}$ blue vertices?

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$$G \models \exists x_1 \ldots \exists x_k \# y\, Red(y) \wedge dom(y, \bar{x}) \geq t_{\text{red}} \wedge \# y\, Blue(y) \wedge dom(y, \bar{x}) \geq t_{\text{blue}}$$

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$$G \models \exists x_1 \ldots \exists x_k \# y\, dom(y, x_1, \ldots, x_k) = t$$

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$$G \models \exists x_1 \ldots \exists x_k \#y\, dom(y, x_1, \ldots, x_k) = t$$

Cannot be expressed as $\exists x_1 \ldots \exists x_k \#y\, \underbrace{\varphi(y, x_1, \ldots, x_k)}_{\text{FO w/o }\#} \geq m$

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Our goal: Lift result to $\exists x_1 \dots \exists x_k \underbrace{\bigvee \bigwedge (\# y\, \varphi_i(y\bar{x}) \geq m_i)}_{\text{boolean combination of } \ell \text{ counting terms}}$

Our Results

Algorithmic Result

Theorem (Our Positive Result)

On classes of bounded expansion, we can decide in time $f(k,\ell)n^{\ell+1}\operatorname{polylog} n$ *whether*

$$G \models \exists x_1 \ldots \exists x_k \mathbf{P}\big(\#y\ \underbrace{\varphi_1(y\bar{x})}_{\text{FO w/o }\#}, \ldots, \#y\, \varphi_\ell(y\bar{x})\big)$$

where $\mathbf{P}$ *is some efficiently computable predicate over* $\mathbb{N}^\ell$.

Moreover, we can count the number of such solutions.

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Moreover, we can count the number of such solutions.

$\implies$ Exact Partial Dominating Set in time $f(k)n^2$ on bounded expansion.

$\implies$ Red Blue Partial Dominating Set in time $f(k)n^3$ on bounded expansion (can be improved to $f(k)n^2$).

From k-Sum to Lower Bounds

k-$\textsc{Sum}$ Problem: given m numbers $x_1, \ldots, x_m$; target T

Find k numbers that add up to exactly T

Algorithms known for k-$\textsc{Sum}$: • $\widetilde{O}(Tm)$ • $O(m^{\lceil k/2 \rceil})$

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On star forests, for formulas of the form

$\exists x_1 \ldots \exists x_k (\#y\, \varphi_1(y, x_1 \ldots x_k) = t_1 \wedge \cdots \wedge \#y\, \varphi_\ell(y, x_1 \ldots x_k) = t_\ell)$

there is no model-checking algorithm in time $f(k, \ell) n^{\ell - \varepsilon}$, for any function f or $\varepsilon > 0$.

Reduction

Example: $T = 100; k = 3$

$x_1 = 32,$ $x_2 = 42,$ $x_3 = 53,$ $x_4 = 15.$

Reduction to model-checking of our fragment (on star forests):

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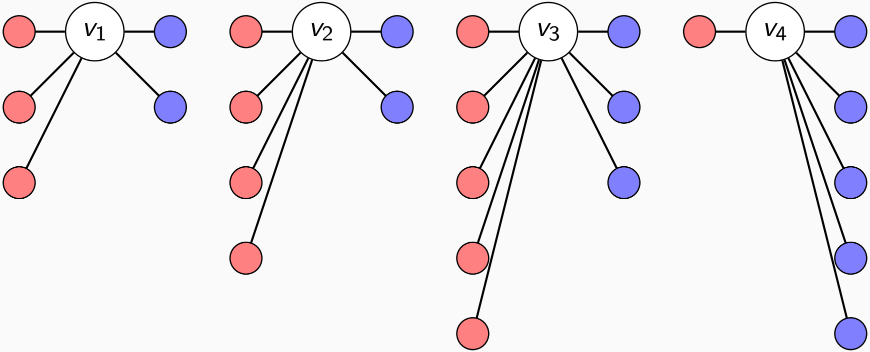

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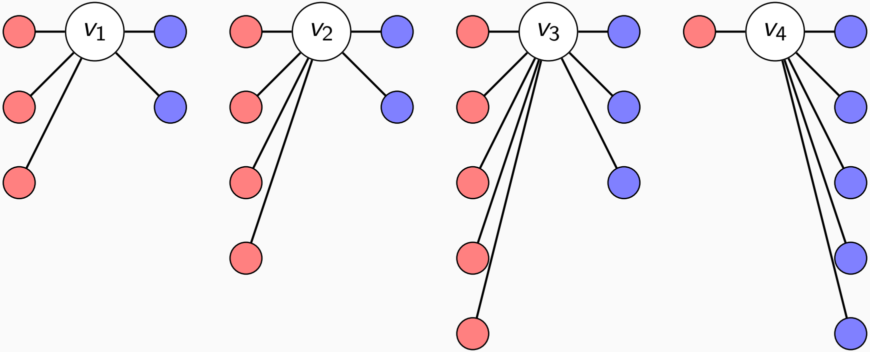

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Conclusion

Our results: Model-checking of FO($\mathbb{P}$) formulas

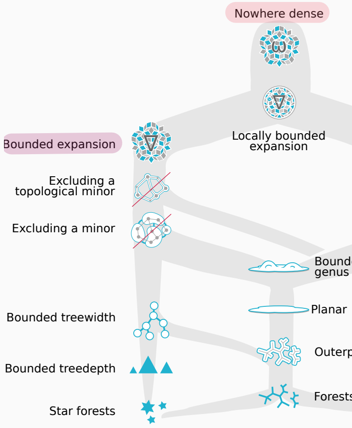
$$G \models \exists x_1 \dots \exists x_k \mathbf{P}(\#y \underbrace{\varphi_1(y\bar{x})}_{\text{first-order}}, \dots, \#y\, \varphi_\ell(y\bar{x}))$$

on classes of bounded expansion

- in time $f(k,\ell)n^{\ell+1}$ polylog n
- not in time $f(k,\ell)n^{\ell-\varepsilon}$ for all $\varepsilon > 0$ under SETH

Outlook:

- Close the gaps
- Lift to (structurally) nowhere dense classes



Thank you!

mock@cs.rwth-aachen.de

Appendix

Partial Dominating Set

PartDomSet

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Parameter:	k

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- PARTDOMSET cannot be expressed as an short FO-formula (requires $\exists y_1 \dots \exists y_t$)

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- W[1]-hard for 2-degenerate graphs
- Can be solved on H-minor free graphs in time $(g(H)k)^k n^{O(1)}$
- Can be solved on classes $\mathcal{C}$ of bounded expansion in time $f_{\mathcal{C}}(k)n$
- Can be solved on nowhere dense classes $\mathcal{C}$ in time $f_{\mathcal{C}}(k)n^{1+\varepsilon}$

Step 1: Reduction to a Simpler Problem

Original Problem:	$\to$	Simpler Problem
$G \in \mathcal{C}$ of bounded expansion	$\to$	$\vec{G} \in \mathcal{C}'$ of bounded expansion
FO-formulas $\begin{cases} \varphi_1 \\ \vdots \\ \varphi_\ell \end{cases}$	$\to$	$\left.\begin{matrix} c_1 \\ \vdots \\ c_\ell \end{matrix}\right\}$ vertex weight fcts

For all $\bar{u} = u_1 \dots u_k \in V(G)^k$:

$$G \models \begin{array}{l} \quad \#y\, \varphi_1(y\bar{u}) \geq t_1 \\ \wedge \quad \vdots \\ \wedge \#y\, \varphi_\ell(y\bar{u}) \geq t_\ell \end{array} \iff \begin{array}{c} c_1(u_1) + \dots + c_1(u_k) \geq t_1 \\ \vdots \\ c_\ell(u_1) + \dots + c_\ell(u_k) \geq t_\ell \\ \vec{G} \models \omega(\bar{u}) \text{ (quantifier-free)} \end{array}$$

Courcelle with Semiring Homomorphisms

Often don't want one satisfying assignment but computing a property of the set of satisfying assignments

Example: Set of all vertex covers $\mapsto$ minimum weight VC, number of VCs, all VCs...

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Definition

A problem P is an *MSO-evaluation problem* if it can be expressed as computing $h(\mathrm{sat}(\varphi, G))$ for some homomorphism h into a semiring and MSO-formula φ.

Example: min. weight VC: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$, h maps set to sum of weights

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Theorem (Courcelle, Mosbah '93)

An MSO-evaluation problem P can be solved in time $f_P(tw)nt$ on graphs of treewidth tw where t is the time complexity of the semiring operations.

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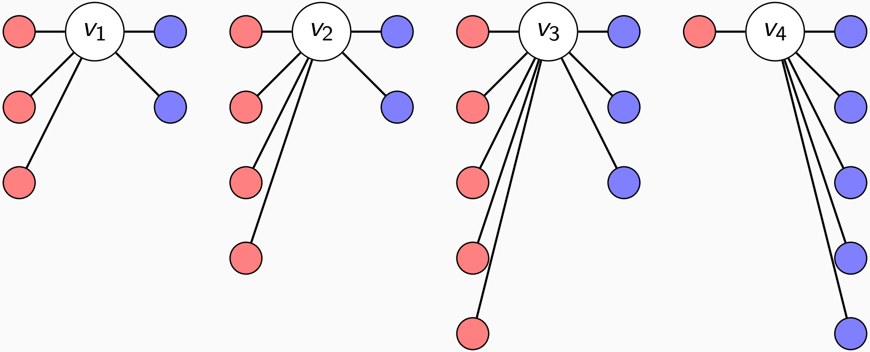

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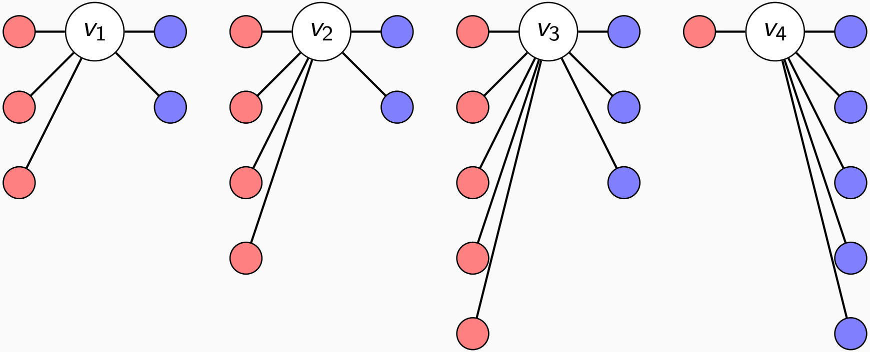

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Reduction: Parameters

Theorem (Reminder)

For every $\varepsilon > 0$, k-$\textsc{Sum}$ is not in time $T^{1-\varepsilon}m^{o(k)}$ (under SETH).

In our example: Parameter $\ell = 2$, size $|G| \leq 2\sqrt{T}m$.

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In general: ℓ freely choosable $\implies$ $|G| = O(\sqrt[\ell]{T}\ell m)$

Have to "guess carry-overs": only $f(k, \ell)$ many choices

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