An approximation scheme for Planar Graph TSP

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The general TSP is **not** approximable [BuK].

Approximation

Given a minimization problem and an optimal solution value OPT,

- the problem is α-**approximable**, if there is $\alpha > 1$ and a polynomial time algorithm that computes a solution with value at most $\alpha \cdot OPT$;
- the problem has a **polynomial-time approximation scheme (PTAS)**, if for any $\epsilon > 0$ it can be approximated in time $n^{\mathcal{O}(1/\epsilon)}$ with solution value at most $(1+\epsilon) \cdot OPT$ using that scheme.
  If $\epsilon = \frac{1}{k}$, the solution is at most k-**optimal**.

Metric TSP (M-TSP)

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The Metric TSP is a TSP with additional conditions:
For adjacent vertices u, v, w:

- $c(u, u) = 0$ (no loops)
- $c(u, v) = c(v, u)$ (symmetry)
- $c(u, w) \leq c(u, v) + c(v, w)$ (triangle inequality)

- more practically relevant than general TSP
- $\frac{3}{2}$-approximable [Christofides '76]
- $\frac{220}{219}$ is lower bound for approximation factor α [PV '06]

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Definition (Planar Graph TSP)

Given an undirected, *planar* graph G with metric cost function c and cost 1 for all (non-loop) edges, find a minimum cost tour that visits all vertices *at least* once and returns to its origin.

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- designers of planar graph PTAS considered it a step towards a PTAS for E-TSP
- major result: there is in fact a PTAS for E-TSP [Arora/Mitchell '98]

What is the goal?

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- run in $n^{\mathcal{O}(1/\epsilon)}$ time and
- compute a TSP tour of length at most $(1+\epsilon) \cdot OPT$. Since $n \leq OPT$, we have $OPT + \epsilon n \leq (1+\epsilon)OPT$, so it suffices to stay within an **additive error** of ϵn.

The algorithm

The algorithm presented here is from the paper *An Approximation Scheme for Planar Graph TSP* by GRIGNI, KOUTSOUPIAS and PAPADIMITRIOU, published in '95.

Remark: Baker's framework (last week) cannot be applied to the TSP; however, there is a PTAS for Planar Graph TSP that modifies the framework and even runs in linear time [Klein, '05/'08].

Circle separators and face-edges

Circle separators, or **simple cycle separators**, partition the graph into

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- an exterior part B
- and a circle C,

w.r.t. some size constraints.

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A **face-edge** is a *virtual* edge through a face.
We will allow separators to use such edges (to some extend).

First approach: Miller's theorem

Theorem (Simple cycle separator, Miller)

Let H be a 2-connected planar graph with n vertices, edge weights and a maximum face size d.
Then H has a simple cycle separator C consisting of $\mathcal{O}(\sqrt{nd})$ edges, the interior and exterior of C both have at most $\frac{2}{3}n$ vertices and C can be found in polynomial time.

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- attempts using this resulted in $n^{\mathcal{O}((log^2 n)/\epsilon^2)}$ complexity
- a "more customizeable" theorem was needed

Final approach: A novel separator theorem

Theorem (Simple cycle separator with vertex weights)

Let H be a connected planar graph with n vertices, vertex weights and parameter f with $1 \leq f \leq \sqrt{n}$.
Then H has a simple cycle separator C through $\mathcal{O}(n/f)$ vertices, the interior and exterior of C both have at most $\frac{2}{3}$ of the total weight, C uses at most f face-edges and C can be found in polynomial (nearly linear) time.

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- f controls trade-off between size of C and amount of face-edges in C
- choice of f is crucial for efficiency of the algorithm

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- more flexible separator parametrization
- occurence of heavy-weighted vertices can be limited in exterior/interior parts

Decomposition - Principle

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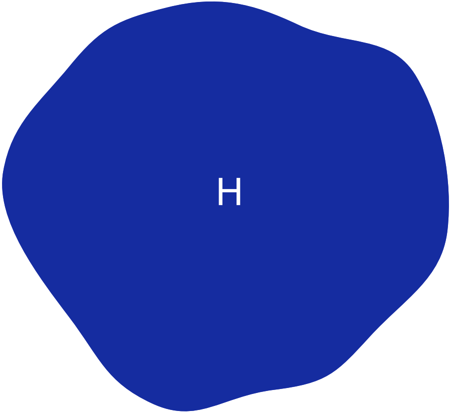
Steps:

1. Applying the separator
2. Contracting path segments
3. Removing face-edges
4. Weighting constraint points
5. Repeating decomposition recursively

Decomposition

1. Applying the separator



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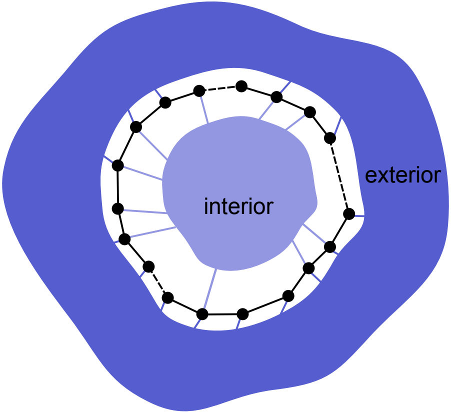
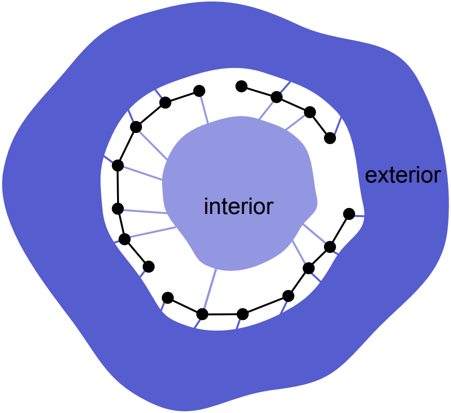
1. Applying the separator



Decomposition

2. Removing face-edges

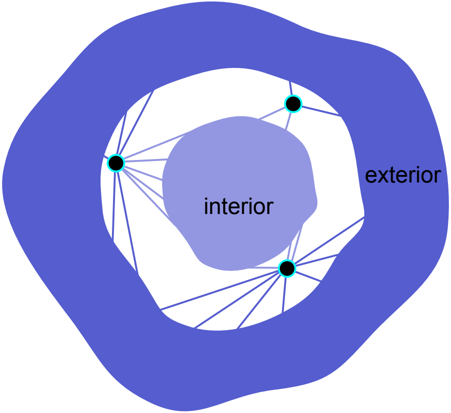
Removing at most f face-edges results in at most f path segments.



Decomposition

3. Contracting path segments

   The resulting nodes are called **constraint points (CPs)**.
   Result: H'



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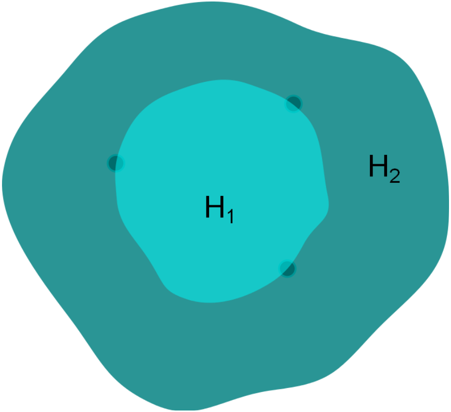
   Contracted subgraphs H_1 and H_2 share the new CPs created in this step.



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   Important: CPs from *previous* decomposition steps are also re-weighted!

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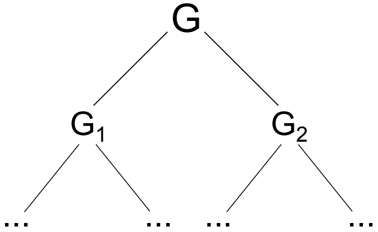
Decomposition tree

The binary **decomposition tree** $\mathcal{T}$ stores the decomposition results.



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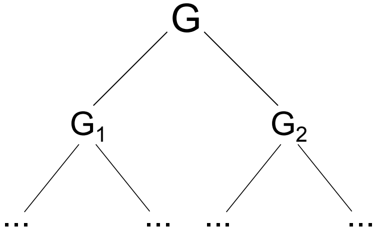

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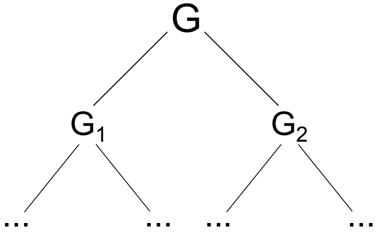
Stopping size:
$S = \Theta(f^2) = \Theta((log^2 n)/\epsilon^2)$
If $|H| \leq S$, stop recursion - H is a leaf.



Tree size and complexity

Some observations:

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$W(H_i) \leq \frac{2}{3}W(H) + f \cdot \frac{W(H)}{6f} = \frac{5}{6}W(H)$. □

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Proof.

Suppose $W(H_i) = x \cdot \frac{W(H)}{6f}$, with x being #CPs in H_i.
Then $x \cdot \frac{W(H)}{6f} \leq \frac{5}{6}W(H) \Leftrightarrow x \leq 5f$. □

Tree size and complexity

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Thus $\mathcal{T}$ has polynomial size, independent of ϵ.

Each decomposition step can be done in polynomial time, so the overall complexity of decomposition is $n^{\mathcal{O}(1)}$.

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Steps (for all inner nodes H):

1. Approximating leaf graph solutions
2. Building solutions in H'
3. Extending H'-solutions to H-solutions
4. Constructing the tour in root G

Path Covering

Definition (Path Cover)

Given a graph H and a set of chosen CPs in H, find the minimum length collection of paths that covers all vertices of H using each chosen CP as a path endpoint exactly once.

A path cover with no endpoints (0 is even) shall be a cycle.

Storing solutions

Every path has two endpoints, so we only consider even subsets of CPs in H.

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Let $c(H) \leq 5f$ be #CPs in H. We identify a choice of CPs with a binary array $x \in \{0,1\}^{c(H)}$ and store the corresponding solution in $T(H)[x]$.

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$T(H)$ is a table of size $2^{c(H)-1} \leq 2^{5f-1} = 2^{5\Theta((logn)/\epsilon)-1} = n^{\mathcal{O}(1/\epsilon)}$.

Approximation in leaf graphs

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Using a simpler approximation scheme based on the Lipton-Tarjan separator theorem and so-called *nonserial dynamic programming*, one can approximate path covers for the leaf graphs in time $2^{\mathcal{O}(\sqrt{|L|})} = n^{\mathcal{O}(1/\epsilon)}$.

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The situation: For an inner graph H, we want to find a path cover approximation that is as small as possible.

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We will look at this process for a given x and H.

Merging in H'

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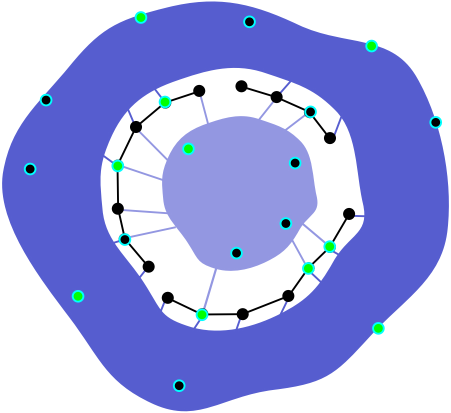
   New CP shall be endpoint in x' iff. path segment contained odd number of endpoints in x.



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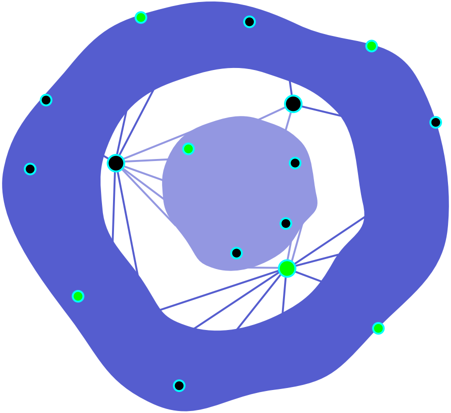

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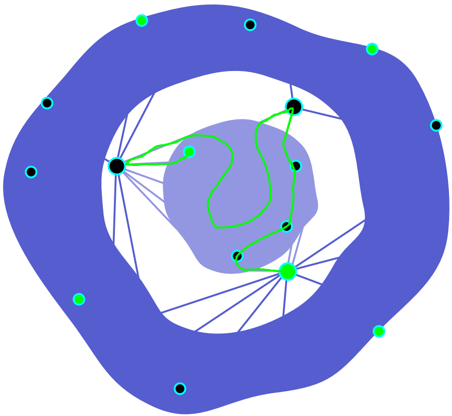
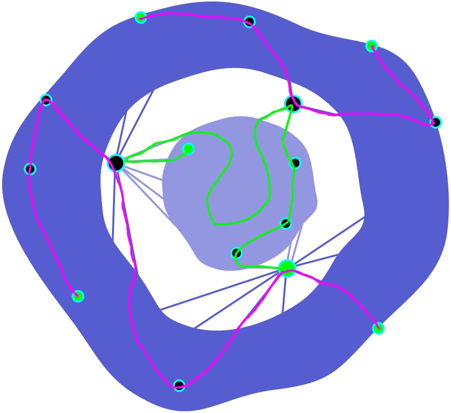
   Approach: Choose x_1 that matches x'..



Merging in H'

2. Find solutions x_1 and x_2 for H_1 and H_2 such that their combination in H' is a minimal length solution matching x'

   Approach: Choose x_1 that matches x'.. .. then pick the shortest matching x_2.



Extending to H

3. Extend the solution in H' to a solution in H

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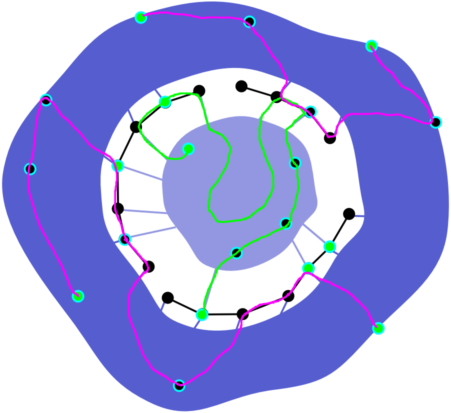

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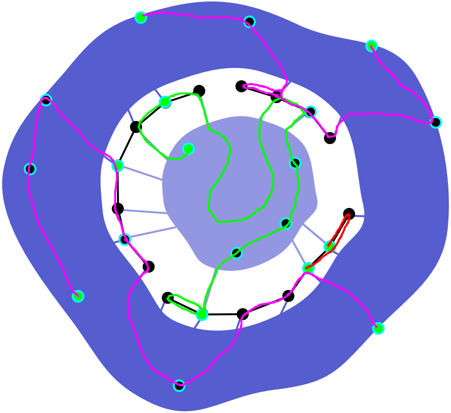
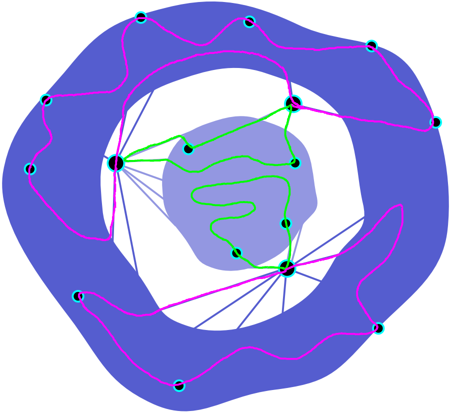


Special case: Input graph G

G' does not contain "old" CPs - build solution for empty x: a tour.

Since G is connected and all vertices are covered, it's possible to connect multiple tours to get a single tour.



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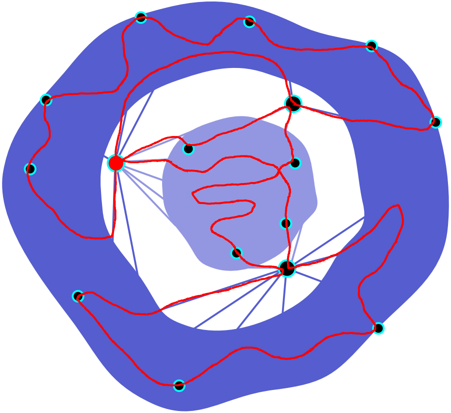

Runtime summary

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- Single leaf approximation step: $n^{\mathcal{O}(1/\epsilon)} \cdot n^{\mathcal{O}(1/\epsilon)}$
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- Approximation complexity in total: $n^{\mathcal{O}(1)} \cdot n^{\mathcal{O}(1/\epsilon)} = n^{\mathcal{O}(1/\epsilon)}$

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- Approximation complexity in total: $n^{\mathcal{O}(1)} \cdot n^{\mathcal{O}(1/\epsilon)} = n^{\mathcal{O}(1/\epsilon)}$

Total runtime: $n^{\mathcal{O}(1/\epsilon)}$

Approximation error

The additive error can be shown to be at most ϵn, but this requires more detailed analysis than we can do here.

All good things come to an end

Thank you for your attention! Questions..?