Shortest Path in Planar Graphs with real lengths

Thomas Prinz

November 26, 2013

Introduction

The Problem

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Assumptions:

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Goal:

Compute the shortest Path from s to all other vertices.

Applications?

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- Route Planing

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- Networks/Routing

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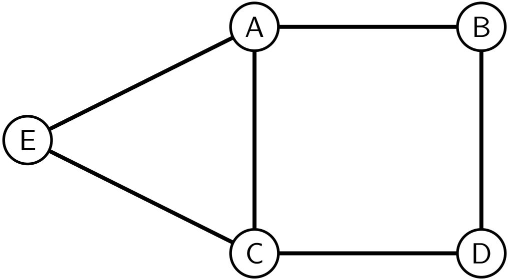
- Networks/Routing
- Computer Vision techniques

Regions of a Graph



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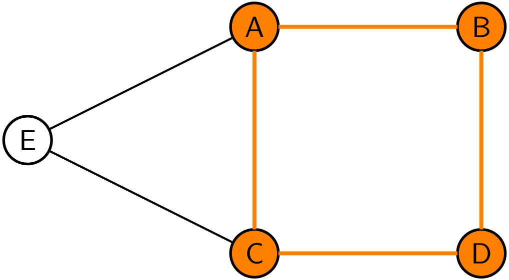

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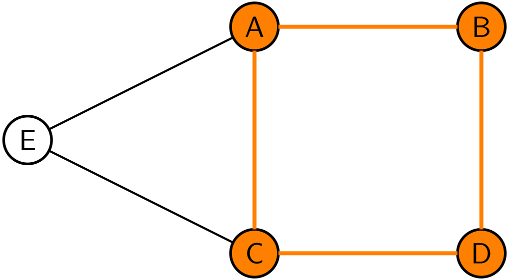
Boundary Vertices



Definition

Vertices of a Region that connect it with the rest of the Graph are called Boundary Vertices (A,C). Other Vertices are called interior Vertices(B,D).

Holes of a Graph

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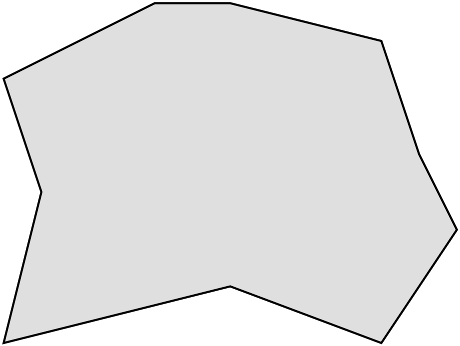
Faces in a Subgraph, that do not exists in the actual Graph are called unnatural faces or holes.



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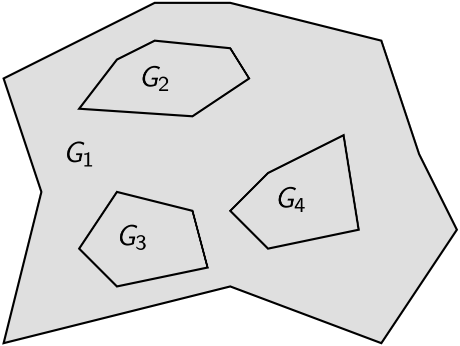

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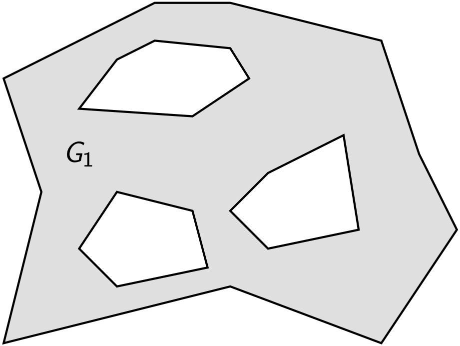

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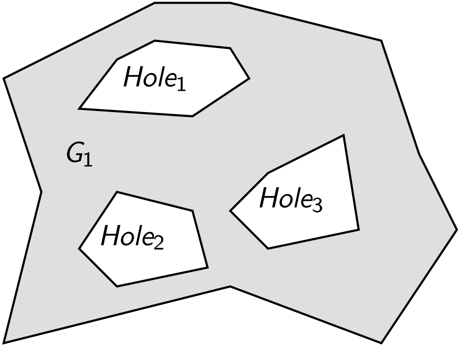

Price functions

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A function $P : V \rightarrow R$ is called a price function.

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Definition

A Price function induces a reduced cost function $w : E \to R$ with w(u,v) = p(u) + l(u,v) - p(v) for every edge e(u,v) where l(u,v) is the length of e(u,v).

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Definition

If the reduced cost function assigns every edge a non-negative value, then we say P is a feasible price function.

Price functions

Lemma

A Single Source Shortest Path distance from one vertex r (root) to all others induces a feasible price function on the Graph G with:
$\phi(v)$ = *(shortest) r-to-v distance*

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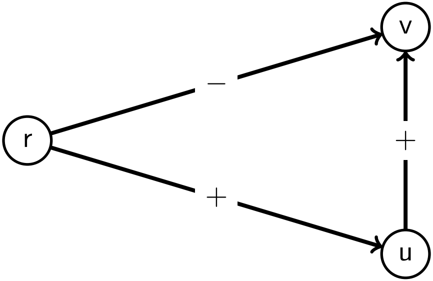

The improved Algorithm of Mozes/Wulff-Nilsen

The Algorithm of Klein (Reminder)

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The Algorithm of Klein (Reminder)

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Assumption:

- No Subgraph has holes.
  (Therefore all boundary nodes lie on a single face)

Millers Cycle Seperator Therorem

Definition

Given a triangulated planar Graph G with n vertices. One can find a cycle seperator (of length $O(\sqrt{n})$) that divides the Graph into 2 Pieces, with each having 2n/3 vertices at most in linear time

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Definition

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Cost creating r-Division: $O(n \cdot log(n))$

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Overall cost: $O(n \cdot log(n))$ ✓

2. SSSP Inter-region Boundary Distances (Bellman Ford)

Let r be the root, B be the Boundary and b=$|B|$.

1. $e_j[v] := \infty$ for all $v \in B$ and j = 0,...,b
2. $e_0[r] := 0$
3. for j = 1,2,...,b
4. let C be the cycle defining the boundary.
5. $e_j[v] := min_{w \in V_C}\{e_{j-1}[w] + \delta_i[w, v]\}$
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5. let C be the cycle defining the boundary of R.
6. $e_j[v] := min\{e_j[v], min\{w \in V_C\ e_{j-1}[w] + d_R(w,v)\}\}\ f.\ all\ v \in V_C$
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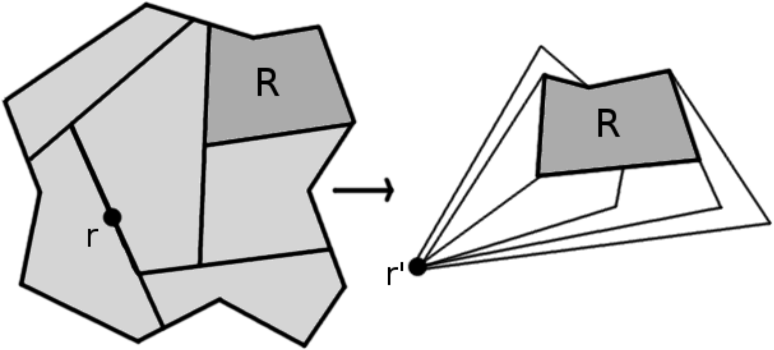

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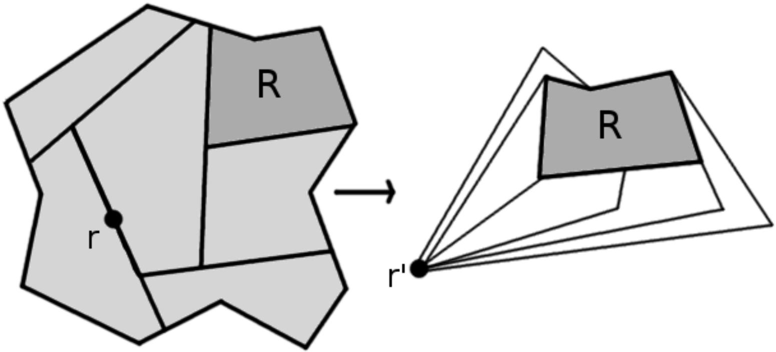

$\phi(v) = d_R(r_R, v)$ *if* $v \neq r'$
$\phi(v) = max\{d_R(r_R, b) - d_G(r, b)\}$ *if* $v = r'$ $(b \in B_R)$
Claim: ϕ is a feasible price function.

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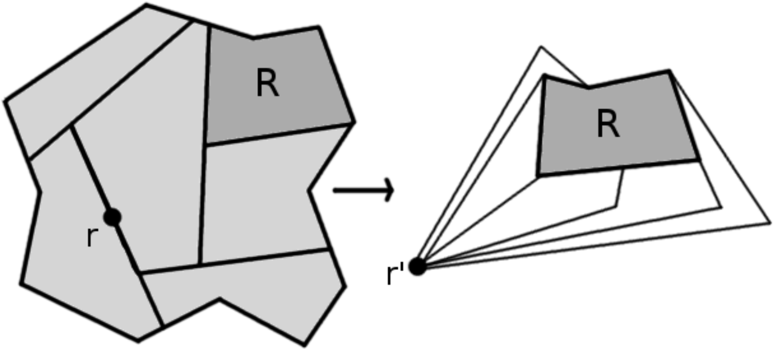

Cost: $O(|V_R| \cdot log(|V_R|))$ (for $|V_R|$ Vertices per Region.)
Overall Cost: $O(n \cdot log(n))$ ✓

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Overall Cost: $O(n)/O(n \cdot log(n))$

Holes?

What if we have Holes?

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2. Inter-region Boundary Distances (Bellman Ford)

Optimized Bellman Ford Pseudocode:

Let r be the root, B be the Boundary and b=$|B|$.

1. initialize vector $e_j[v]$ for j = 0,...,b and $v \in B$
2. $e_j[v] := \infty$ for all $v \in B$ and j = 0,...,b
3. $e_0[r] := 0$
4. for j = 1,...,b
5. for each region $R \in R_{all}$
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7. $e_j[v] := min\{e_j[v], min\{w \in V_C\ e_{j-1}[w] + d_R(w, v)\}\}\ f.\ all\ v \in V_C$
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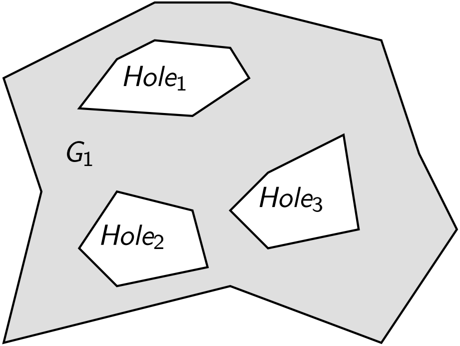

Problem: We have no Cylce C that defines the boundary. There are holes, each defining some of the boundary.

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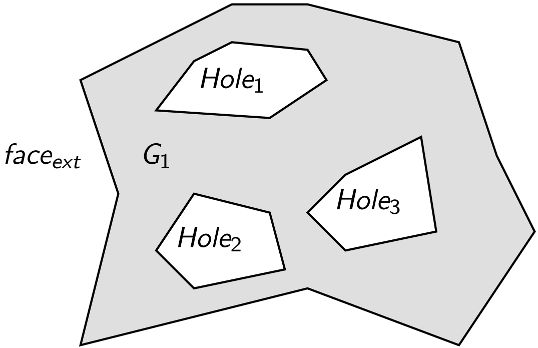
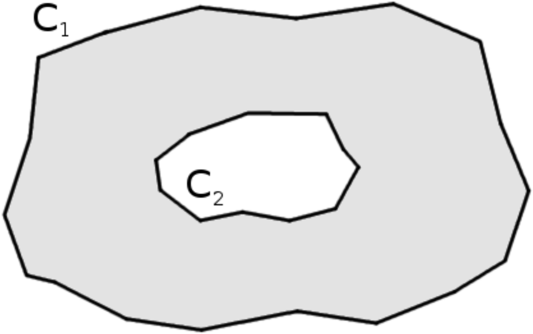


Consider: (H1,H2) (H1,H3) (H1,face) (H2,H1) (H2,H3) (H2,face) ... $(O(h^2) = O(1))$

2. Inter-region Boundary Distances (Bellman Ford)

1. Let (C_1, C_2) be a pair of holes/external face.



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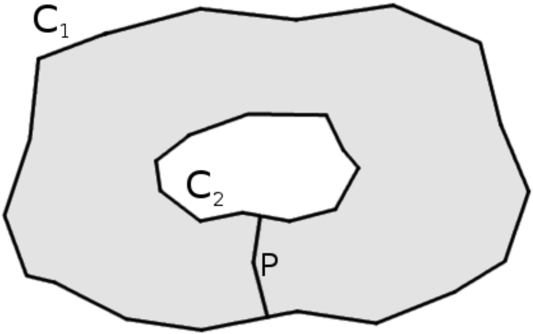
2. Let P be some Path starting in C_1 and ending in C_2.



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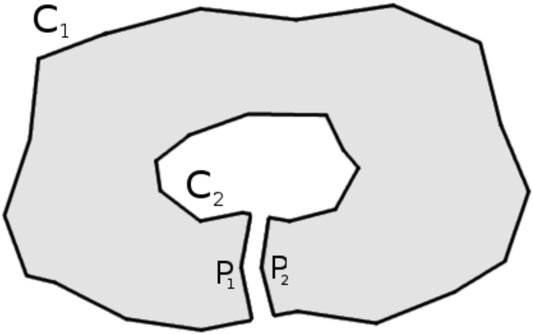
3. Cut the graph along P. Now all node lie on the same face again.



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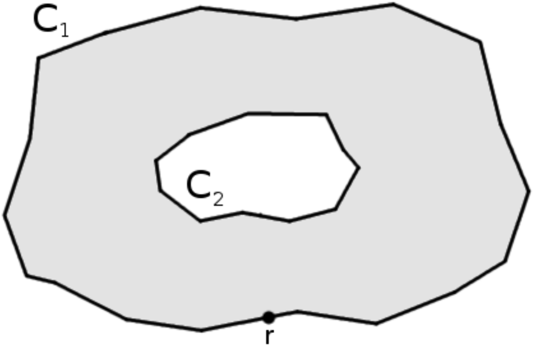
3. Choose some node r on C_1.



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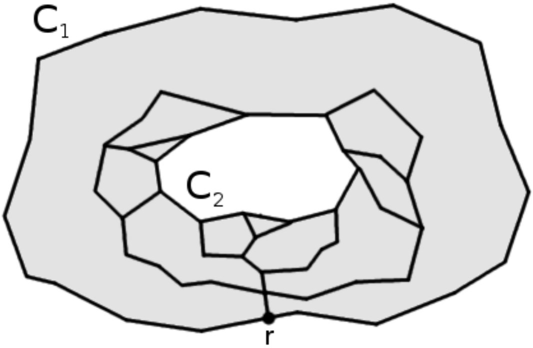
4. Compute the shortest Path Tree from r to all nodes on C_2.



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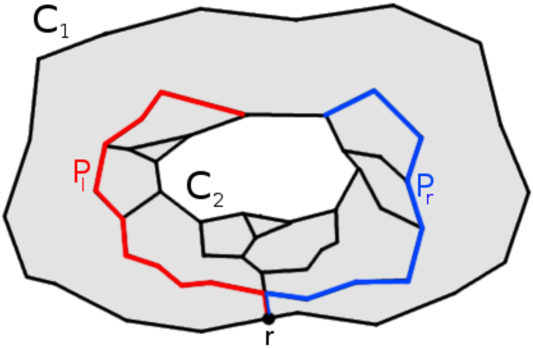
Lemma

There is always a shortest Path from $v \in C_1$ to $u \in C_2$ which does not cross both, P_l and P_r.



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Assume there is a shortest Path, say from u to v, that crosses both:
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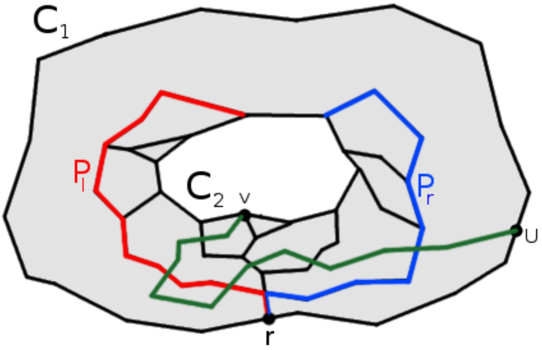

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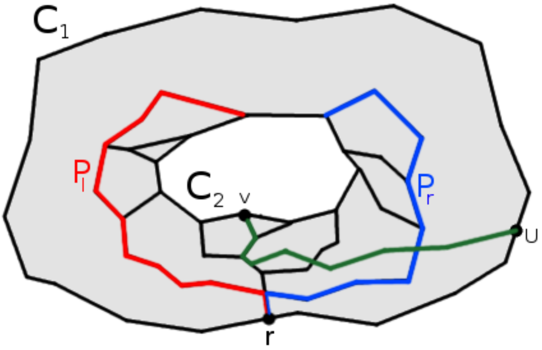

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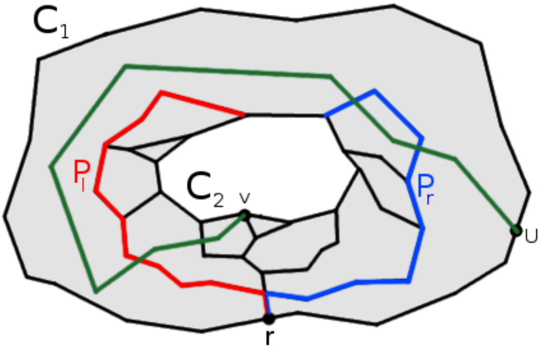

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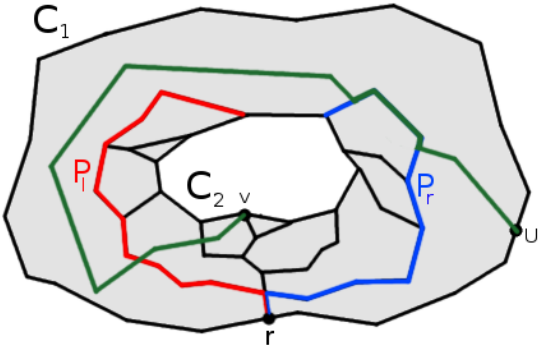

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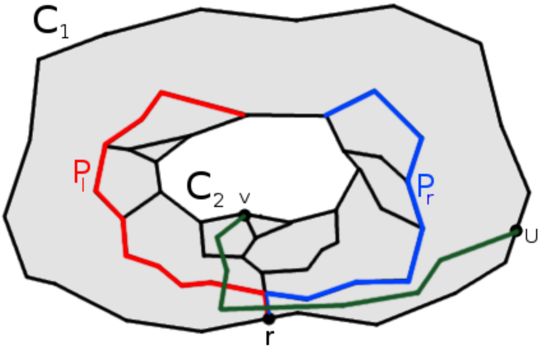

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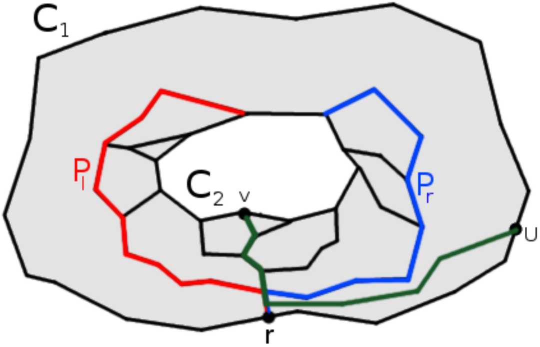
Assume there is a shortest Path, say from u to v, that crosses both:
Case 3:



New modified Bellman Ford

Let r be the root, B be the Boundary and b=$|B|$.

1. initialize vector $e_j[v]$ for j = 0,...,b and $v \in B$
2. $e_j[v] := \infty$ for all $v \in B$ and j = 0,...,b
3. $e_0[r] := 0$
4. for j = 1,...,b
5. for each region $R \in R_{all}$
6. for each Pair (C_1, C_2)
7. IF $(C_1 = C_2)$ THEN continue as before
7. ELSE (assume C_1 is external and d_{P_i} have been precomputed)

$e_j[v] := min\{e_j[v], min_{w \in C_1}\{e_{j-1}[w] + d_{P_r}(w', v')\}\}$ $f.\ all\ v \in C_2$
$e_j[v] := min\{e_j[v], min_{w \in C_1}\{e_{j-1}[w] + d_{P_l}(w', v')\}\}$ $f.\ all\ v \in C_2$

8. $D[v] := e_b[v]$ for all $v \in B$

New modified Bellman Ford time complexity

We have:

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- h^2 Pairs of holes (=O(1))
- 2 Iterations per hole pair (=O(1))

Overall Cost: $O(2 \cdot h^2 \cdot p \cdot n \cdot \alpha(n)) = O(n \cdot log(n))$

Overall Cost of Improved algorithm

1. Dividing into p pieces	$\Rightarrow$	$O(n \cdot log(n))$
2. Recursive Call	$\Rightarrow$	$O(\frac{log(n)}{log(log(n))})$
3. MSSP	$\Rightarrow$	$O(n \cdot log(n))$
4. Optimized Bellman Ford	$\Rightarrow$	$O(n \cdot log(n))$
5. Price function/Dijkstra	$\Rightarrow$	$O(n \cdot log(n))$
6. Price function/Dijkstra	$\Rightarrow$	$O(n \cdot log(n))$
Overall Cost:	$\Rightarrow$	$O(\frac{n \cdot log(n) \cdot log(n)}{log(log(n))})$

Conclusion

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- Most of the time is used to find a feasible price function, we then simply use djistra.
- Is it possible to find a feasible price function in $O(n \cdot log(n))$ time?
- What about Constanst hidden in $O - Notation$?

Thank You!