Preferential Attachment Graphs are somewhere-dense

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RWTH Aachen University

Motivation

Sparsity

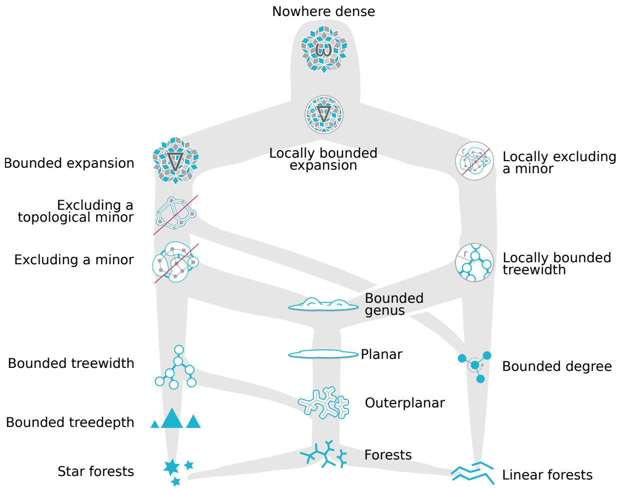
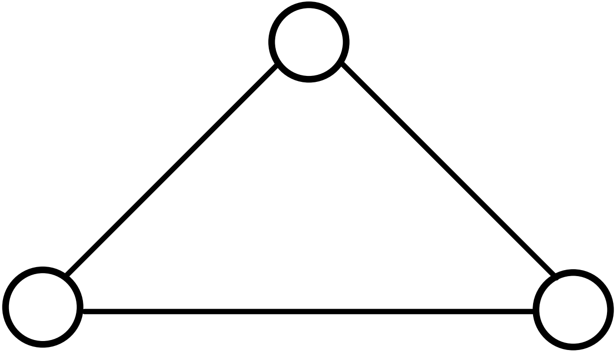


Image by Felix Reidl

Sparsity: r-shallow topological minor



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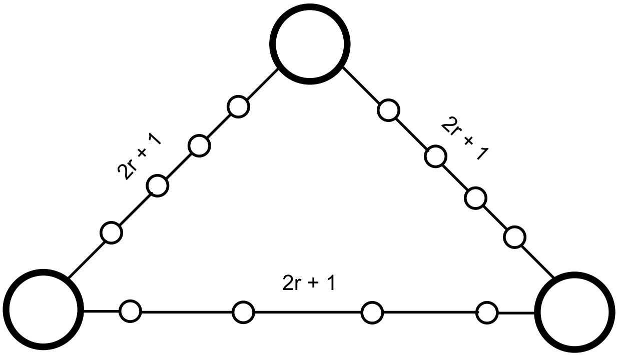

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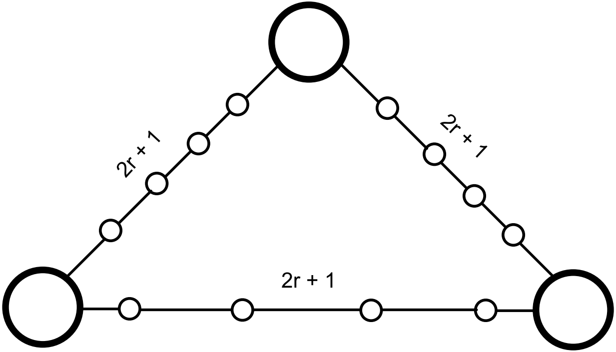
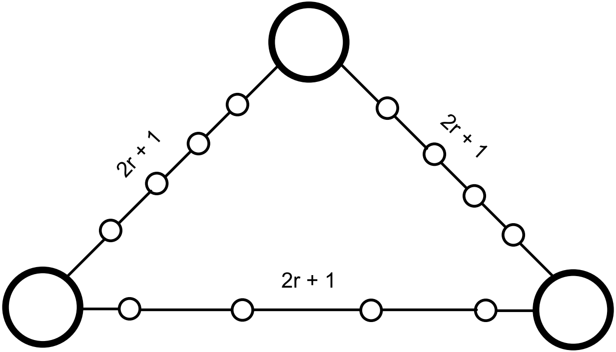


$G \,\widetilde{\nabla}\, r$:= The set of all r-shallow topological minors of G.

Sparsity: r-shallow topological minor



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$$\omega(G \mathbin{\widetilde{\nabla}} r) = \max_{H \in G \mathbin{\widetilde{\nabla}} r} \omega(H) \quad \text{(clique size)}$$

Definition (Nowhere-dense)

A graph class $\mathcal{G}$ is nowhere-dense if there exists a function f, such that for all r and all $G \in \mathcal{G}$, $\omega(G \mathbin{\widetilde{\nabla}} r) \leq f(r)$.

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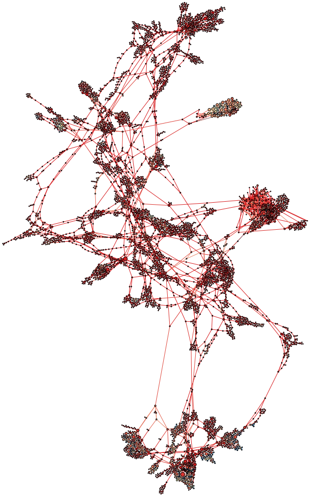
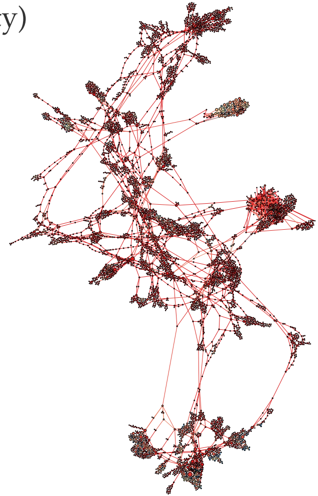
$\mathcal{G}$ is not nowhere-dense $\Leftrightarrow$ $\mathcal{G}$ is somewhere-dense

Typical Properties of Complex Networks



Typical Properties of Complex Networks

- low diameter (small world property)



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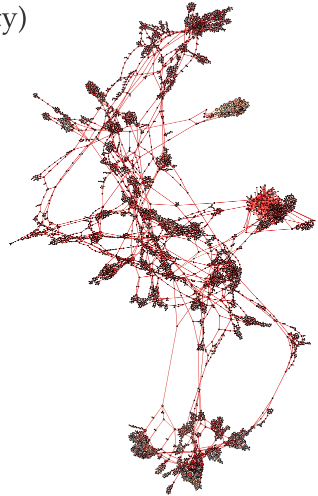

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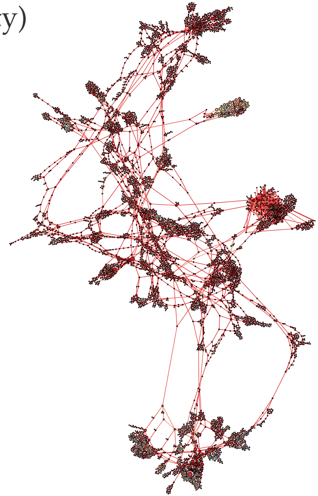

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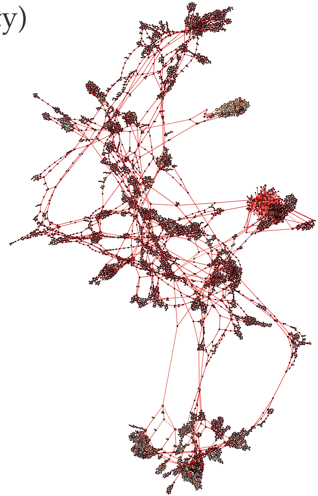

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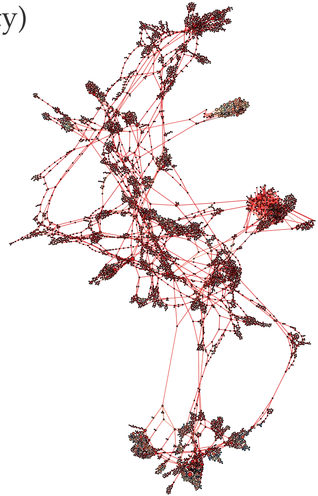

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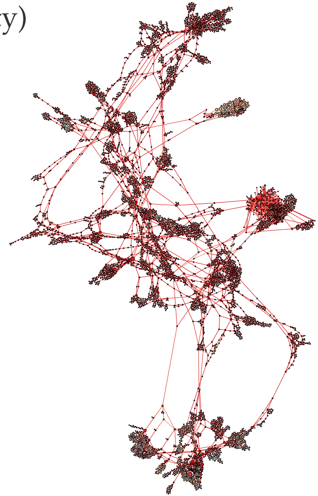
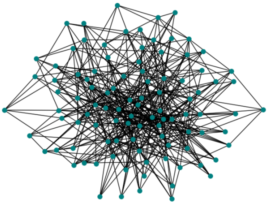
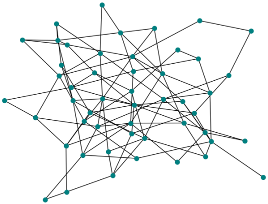


Random Graphs





Random graphs with the goal of modeling real-world data:

- Mathematically analyzable
- Generation of infinite instances

Sparse in the limit

Definition (a.a.s. nowhere-dense)

A random graph model $\mathcal{G}$ is a.a.s. nowhere-dense if there exists a function f such that for all r

$$\lim_{n \to \infty} \mathrm{P}[\omega(G_n \,\widetilde{\nabla}\, r) \leq f(r)] = 1$$

where G_n is a random variable modeling a graph with n vertices randomly drawn from $\mathcal{G}$.

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3. $p = 1/2$

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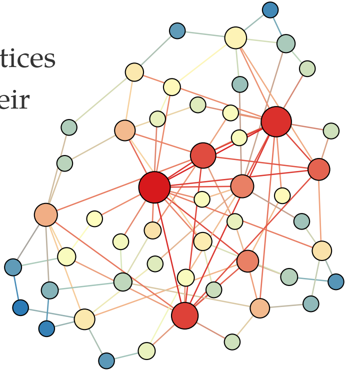
1. $p = 1/n$ $\rightarrow$ a.a.s. nowhere-dense
2. $p = 1 - 1/n$ $\rightarrow$ a.a.s. somewhere-dense
3. $p = 1/2$ $\rightarrow$ Neither!

Preferential attachment graphs

"the rich get richer", "preferential attachment", "Barabási–Albert graphs"

Start with some small fixed graph.

Add vertices. Connect them to m vertices with a probability proportional to their degrees.

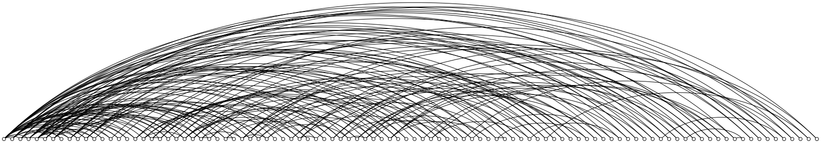


Interesting properties:

- power law degree distribution
- scale free

Preferential attachment graphs

$m = 2, n = 100$:



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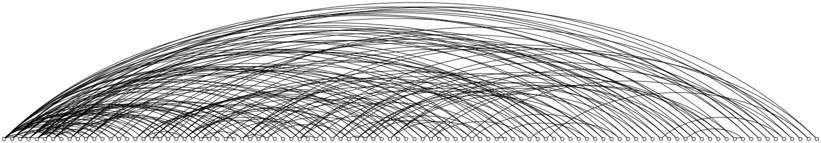

$$\mathrm{E}[d_m^n(v_i)] \sim m\sqrt{n/i}$$

Tail bounds

Degree Concentrations

- Tail bounds exists for number of vertices with degree d. [Bollobás et al. 2001]

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Degree Concentrations

- Tail bounds exists for number of vertices with degree d. [Bollobás et al. 2001]
- Via Martingales + Azuma-Hoeffding inequality
- Does not work for large d (i.e. order $\sqrt{n}$)
- But we need high degree vertices!

Concentration of a single vertex

No vertex is sharply concentrated!

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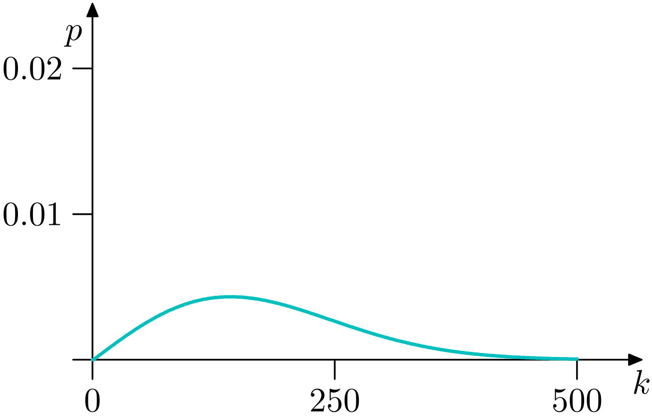
We can not hope for general *exponential* bounds.

Concentration of a single vertex



Distribution of $d_1^{10000}(v_1)$.

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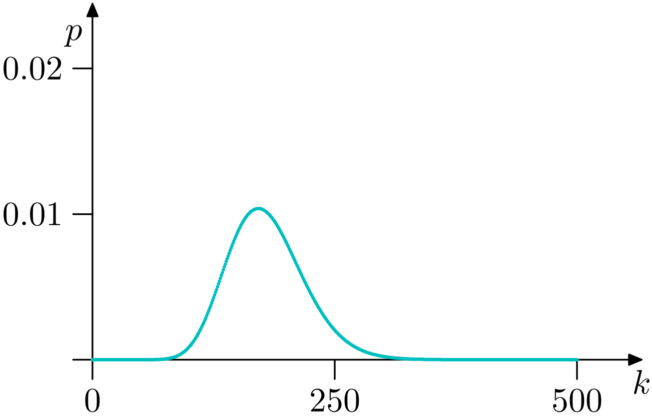

Distribution of $d_1^{10000}(v_1)$ conditioned under $d_1^{100}(v_1) = 18$.

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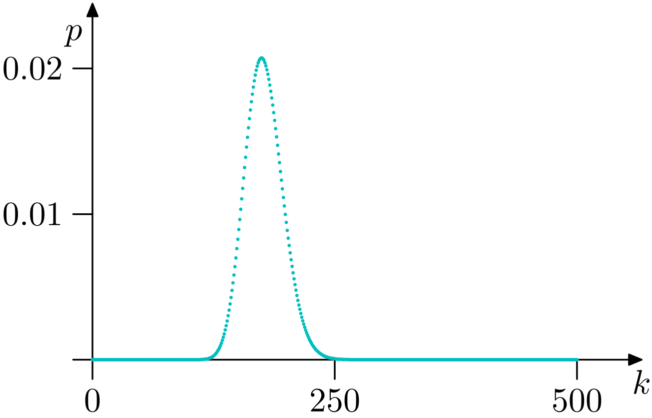

Distribution of $d_1^{10000}(v_1)$ conditioned under $d_1^{1000}(v_1) = 56$.

The rich stay rich

Theorem

Let $0 < \varepsilon \leq 1/40$, $t, m, n \in \mathbf{N}$, $t > \frac{1}{\varepsilon^6}$ *and* $S \subseteq \{v_1, \ldots, v_t\}$. *Then*

$$\mathrm{P}\Big[(1-\varepsilon)\sqrt{\frac{n}{t}}d_m^t(S) < d_m^n(S) < (1+\varepsilon)\sqrt{\frac{n}{t}}d_m^t(S) \text{ for all } n \geq t \ \Big|\ d_m^t(S)\Big]$$
$$\geq 1 - \ln(15t)e^{\varepsilon^{-O(1)}d_m^t(S)}.$$

The rich stay rich

Theorem (The approximate version)

Let $\varepsilon \geq 0$*,* $t, m, n \in \mathbf{N}$*, and* $S \subseteq \{v_1, \ldots, v_t\}$*:*

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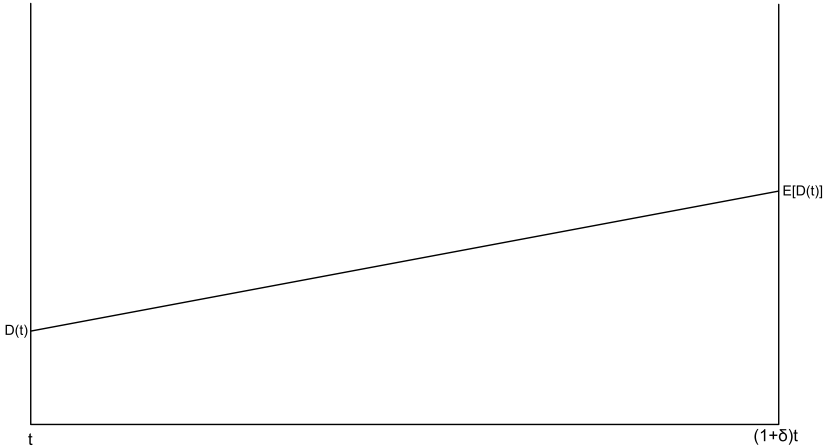
- The rich stay rich
- At first there is chaos
- If we have information for t we can better predict $n > t$

Proving the theorem



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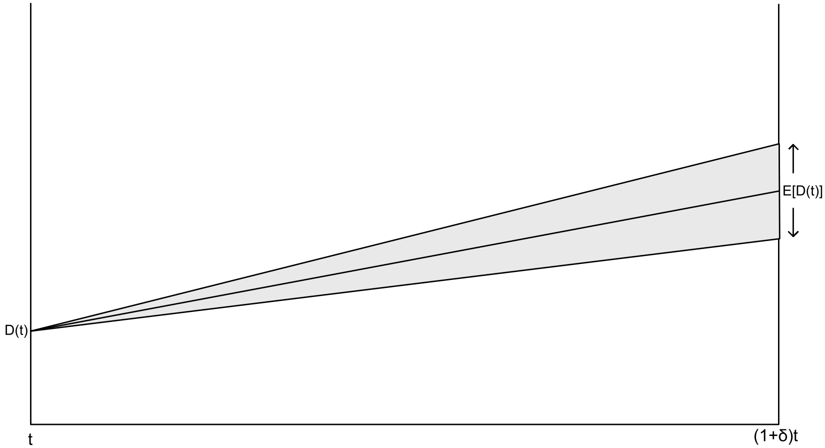

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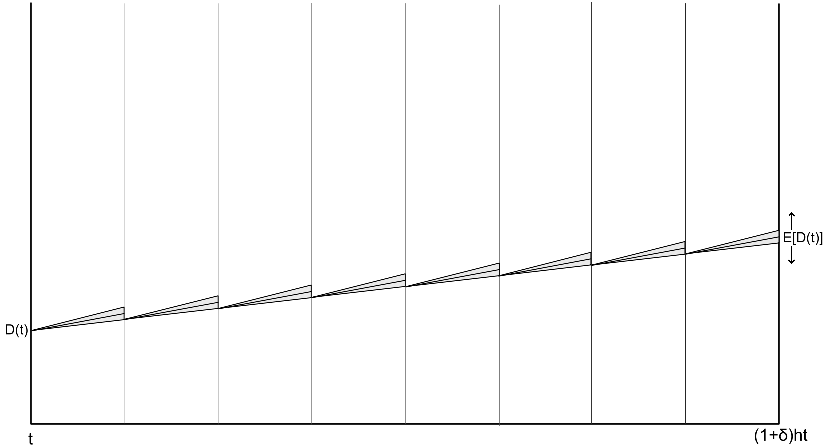

Somewhere-Dense

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Theorem

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Corollary

G_m^n is a.a.s. somewhere-dense for $m \geq 2$.

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k sets of vertices

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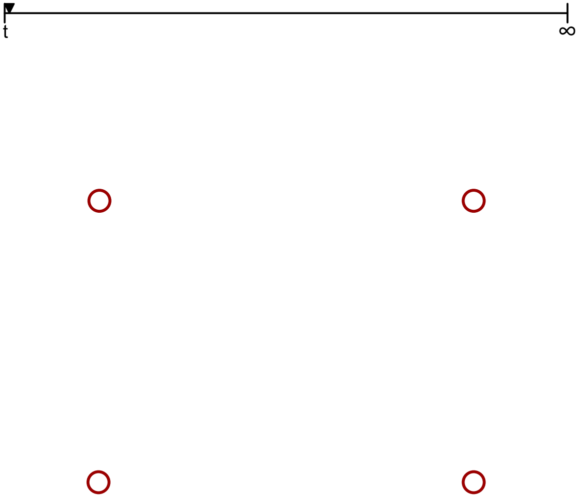
$\rightarrow$ Ensure with tail bounds it also has high degree in the future.

Building cliques



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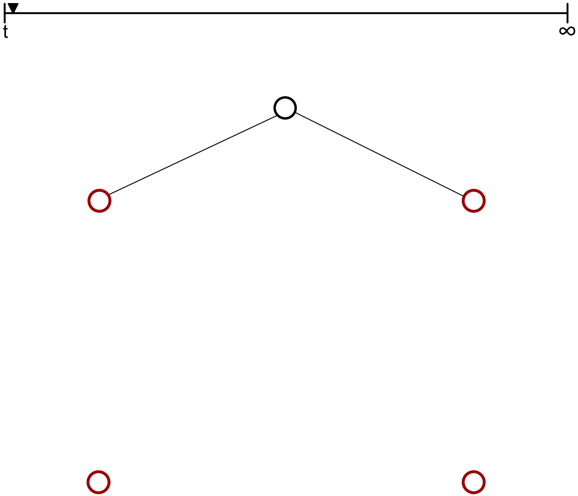

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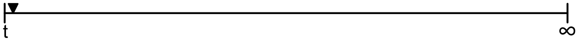

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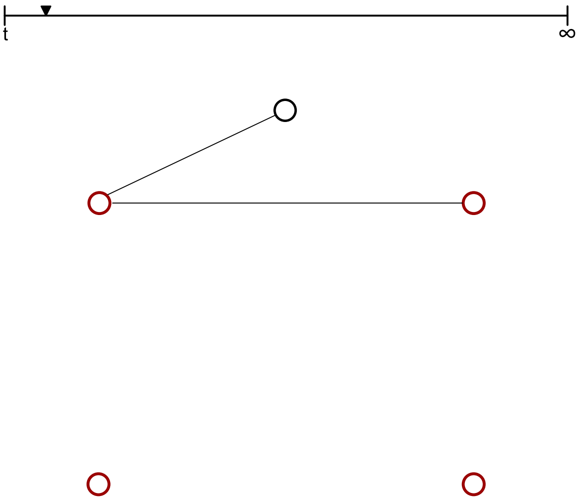

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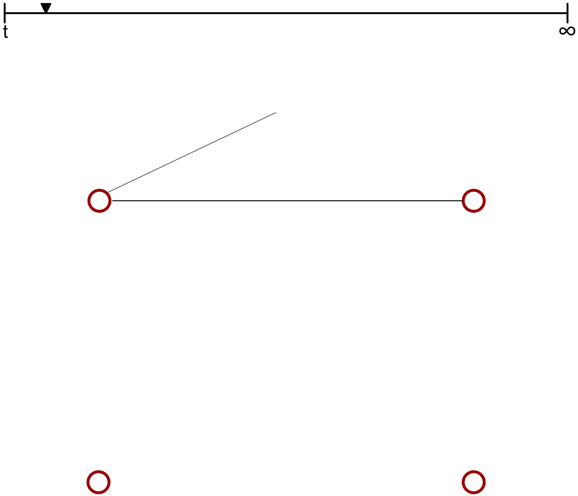

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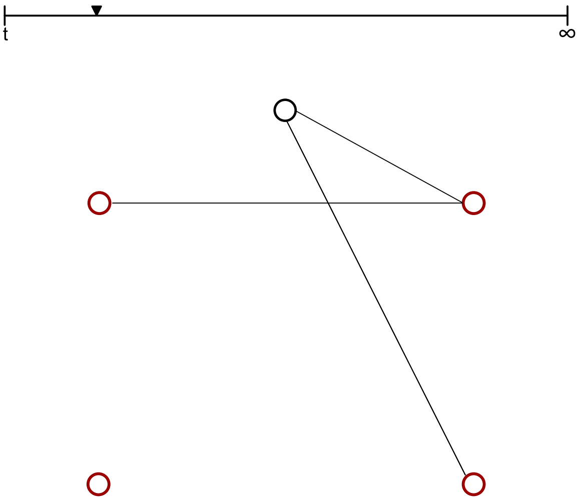

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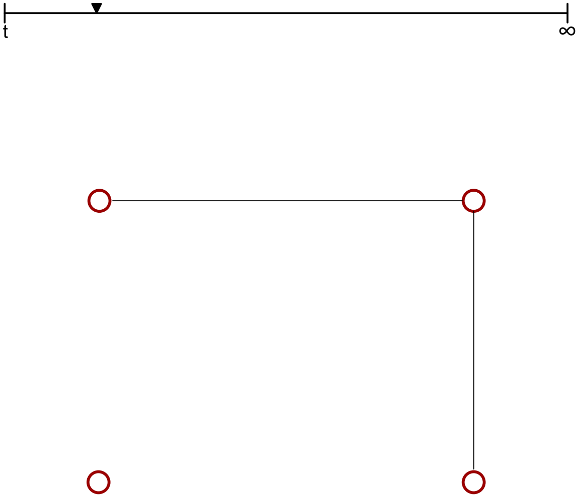

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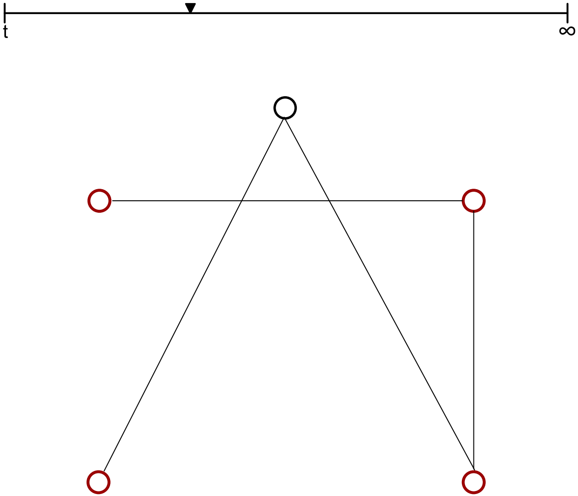

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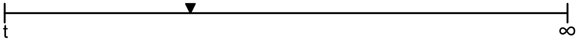

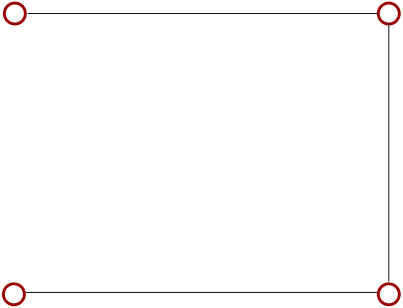

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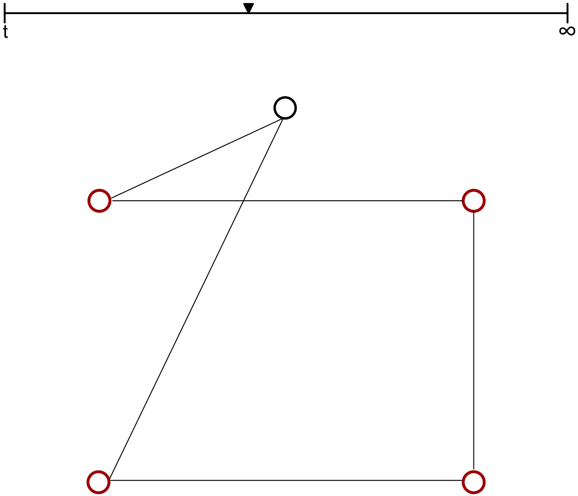

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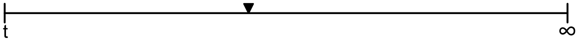

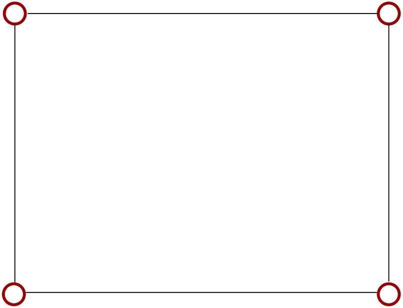

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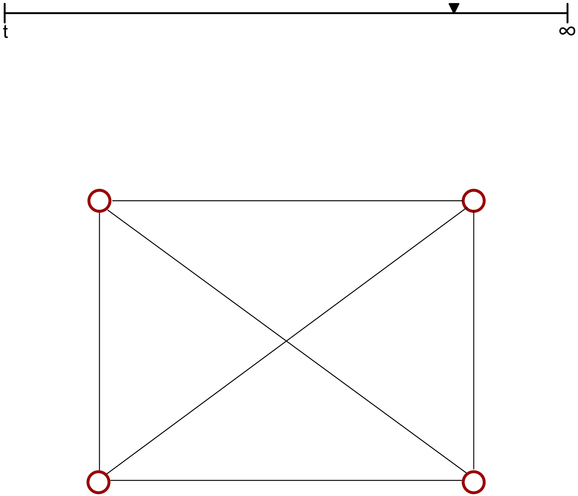

Connecting principals: Why we need $\sqrt{i}$

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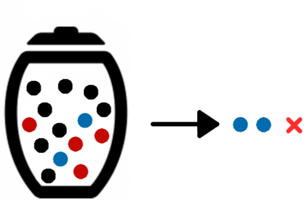
Step i:

- $\sqrt{i}$ red
- $\sqrt{i}$ blue
- remainder black

Two balls drawn, success if red and blue

Connecting principals: Why we need $\sqrt{i}$



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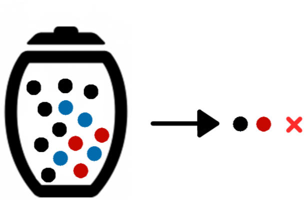

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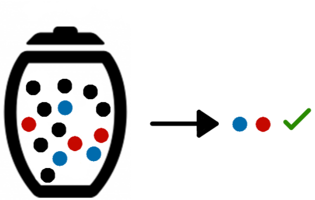

Connecting principals: Why we need $\sqrt{i}$

$$1 - \prod_{i=10}^{\infty}\left(1 - 2\left(\frac{\sqrt{i}}{i}\right)^2\right) = 1$$

Connecting principals: Why we need $\sqrt{i}$

$$1 - \prod_{i=10}^{\infty}\left(1 - 2\left(\frac{\sqrt{i}/\log(i)}{i}\right)^2\right) \neq 1$$

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- What about more general PA-graphs with δ parameter?
  - $\delta = 0$: Our model
  - $\delta = \infty$: Uniform attachment

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