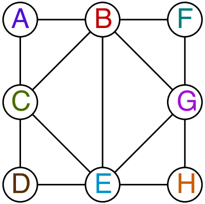
Formal Language Techniques for Space Lower Bounds

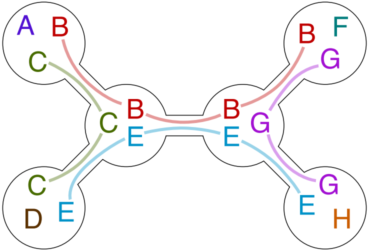
Philipp Kuinke

February 23, 2018

Contained in Sánchez Villaamil's Phd Thesis 2017

Treewidth

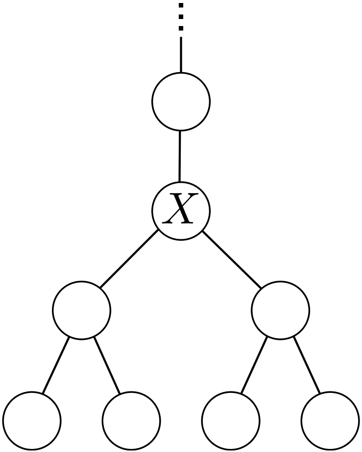




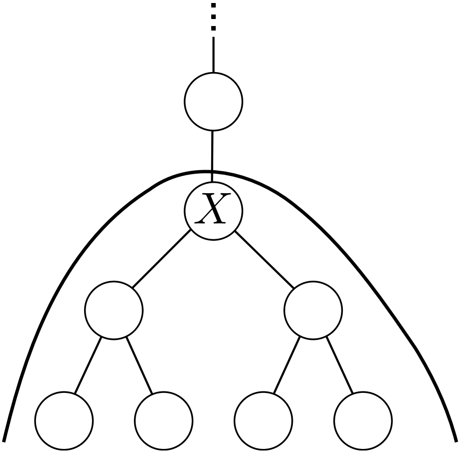
Dynamic Programming

Use treewidth structure to traverse the graph

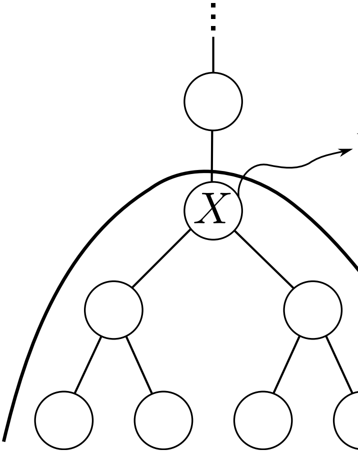
Dynamic Programming



Dynamic Programming

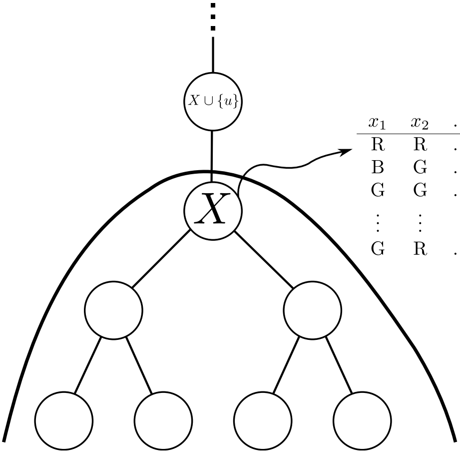


Dynamic Programming



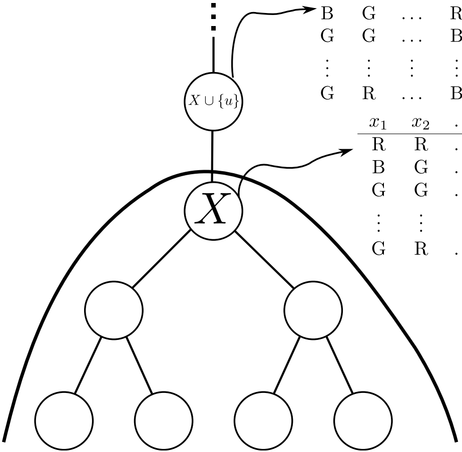
x_1	x_2	$\dots$	x_w
R	R	$\dots$	G
B	G	$\dots$	R
G	G	$\dots$	B
$\vdots$	$\vdots$	$\vdots$	$\vdots$
G	R	$\dots$	B

Dynamic Programming



x_1	x_2	$\dots$	x_w
R	R	$\dots$	G
B	G	$\dots$	R
G	G	$\dots$	B
$\vdots$	$\vdots$	$\vdots$	$\vdots$
G	R	$\dots$	B

Dynamic Programming



x_1	x_2	…	x_w	u
R	R	…	G	R
B	G	…	R	R
G	G	…	B	G
⋮	⋮	⋮	⋮	⋮
G	R	…	B	G

x_1	x_2	…	x_w
R	R	…	G
B	G	…	R
G	G	…	B
⋮	⋮	⋮	⋮
G	R	…	B

Dynamic Programming

The runtime of dynamic programming algorithms depends on the table sizes!

Dynamic Programming

Common properties of DP-algorithms we formalize

Dynamic Programming

Common properties of DP-algorithms we formalize

1. They do a single pass over the decomposition;

Dynamic Programming

Common properties of DP-algorithms we formalize

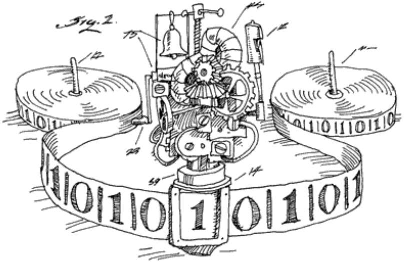
1. They do a single pass over the decomposition;
2. they use $O(f(w) \log^{O(1)} n)$ space; and

Dynamic Programming

Common properties of DP-algorithms we formalize

1. They do a single pass over the decomposition;
2. they use $O(f(w) \log^{O(1)} n)$ space; and
3. they do not modify or rearrange the decomposition.

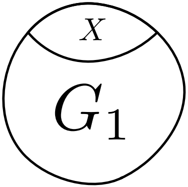
Dynamic Programming



Definition (DPTM)

A Dynamic Programming Turing Machine (DPTM) is a Turing Machine with an input read-only tape, whose head moves only in one direction and a separate working tape. It only accepts well-formed instances as inputs.

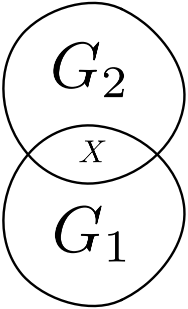
Boundaried Graphs



Definition

An s-boundaried graph G is a graph with s distinguished vertices, called the boundary.

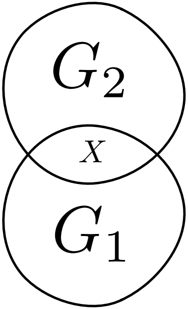
Boundaried Graphs



Definition

$G_1 \oplus G_2$ is the disjoint union of two s-boundaried graphs merged at the boundary.

Boundaried Graphs



Definition

$\mathcal{G}_s$ is the set of all s-boundaried graphs.

Formal Languages

Interpret Problem as a language Π, i.e. $G \in \Pi$ if and only if G is a yes-instance.

Myhill-Nerode Families

Myhill-Nerode Families

Definition (Myhill-Nerode family)

A set $\mathcal{H} \subseteq \mathcal{G}_s$ is an s-Myhill-Nerode family for a DP language Π if

Myhill-Nerode Families

Definition (Myhill-Nerode family)

A set $\mathcal{H} \subseteq \mathcal{G}_s$ is an s-Myhill-Nerode family for a DP language Π if

1. For every subset $\mathcal{I} \subseteq \mathcal{H}$ there exists an s-boundaried graph $G_{\mathcal{I}}$ with bounded size, such that for every $H \in \mathcal{H}$ it holds that

$$G_{\mathcal{I}} \oplus H \notin \Pi \Leftrightarrow H \in \mathcal{I}$$

Myhill-Nerode Families

Definition (Myhill-Nerode family)

A set $\mathcal{H} \subseteq \mathcal{G}_s$ is an s-Myhill-Nerode family for a DP language Π if

1. For every subset $\mathcal{I} \subseteq \mathcal{H}$ there exists an s-boundaried graph $G_{\mathcal{I}}$ with bounded size, such that for every $H \in \mathcal{H}$ it holds that

$$G_{\mathcal{I}} \oplus H \notin \Pi \Leftrightarrow H \in \mathcal{I}$$

2. For every $H \in \mathcal{H}$ it holds that H has bounded size.

Myhill-Nerode Families

Definition (Myhill-Nerode family)

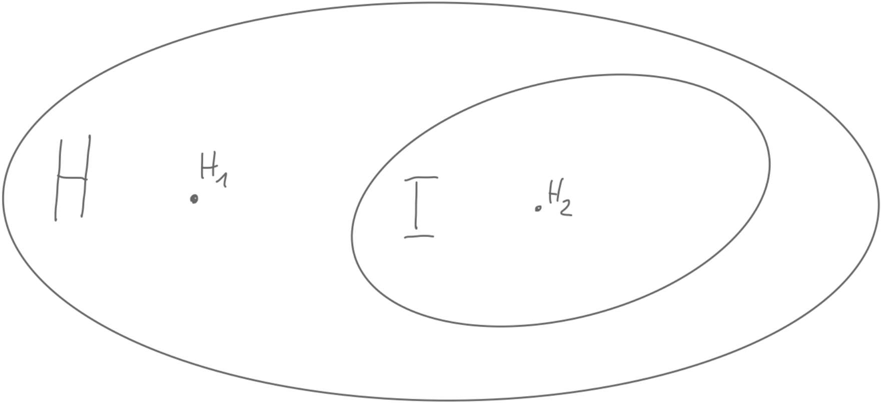
A set $\mathcal{H} \subseteq \mathcal{G}_s$ is an s-Myhill-Nerode family for a DP language Π if

1. For every subset $\mathcal{I} \subseteq \mathcal{H}$ there exists an s-boundaried graph $G_{\mathcal{I}}$ with $|G_{\mathcal{I}}| = |\mathcal{H}| \log^{O(1)} \mathcal{H}$, such that for every $H \in \mathcal{H}$ it holds that

$$G_{\mathcal{I}} \oplus H \notin \Pi \Leftrightarrow H \in \mathcal{I}$$

2. For every $H \in \mathcal{H}$ it holds that $|H| = |\mathcal{H}| \log^{O(1)} \mathcal{H}$.

Myhill-Nerode Families



$G_{\mathcal{I}} \oplus H_1 \in \Pi$ $\qquad$ $G_{\mathcal{I}} \oplus H_2 \notin \Pi$

DPTM bounds

Lemma ([Sánchez Villaamil '17])

Let $\epsilon > 0$ and Π be a DP decision problem such that for every s there exists an s-Myhill-Nerode family $\mathcal{H}$ for Π of size c^s and width $\mathbf{tw}(\mathcal{H}) = s$. Then no DPTM can decide Π using space $O((c - \epsilon)^k \log n)$, where n is the size of the input and k the treewidth of the input.

DPTM bounds

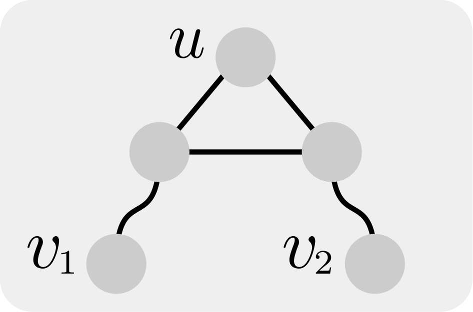
Lemma ([Sánchez Villaamil '17])

Let $\epsilon > 0$ and Π be a DP decision problem such that for every s there exists an s-Myhill-Nerode family $\mathcal{H}$ for Π of size $c^s/f(s)$, where $f(s) = s^{O(1)} \cap \Theta(1)$ and width $\mathbf{tw}(\mathcal{H}) = s + o(s)$. Then no DPTM can decide Π using space $O((c-\epsilon)^k \log^{O(1)} n)$, where n is the size of the input and k the treewidth of the input.

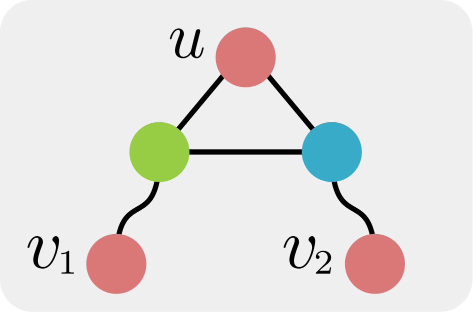
3-Coloring

- Input: A Graph G
- k: The treewidth of G
- Question: Can G be colored with 3 colors?

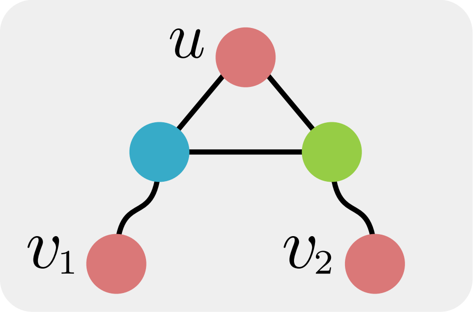
Coloring Gadget



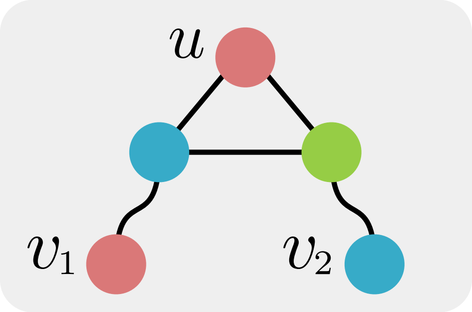
Coloring Gadget



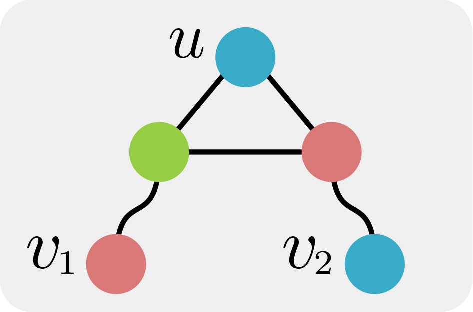
Coloring Gadget



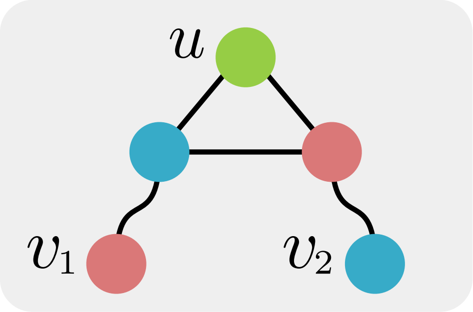
Coloring Gadget



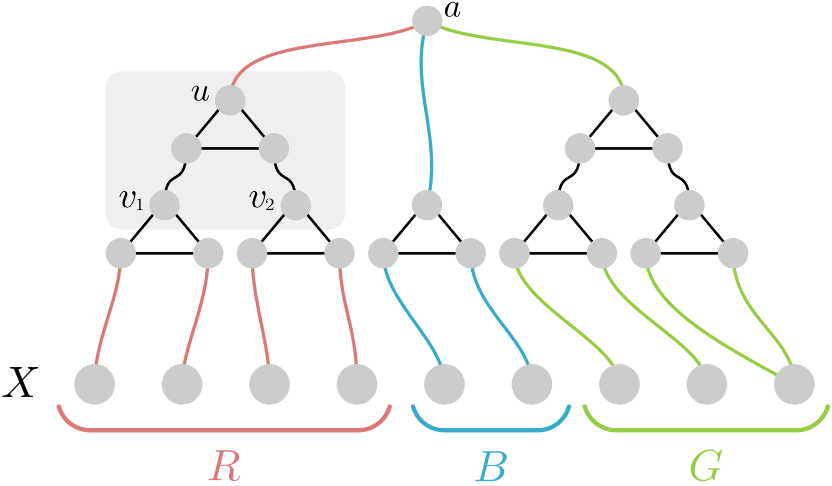
Coloring Gadget



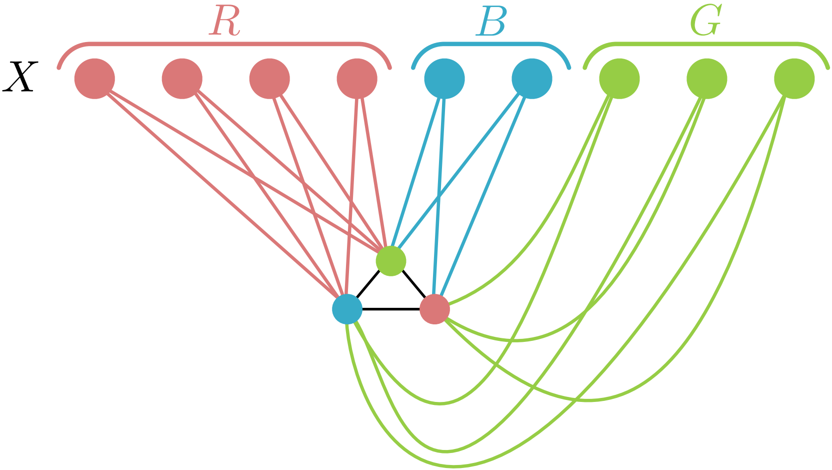
Coloring Gadget



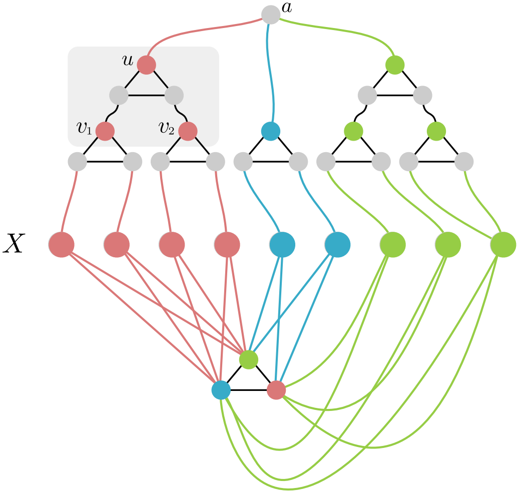
The Graph Γ_X



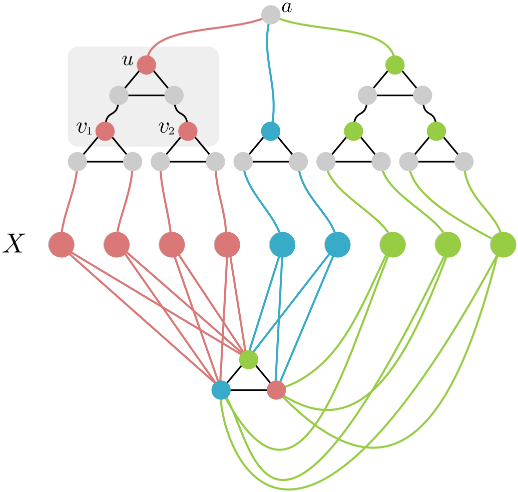
Enforcing Colorings with H_X



No-Instances

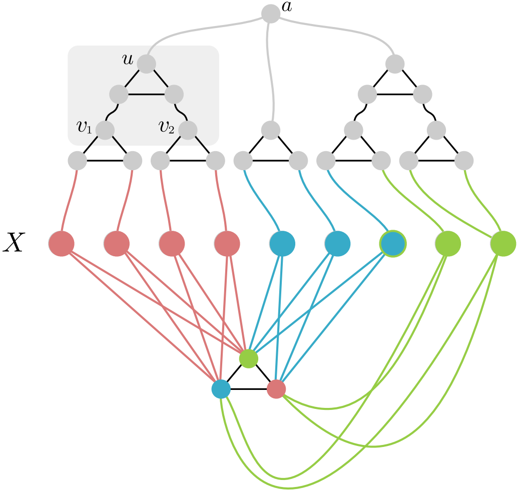


No-Instances



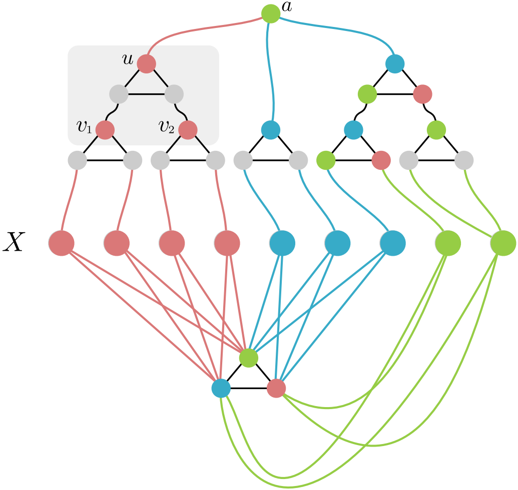
This is **not** 3-colorable.

Yes-Instances



This **is** 3-colorable.

Yes-Instances



This **is** 3-colorable.

Myhill-Nerode Families

- $\Gamma_X \oplus H_X \notin \Pi$

Myhill-Nerode Families

- $\Gamma_X \oplus H_X \notin \Pi$
- $\Gamma_X \oplus H_{X'} \in \Pi$, for $(X \neq X')$

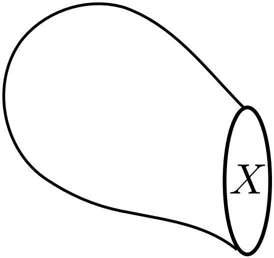
Myhill-Nerode Families

- $\Gamma_X \oplus H_X \notin \Pi$
- $\Gamma_X \oplus H_{X'} \in \Pi$, for $(X \neq X')$
- $G_{\mathcal{I}} = \bigoplus_{H_X \in \mathcal{I}} \Gamma_X$

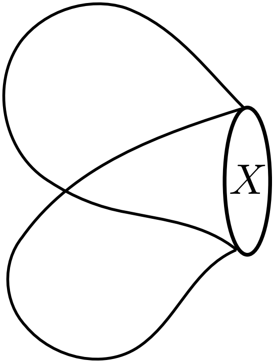
Myhill-Nerode Families

- $\Gamma_X \oplus H_X \notin \Pi$
- $\Gamma_X \oplus H_{X'} \in \Pi$, for $(X \neq X')$
- $G_{\mathcal{I}} = \oplus_{H_X \in \mathcal{I}} \Gamma_X$
- $G_{\mathcal{I}} \oplus H_X \in \Pi \Leftrightarrow H_X \notin \Pi$

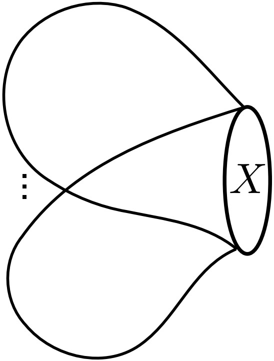
Myhill-Nerode Families



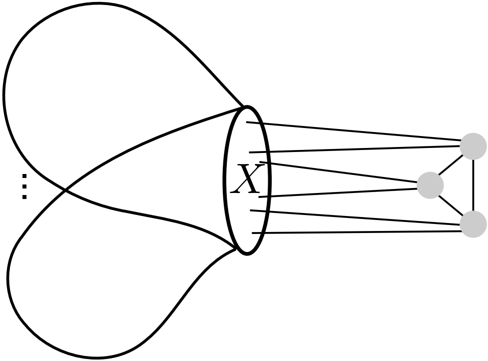
Myhill-Nerode Families



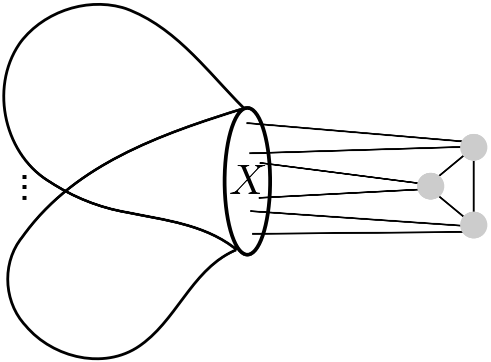
Myhill-Nerode Families



Myhill-Nerode Families

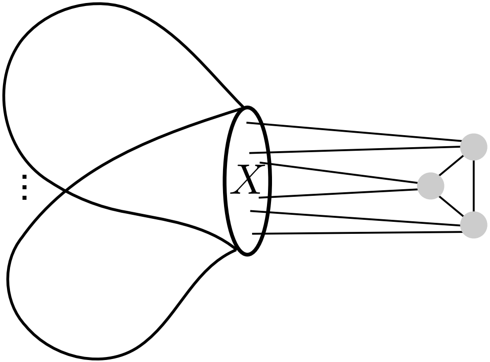


Myhill-Nerode Families



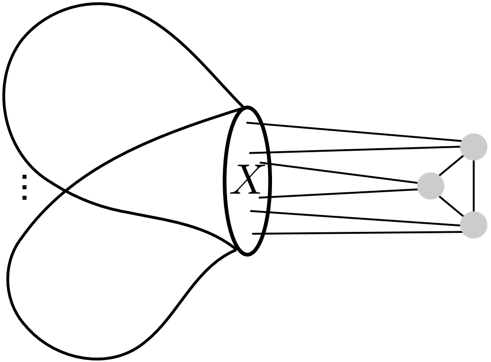
$$G_{\mathcal{I}} \oplus H_X \in \Pi \Leftrightarrow H_X \notin \Pi$$

Myhill-Nerode Families



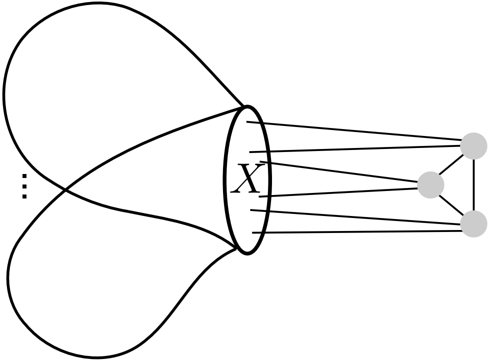
We can generate $3^w/6$ such graphs.

Myhill-Nerode Families



We can generate a Myhill-Nerode family of index $3^w/6$.

Myhill-Nerode Families



We cannot use $O((3-\epsilon)^w \cdot \log n)$ space for a dynamic programming algorithm.

Obtained result

Theorem ([Sánchez Villaamil '17])

No DPTM solves 3-Coloring *on a treewidth-decomposition of width w with space bounded by $O((3-\epsilon)^w \cdot \log^{O(1)} n)$.*

Further results

Theorem ([Sánchez Villaamil '17])

No DPTM solves VERTEX COVER *on a treewidth-decomposition of width* w *with space bounded by* $O((2-\epsilon)^w \cdot \log^{O(1)} n)$.

Theorem ([Sánchez Villaamil '17])

No DPTM solves DOMINATING SET *on a treewidth-decomposition of width* w *with space bounded by* $O((3-\epsilon)^w \cdot \log^{O(1)} n)$.

Not Captured

- Compression.
- Algebraic techniques.
- Preprocessing to compute optimal traversal.
- Branching instead of DP

The end