Shortest Paths in Directed Planar Graphs with Negative Lengths

Authors: Philip N. Klein, Shay Mozes, Oren Weimann
presented by Sebastian Addicks

19. November 2013

Find an $O(n\ log^2 n)$-time, linear-space algorithm to find shortest paths in planar directed graphs with negative lengths.

requirement: no `negative circles`

Applications

- **perfect matching**
  $\rightarrow$ matching which contains n/2 edges

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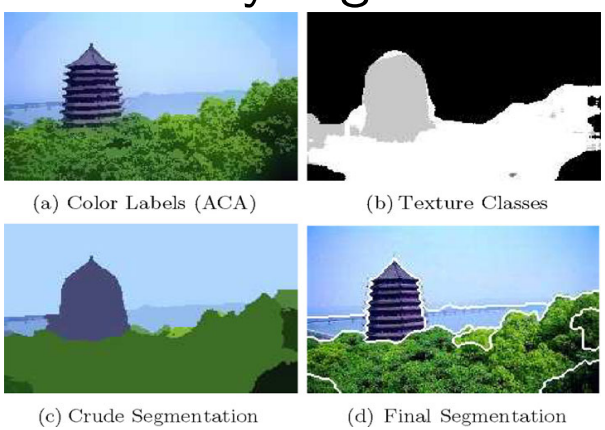
  $\rightarrow$ for a point in image1 find the corresponding one in image2
- **image segmentation**
  $\rightarrow$ identify segments of an image



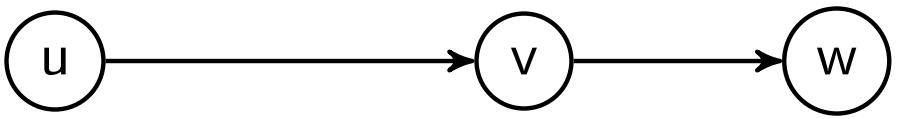
(a) Color Labels (ACA) (b) Texture Classes

(c) Crude Segmentation (d) Final Segmentation

- divide and conquer $\Rightarrow$ separator
- transform the negative lengths $\Rightarrow$ price function

Price Function and Reduced Lengths

Price function ϕ to eliminate negative lengths, $\phi : N \longrightarrow \mathbb{R}$
$l_\phi(uv) = l(uv) + \phi(u) - \phi(v)$



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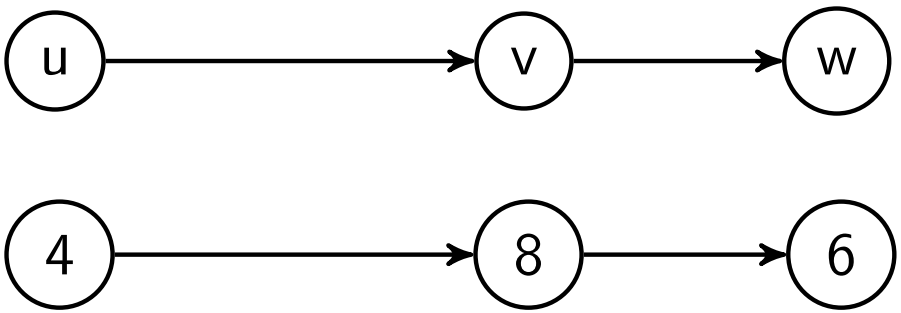

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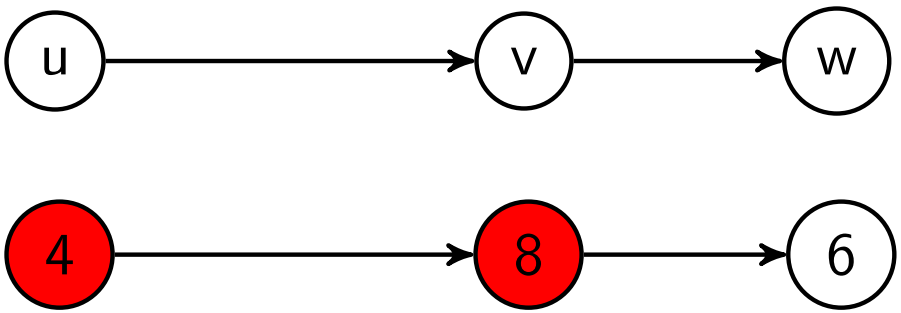

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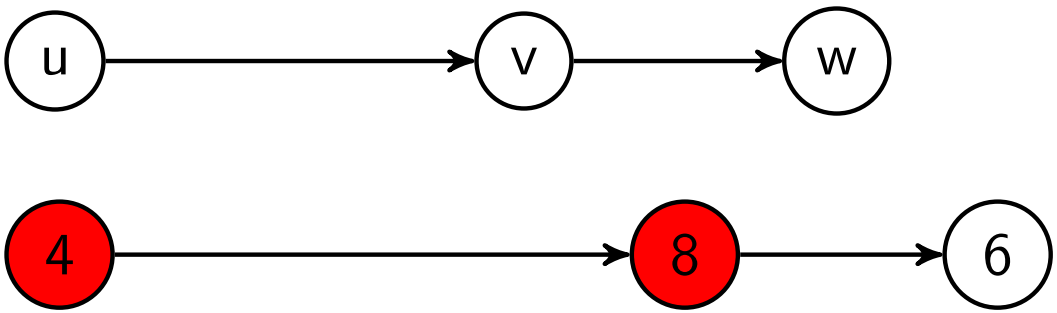

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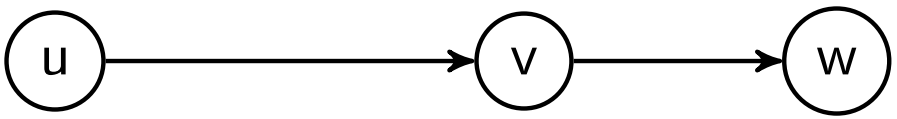

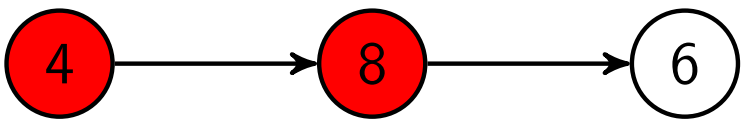

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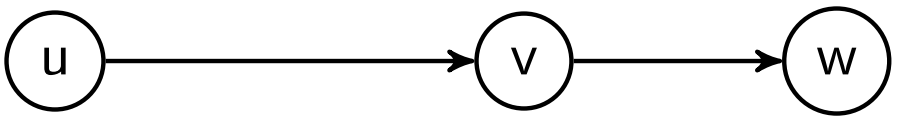

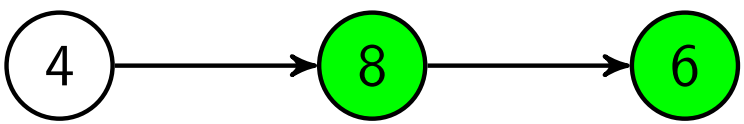

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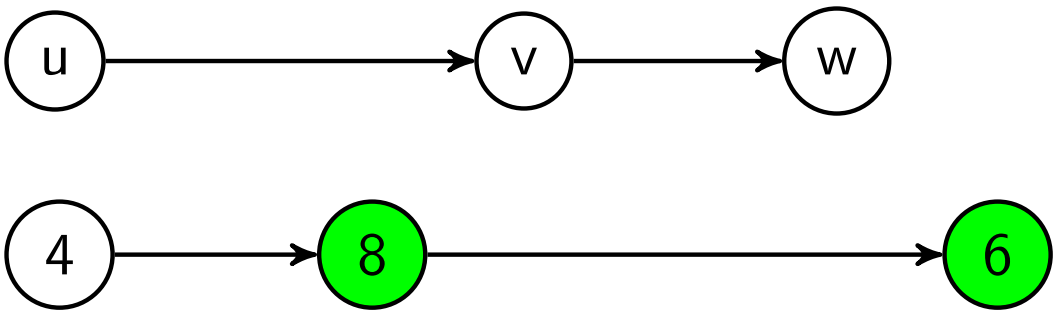

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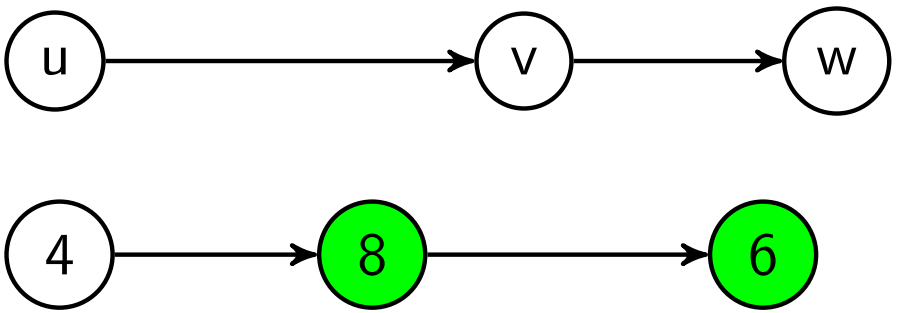

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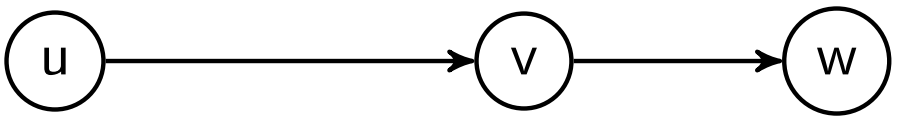

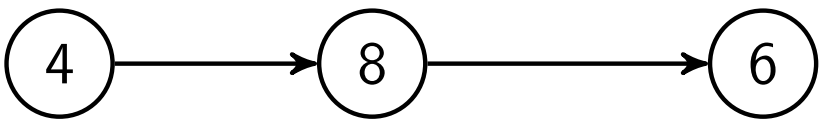

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$\Longrightarrow \phi$ reserves shortest paths

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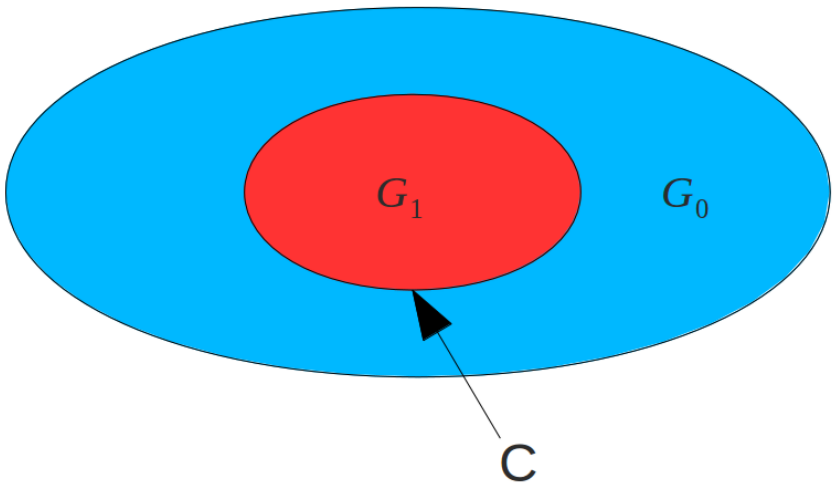

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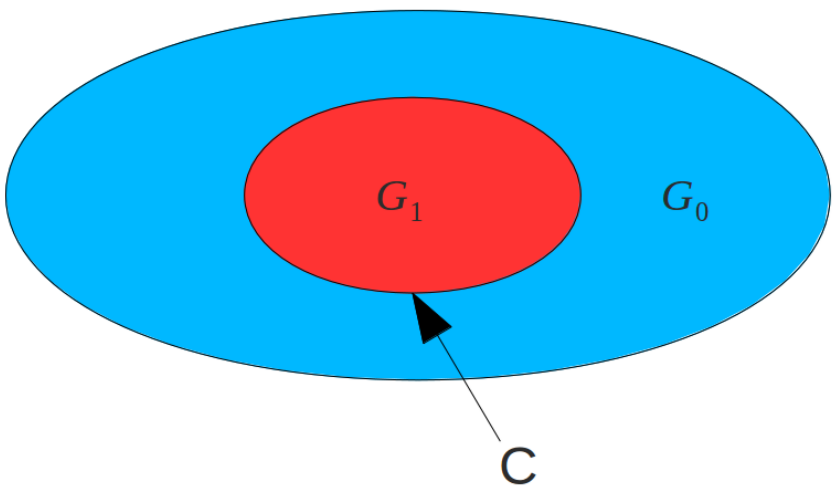

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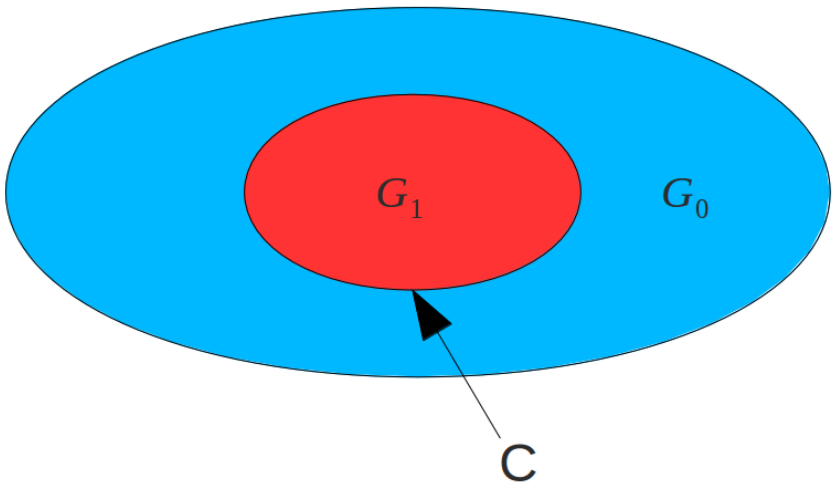

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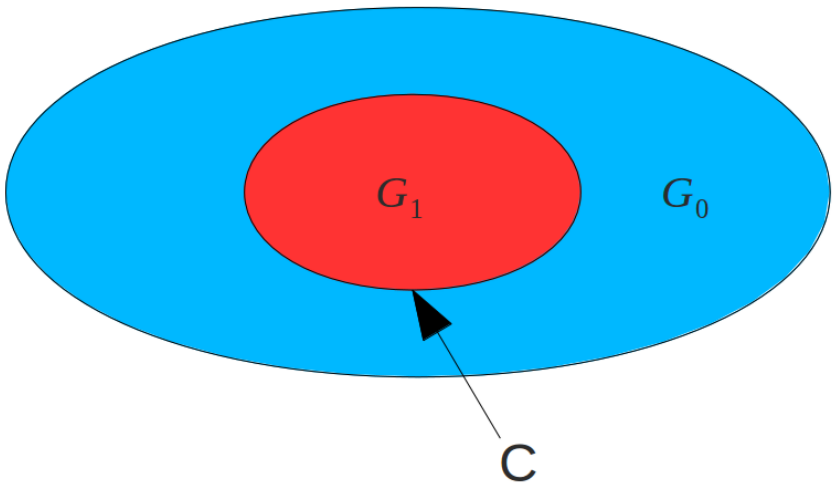

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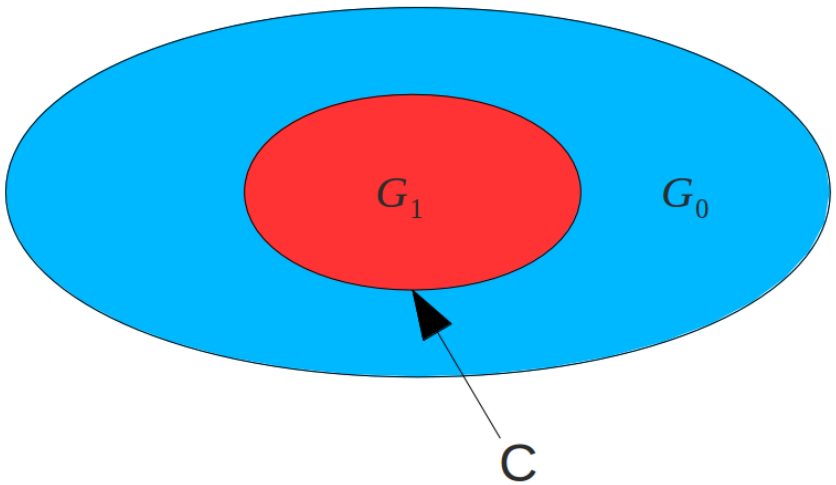

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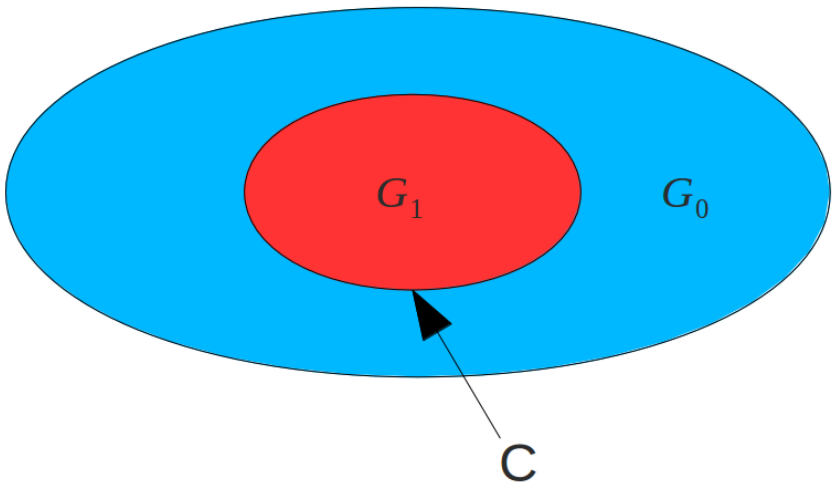

G_1 and G_0 denote the internal and external part of G with respect to C

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   - find a jordan separator C
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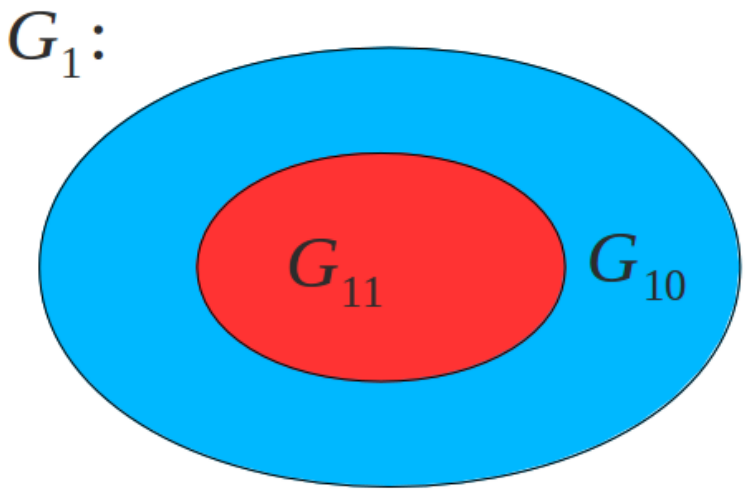
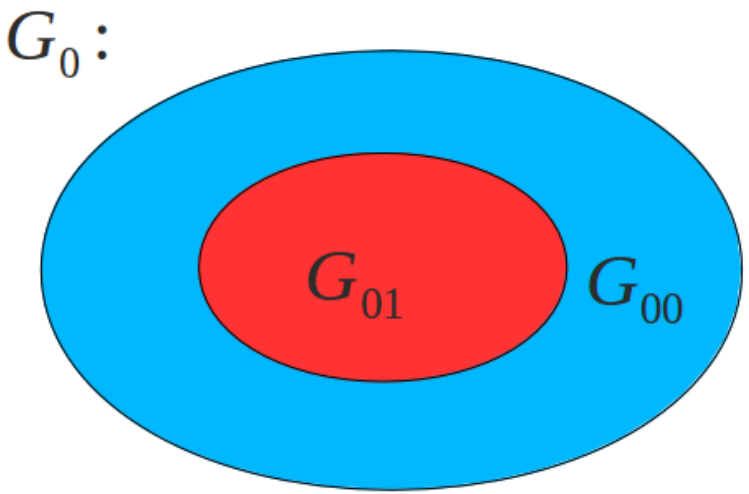
2. recursive call
   - for $i = 0, 1$: let d_i shortest-paths(G_i, r)





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- stored in δ_1 and δ_0 such that $\delta_i[uv]$ is the u-to-v distance in G_i

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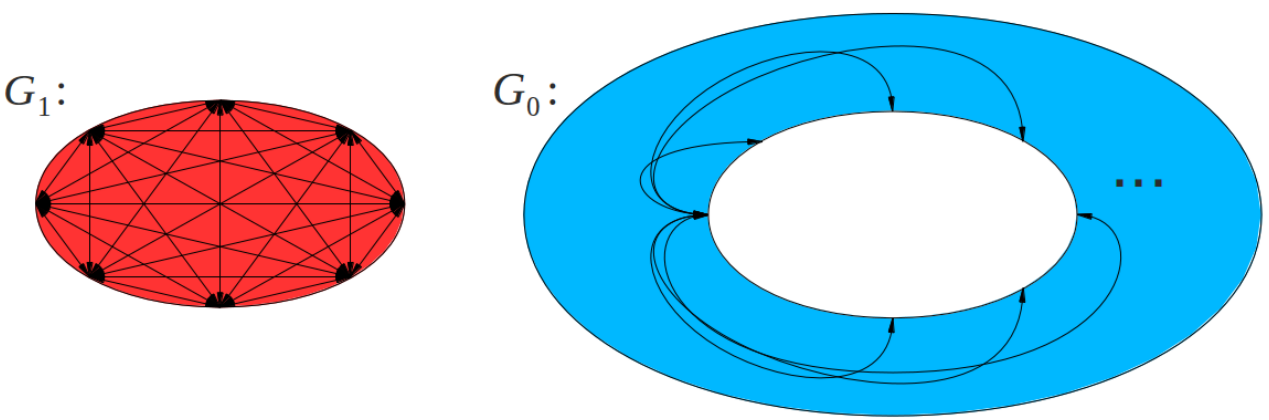

4. single-source inter-part boundary distances
   $\longrightarrow O(n^{3/2})$
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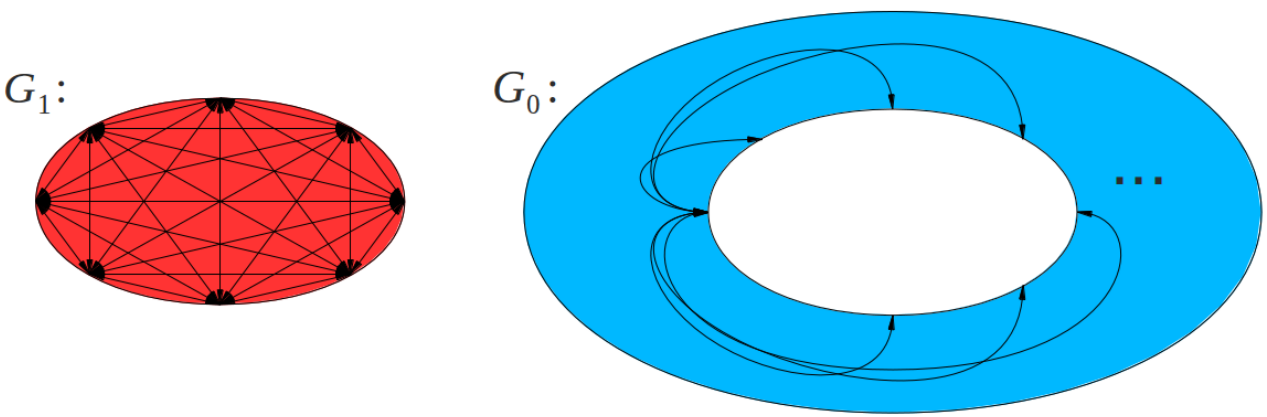

5. single-source inter part distances
   - use d_i and B and Dijkstra's Algorithm to compute d'_i such that $d'_i[v]$ are the r-to-v distances in G for every node v in G_i

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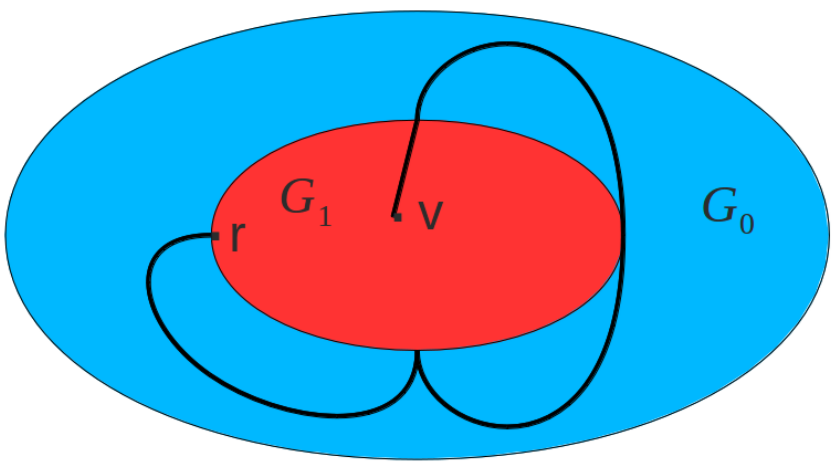

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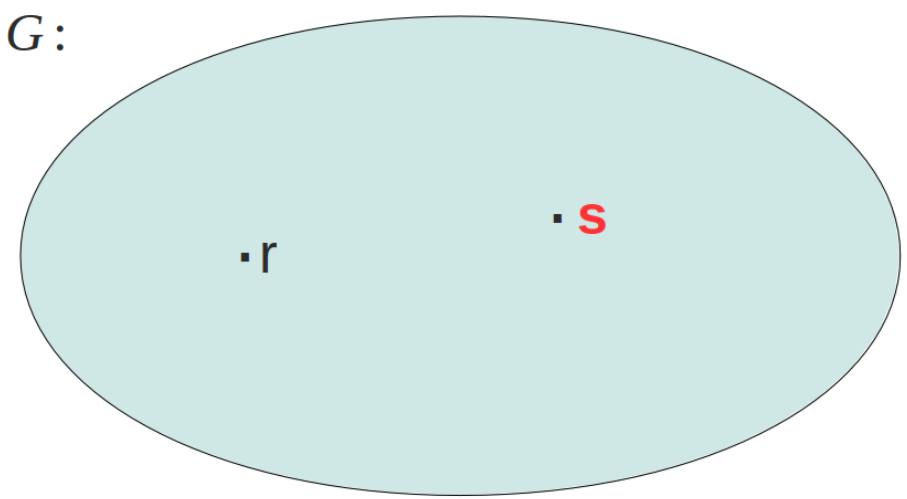

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step 5 and 6 can be done in $O(n)$

to get $O(n\ log^2 n)$ as overall time improve step 4

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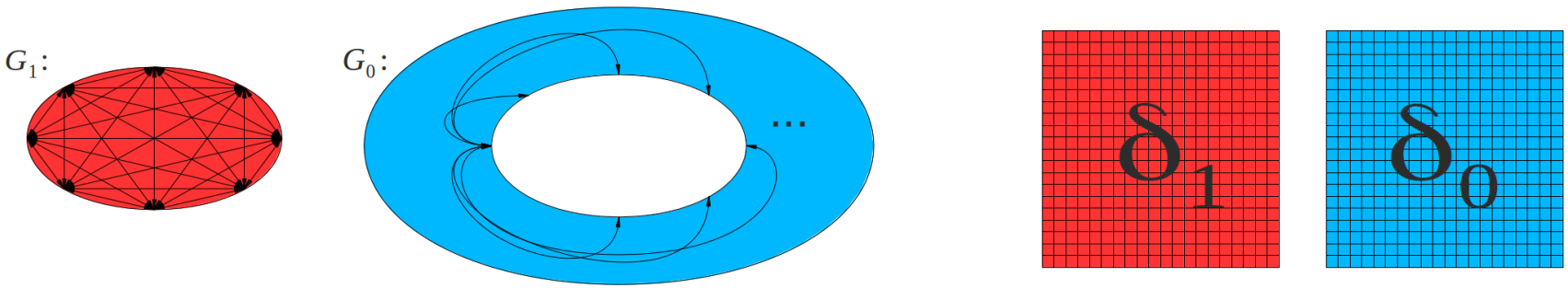

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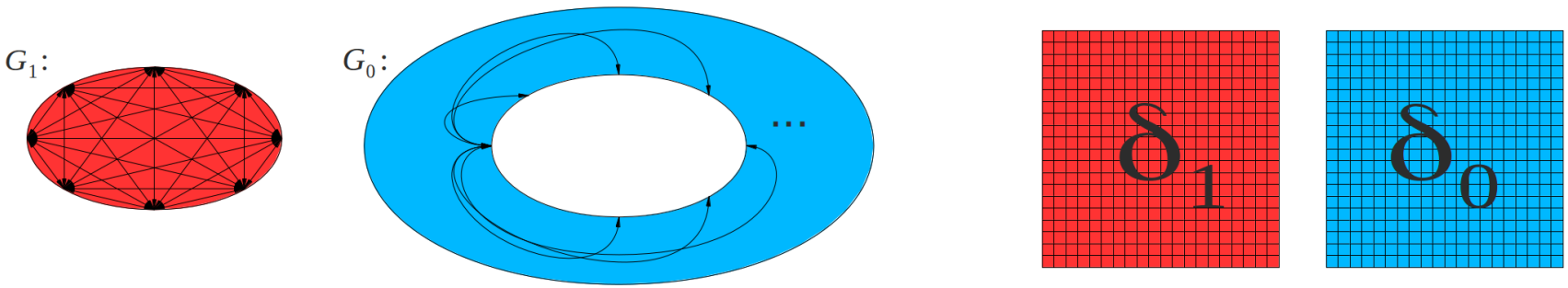

$$\forall v \in C \; d(v) := min_{u \in C} \left\{ \begin{array}{l} d(u) + \delta_1(u, v) \\ d(u) + \delta_0(u, v) \end{array} \right\}$$

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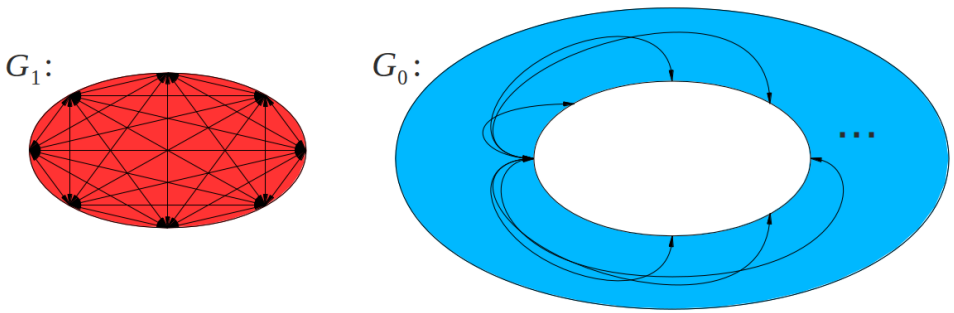

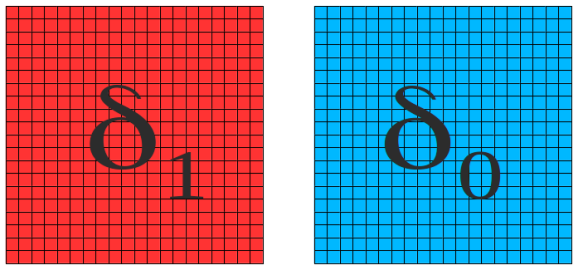

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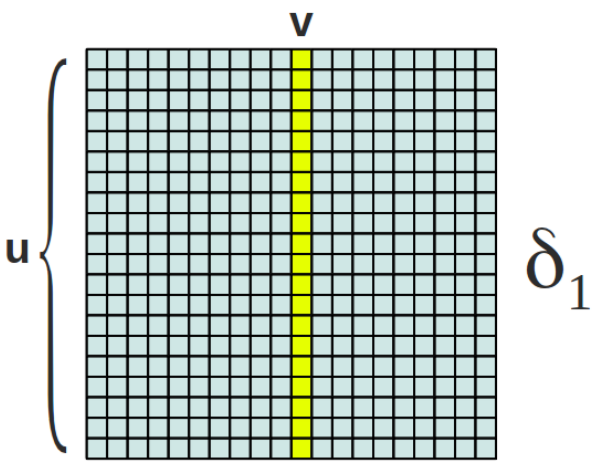

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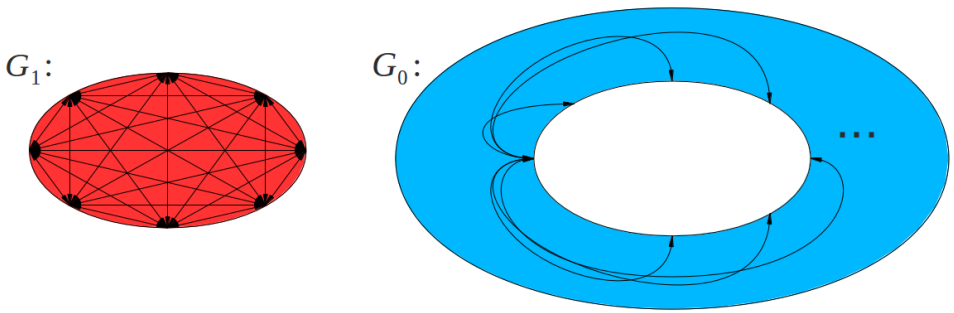

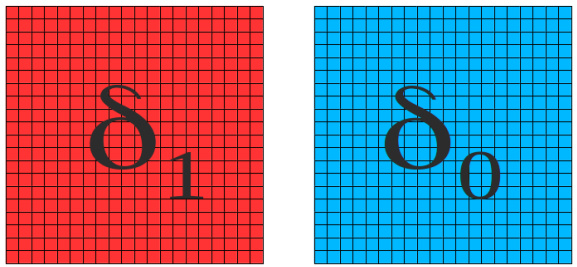

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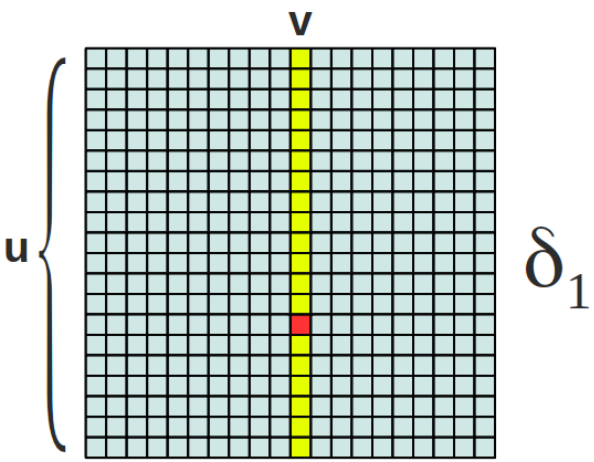

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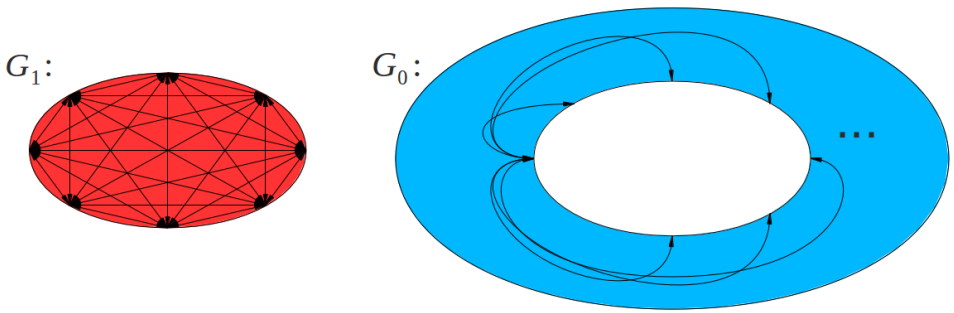
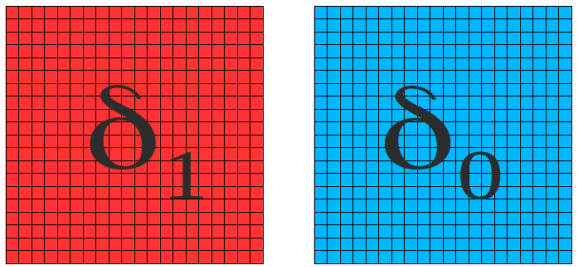
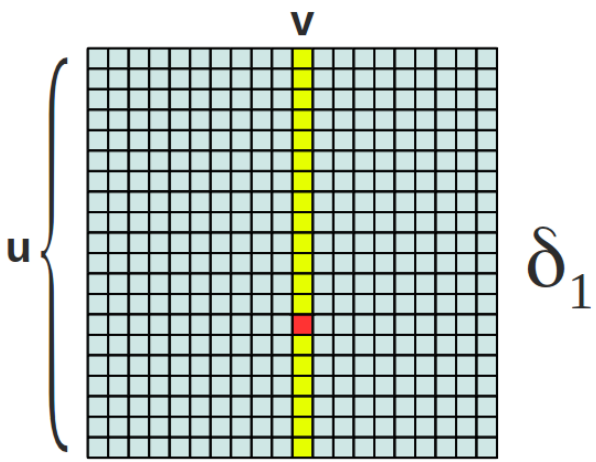




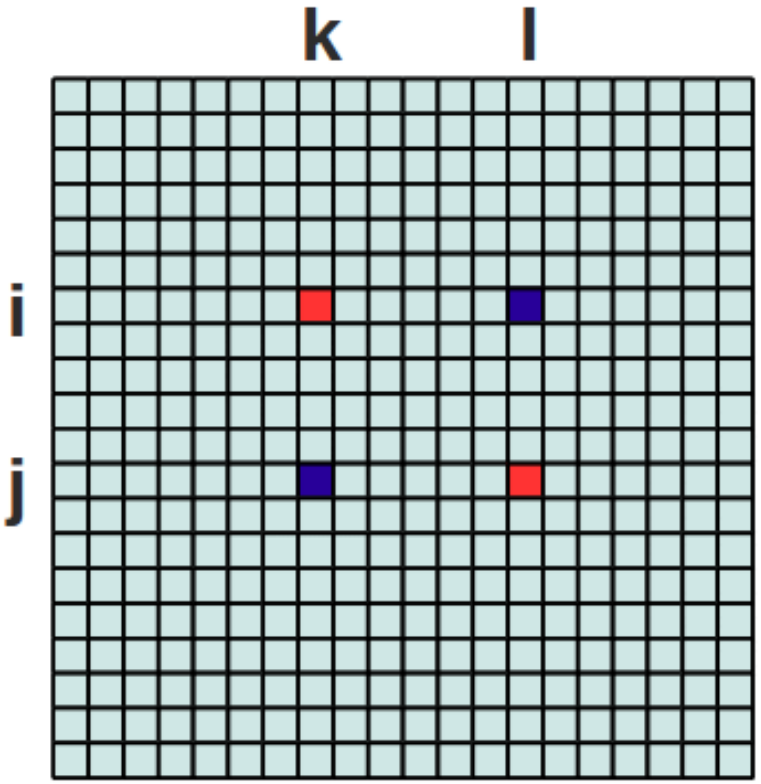
$$\forall v \in C \; d(v) := min_{u \in C} \left\{ \begin{array}{l} d(u) + \delta_1(u, v) \\ d(u) + \delta_0(u, v) \end{array} \right\}$$



finding column minima!

Monge Matrices

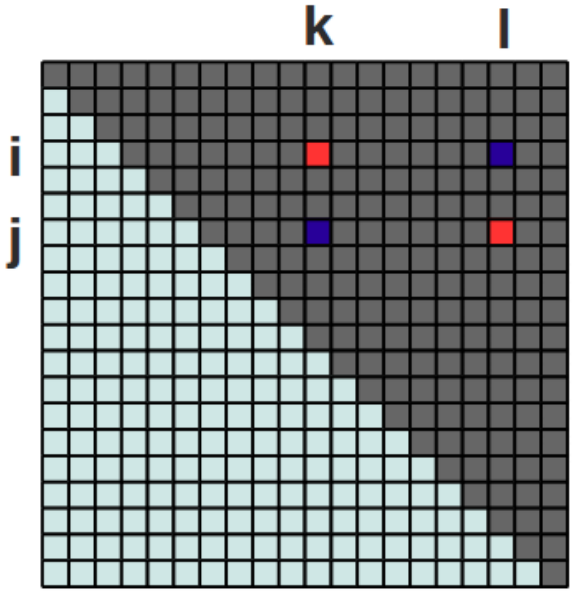
A:



- a matrix A is Monge if for any $i \leq j, k \leq l$ $A(i,k) + A(j,l) \geq A(i,l) + A(j,k)$

Monge Property

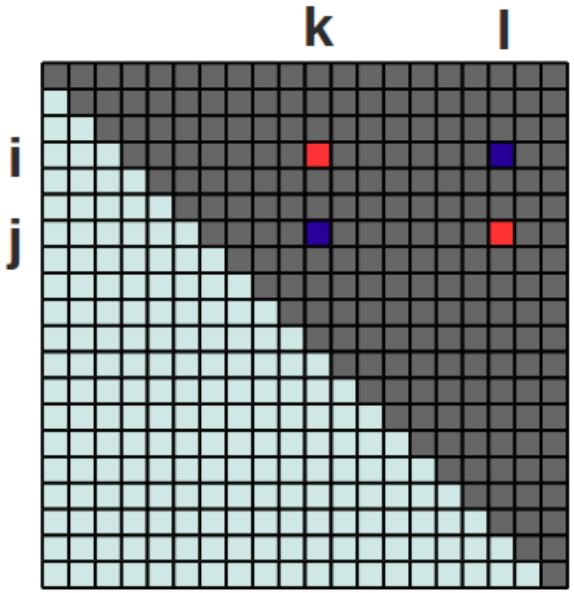
δ :

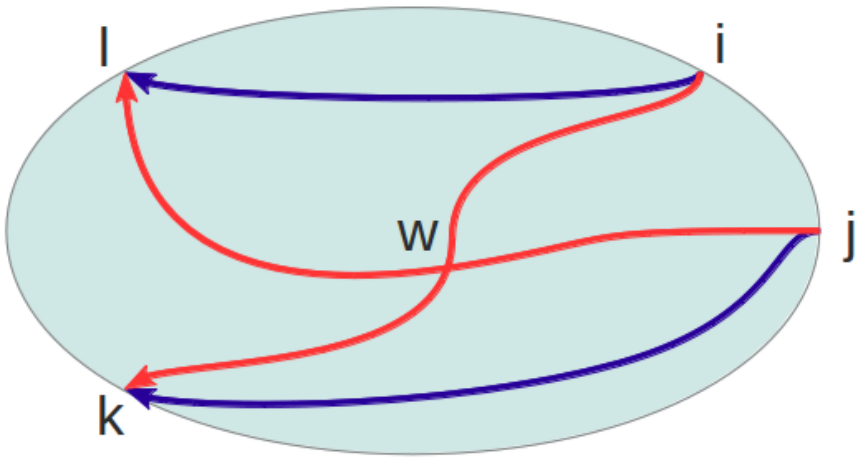


$$A(i,k) + A(j,l) \geq A(i,l) + A(j,k)$$

Monge Property

δ :





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Monge Property

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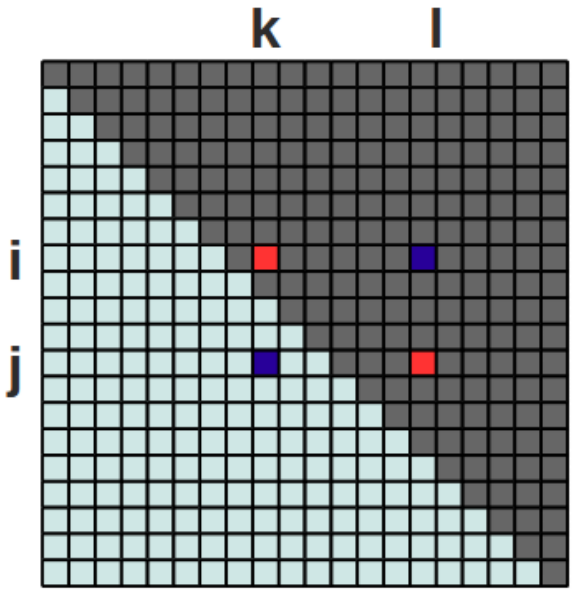

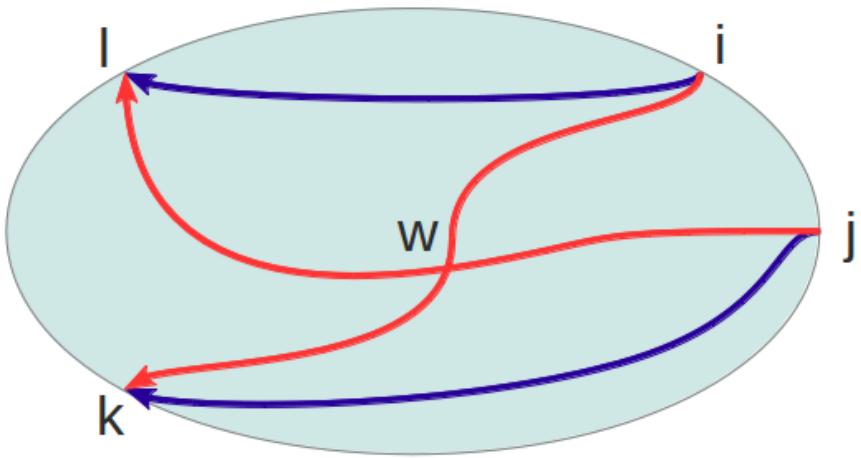

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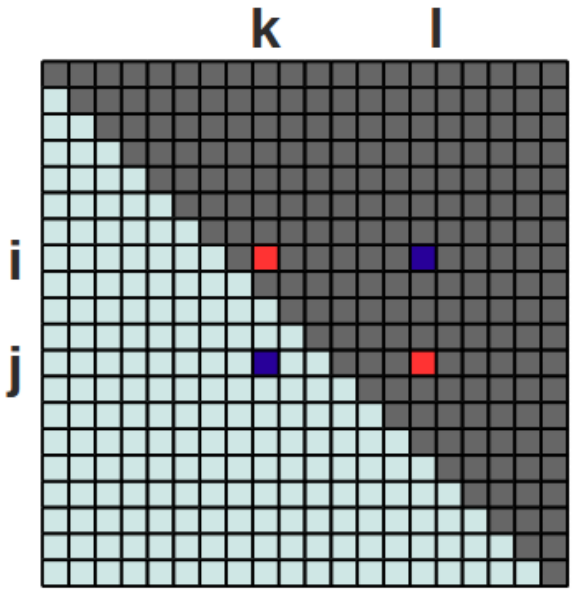
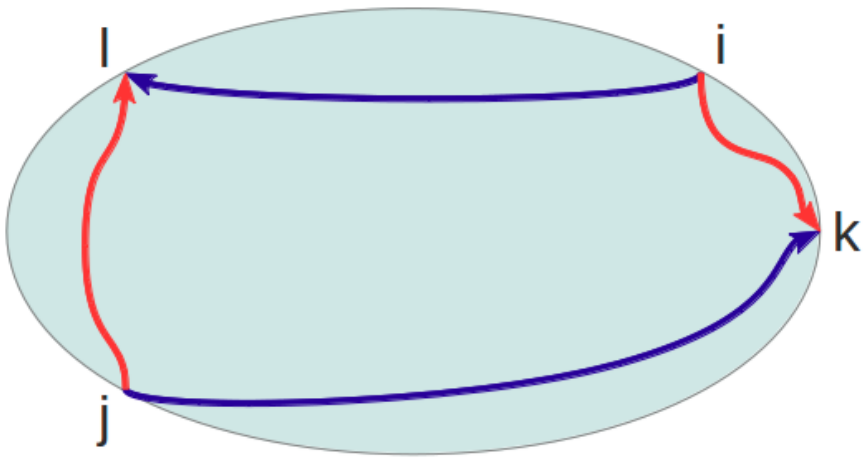
Monge Property

δ :





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Monge Property

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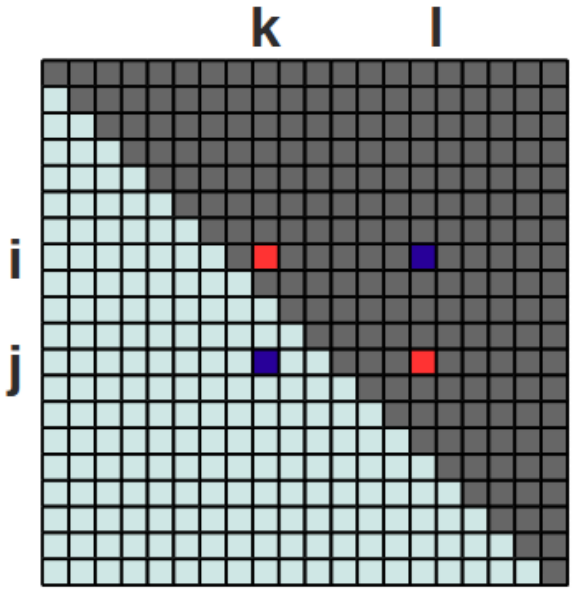

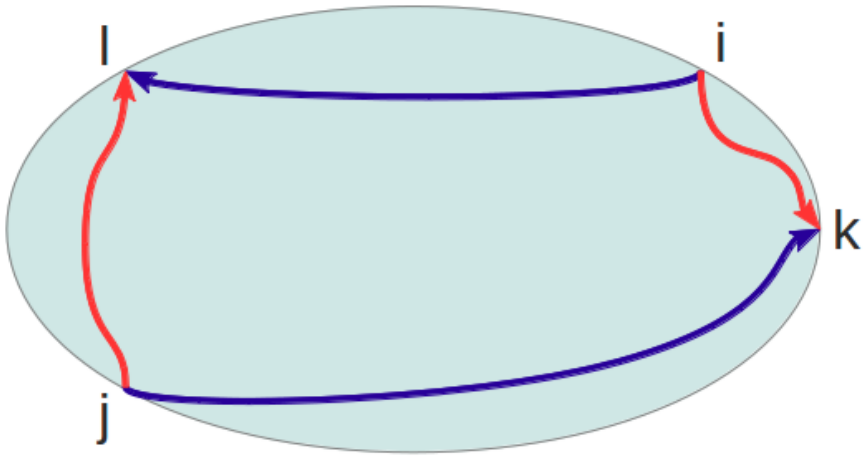

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Monge Property

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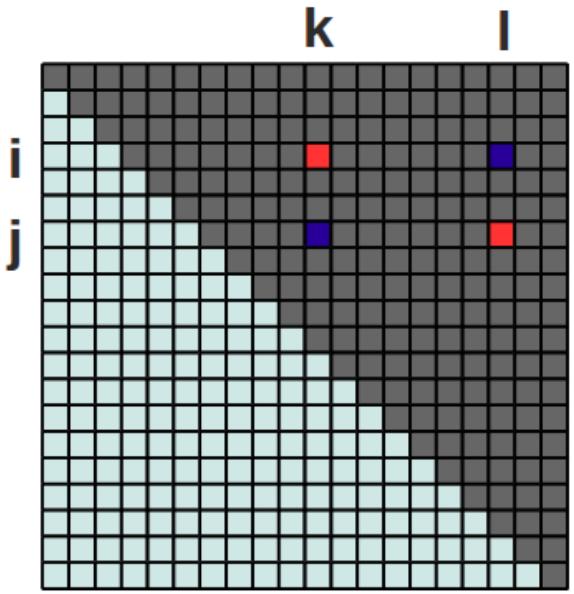
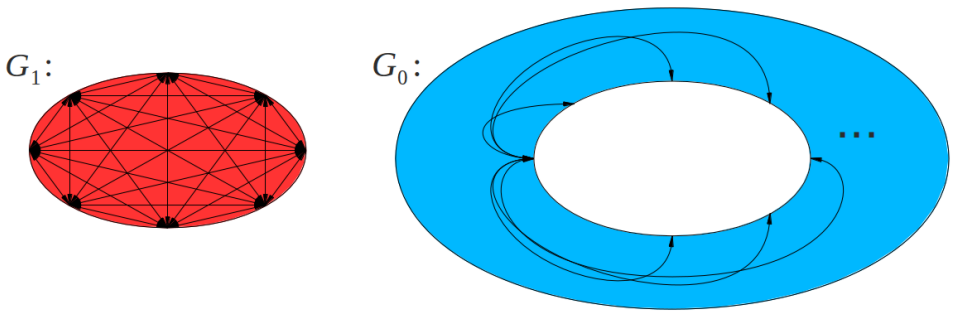
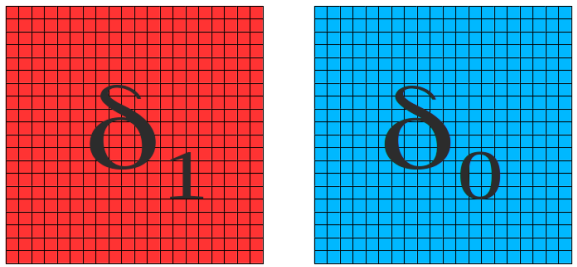


- it is possible to find Column Minima in a triangular Monge matrix in $O(n\alpha(n))$ [Klawe-Kleitman 1990]

4. single-source inter-part boundary distances





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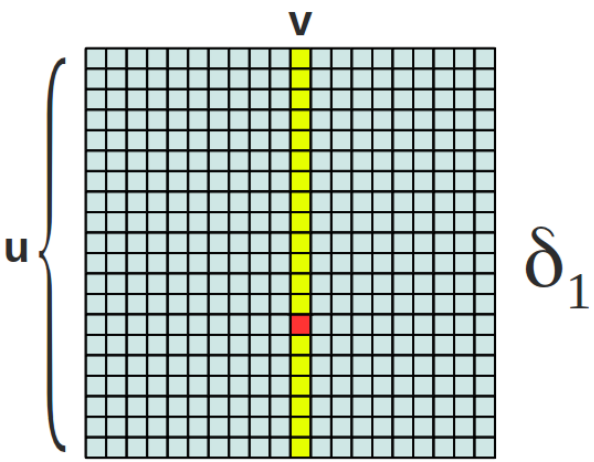

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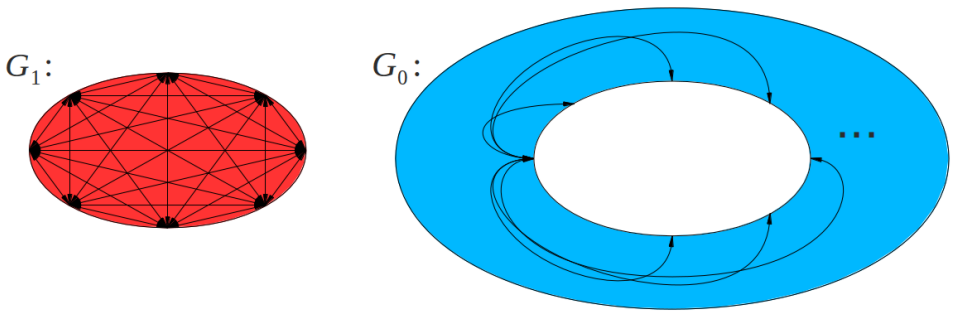

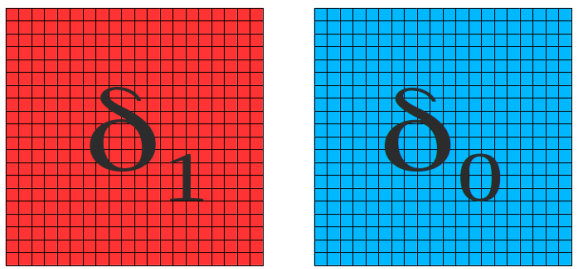

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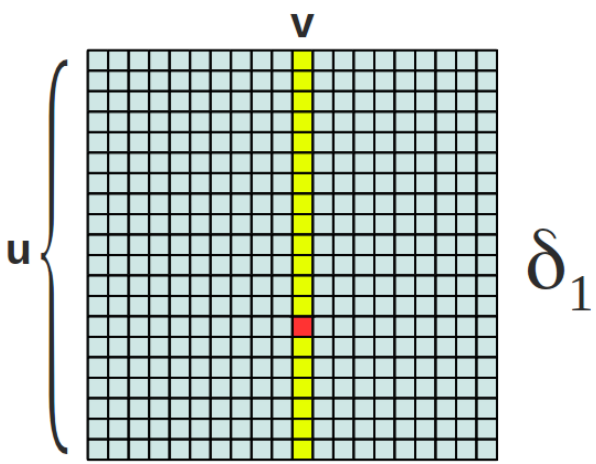

- each iteration takes $O(\sqrt{n}\alpha(\sqrt{n}))$
- $\sqrt{n}$ iterations

  overall: $O(n\alpha(n))$

1. initialization $\longrightarrow O(n)$
2. recursive call $\longrightarrow log(n)$ levels
3. MSSP $\longrightarrow O(n\ log(n))$
4. Bellman Ford $\longrightarrow O(n^{3/2})$
5. feasible price function, Dijkstra $\longrightarrow O(n\ log(n))$
6. feasible price function, Dijkstra $\longrightarrow O(n\ log(n))$

---

step 5 and 6 can be done in $O(n)$

1. initialization $\longrightarrow O(n)$
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6. feasible price function, Dijkstra $\longrightarrow O(n\ log(n))$

---

step 5 and 6 can be done in $O(n)$

overall: $O(n\ log^2 n)$

Sources:

[Image 1: image segmentation]
http://spie.org/x8899.xml?pf=true&ArticleID=x8899

http:
//www.leda-tutorial.org/de/offiziell/ch05s02s06.html

Thank You