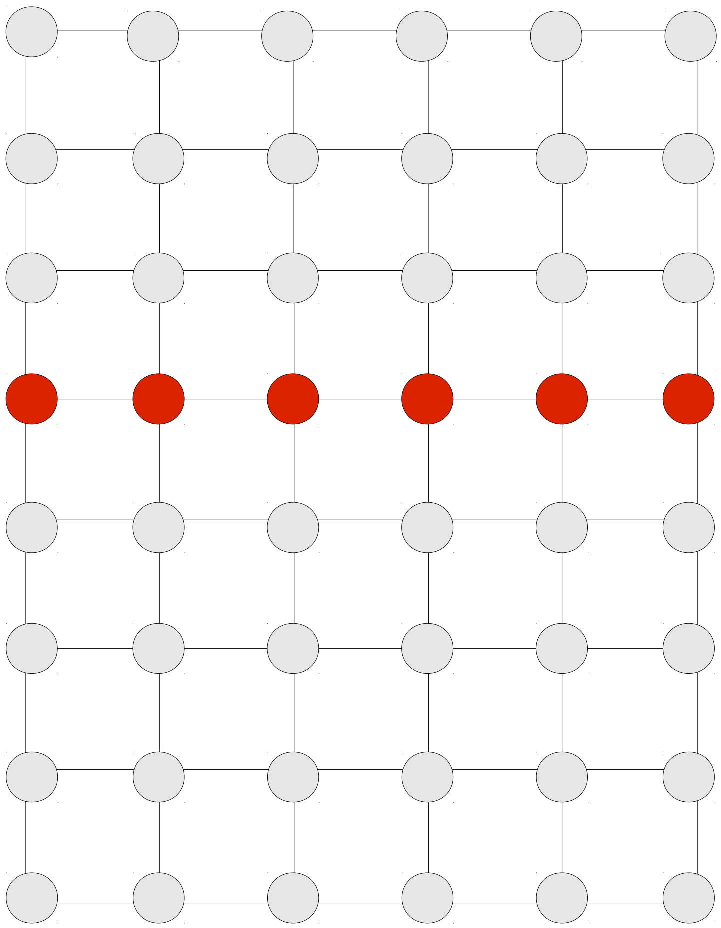
The Planar Separator Theorem

by Jan Dreier

- **Separator:** A set of vertices, that when removed divide the graph in two non-connected components.

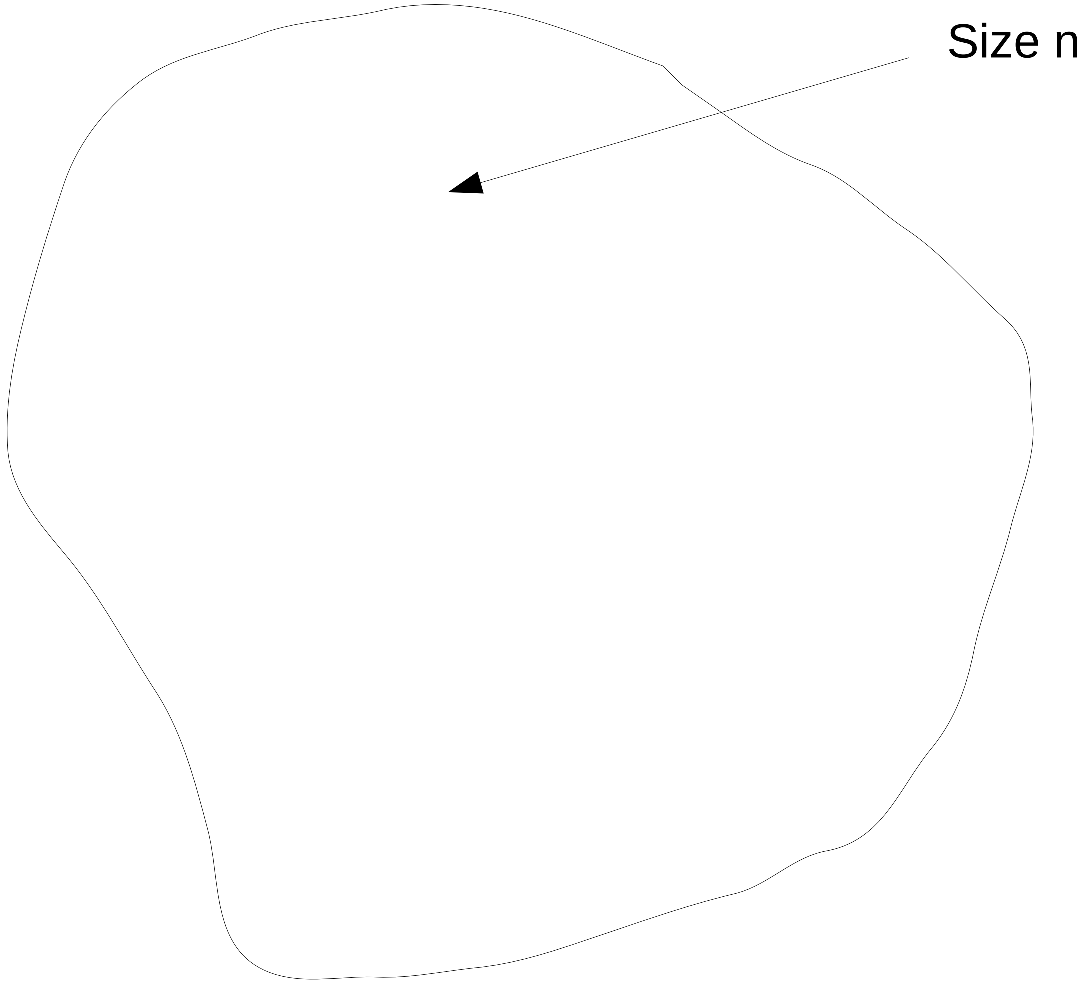
- **Goals:**
  - Equally-sized components
  - Small Separator

- **Applications:**
  - "Divide-and-conquer" Algorithms



- **Application-Example**: Approximation of Maximum Independent Set

Problem is NP-complete



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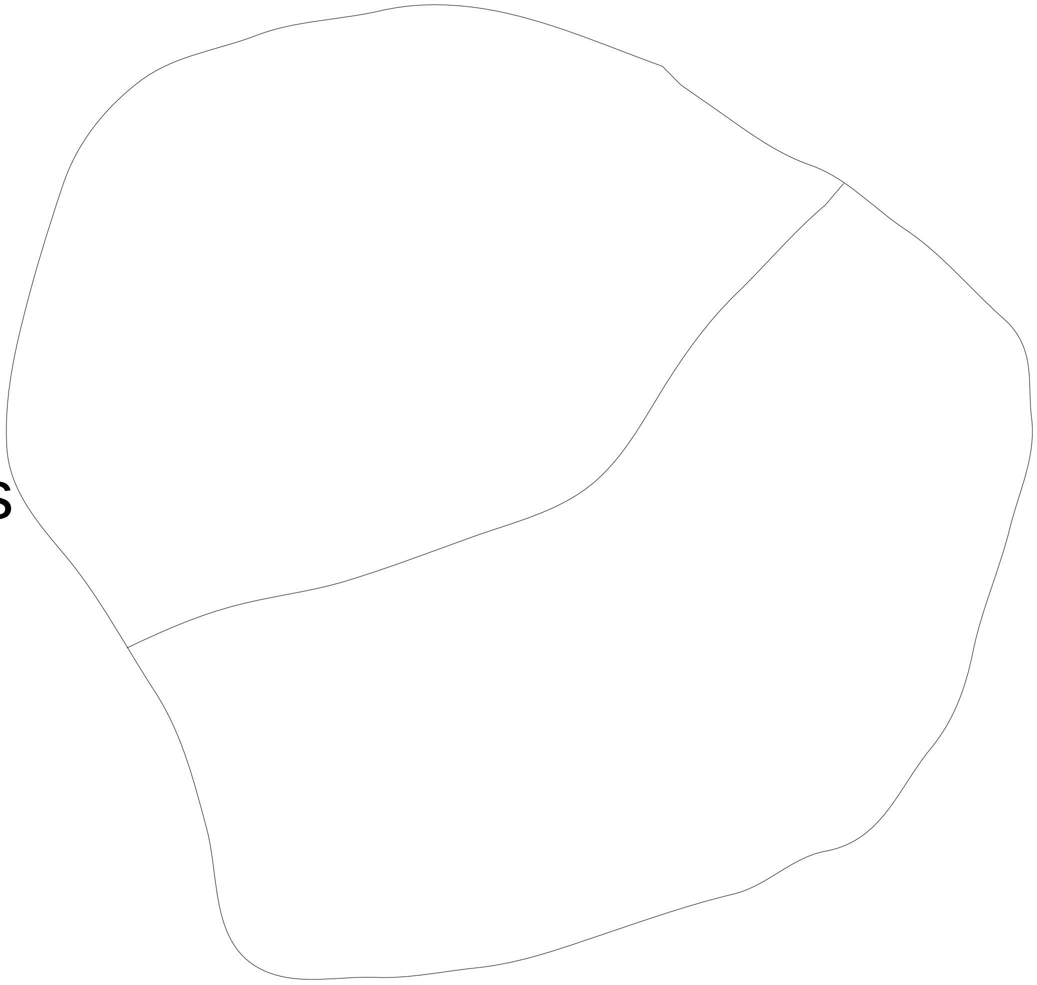
Separate into small connected components



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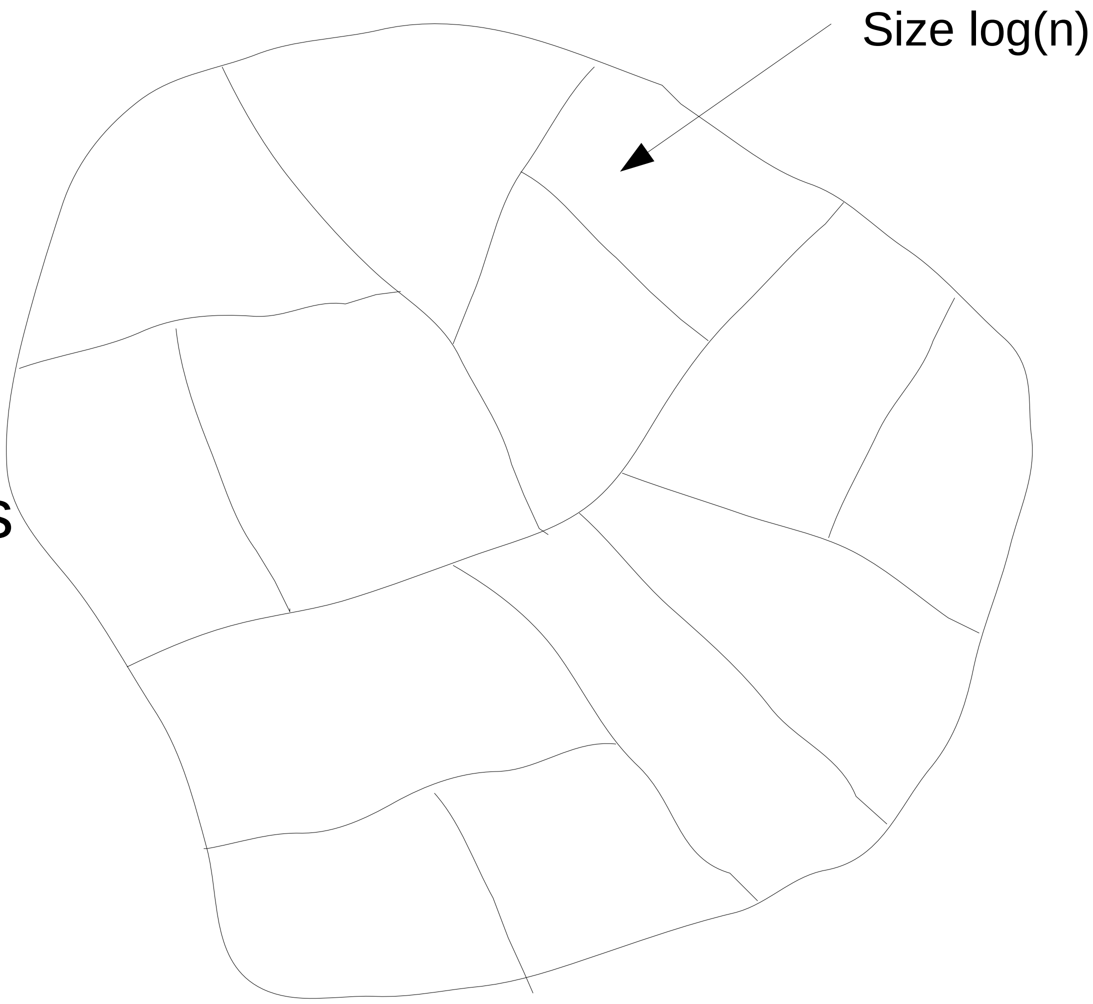

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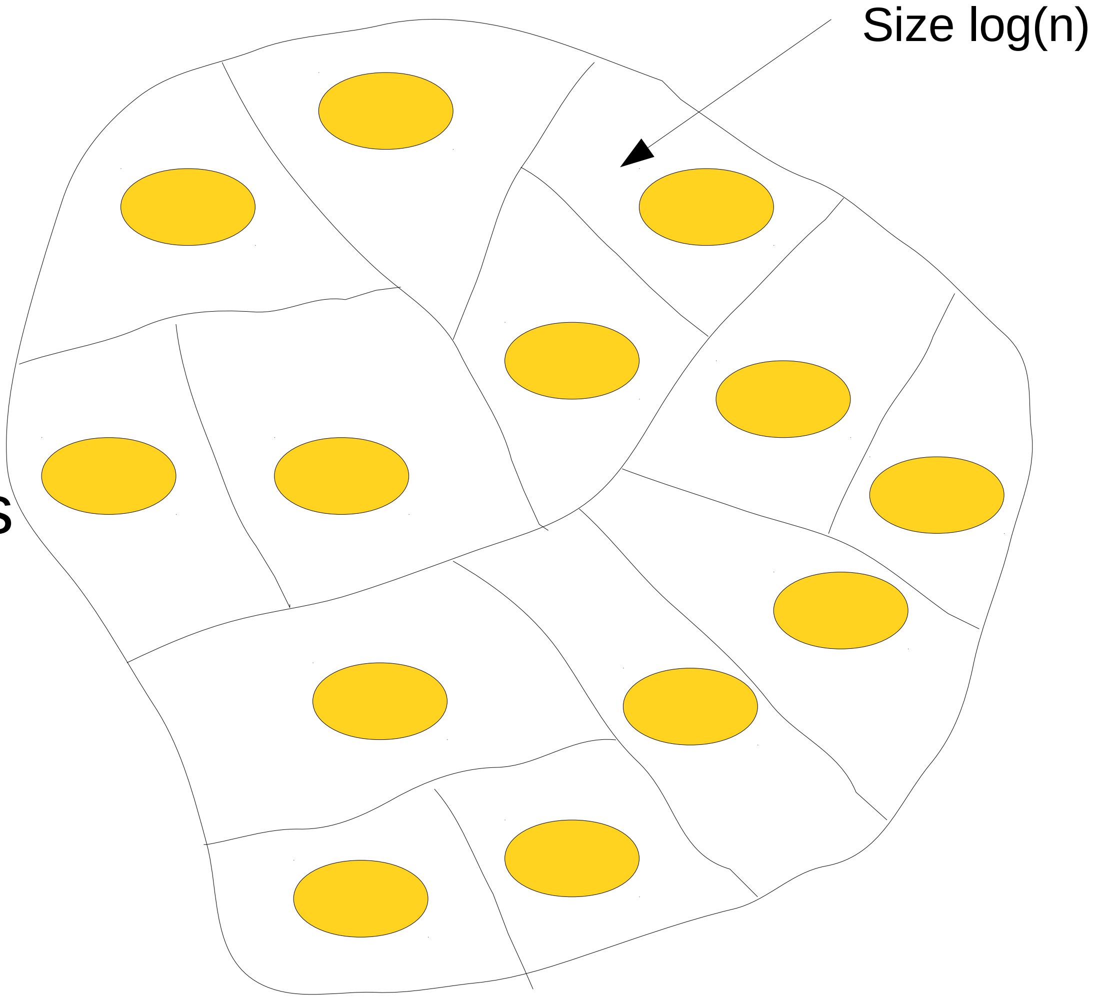
Problem is NP-complete

Separate into small connected components

Find Maximum Independent Set in each component



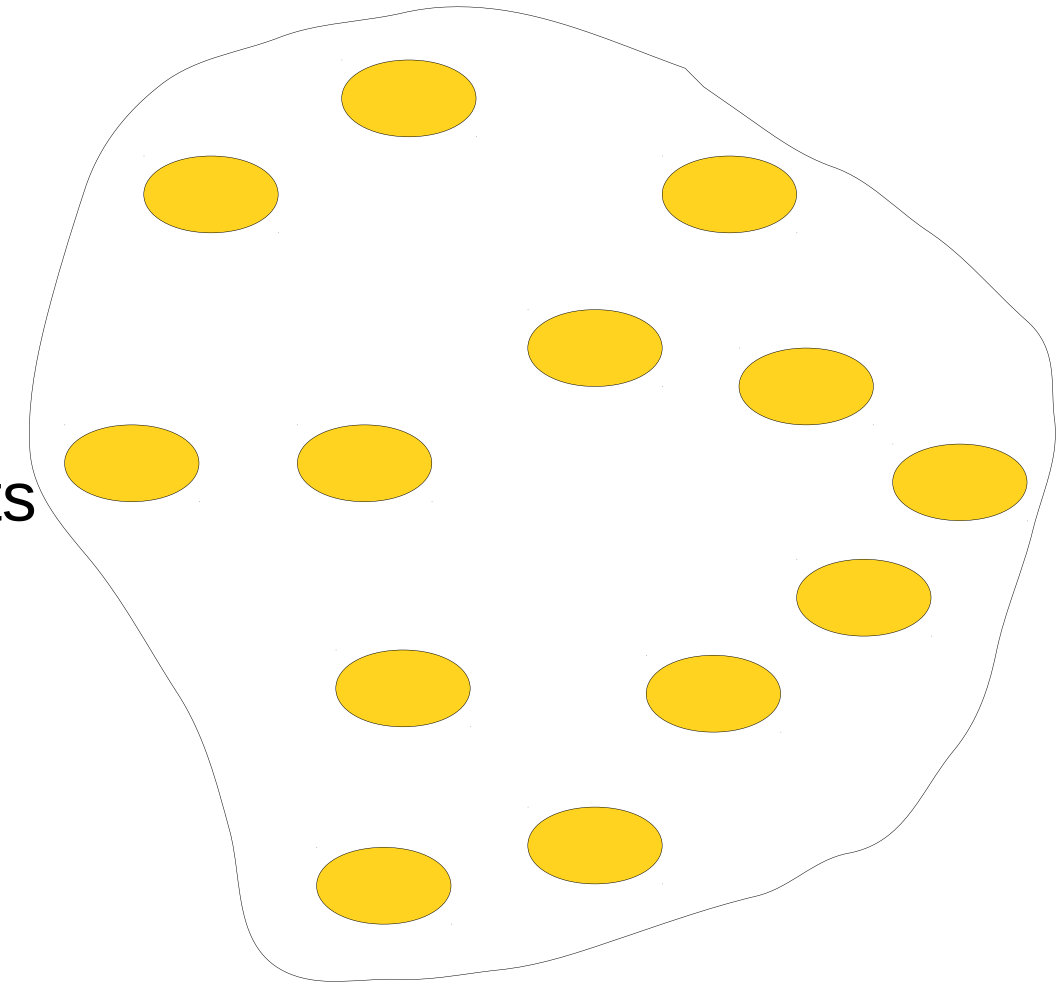
- **Application-Example**: Approximation of Maximum Independent Set

Problem is NP-complete

Separate into small connected components

Find Maximum Independent Set in each component

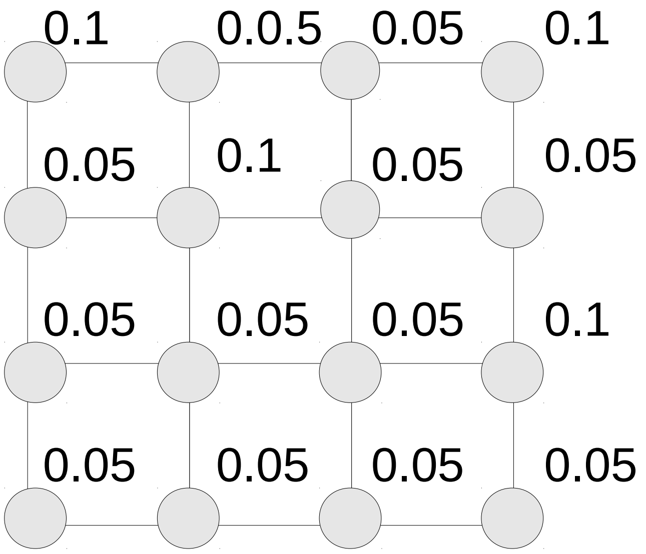
Return union of all Independent Sets



- **Planar Separator Theorem (Lipton and Tarjan, 1979):**
  - For a weighted planar graph with n nodes and a summed weight of no more than 1, there is a separator of size $2\sqrt{(2)}\sqrt{(n)}$ dividing the graph in two components with weight less than $^2/_3$.

- **Planar Separator Theorem (Lipton and Tarjan, 1979):**

- **Example**:
  - planar graph
  - n nodes
  - weight of 1

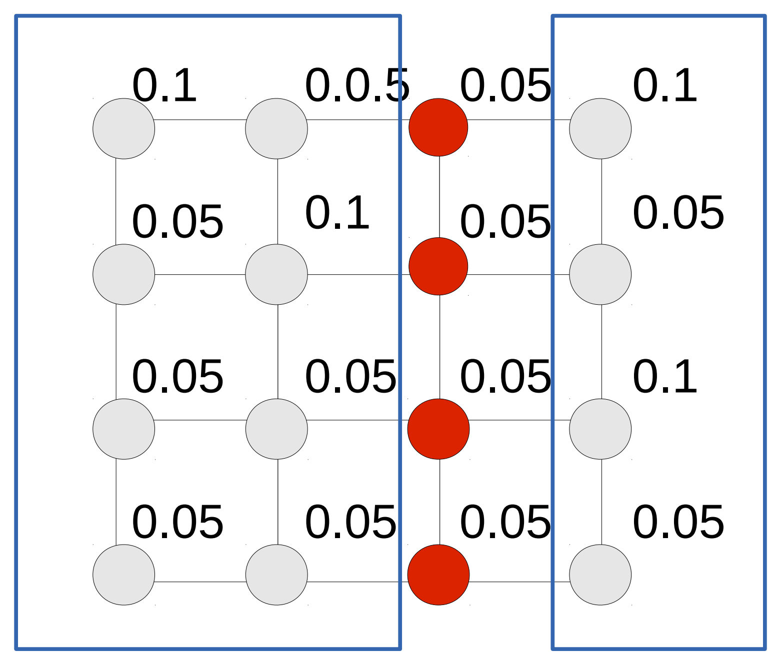


The Planar Separator Theorem

- **Planar Separator Theorem (Lipton and Tarjan, 1979):**

- **Example**:
  - planar graph
  - n nodes
  - weight of 1

Weight: $0.5 \leq 2/3$



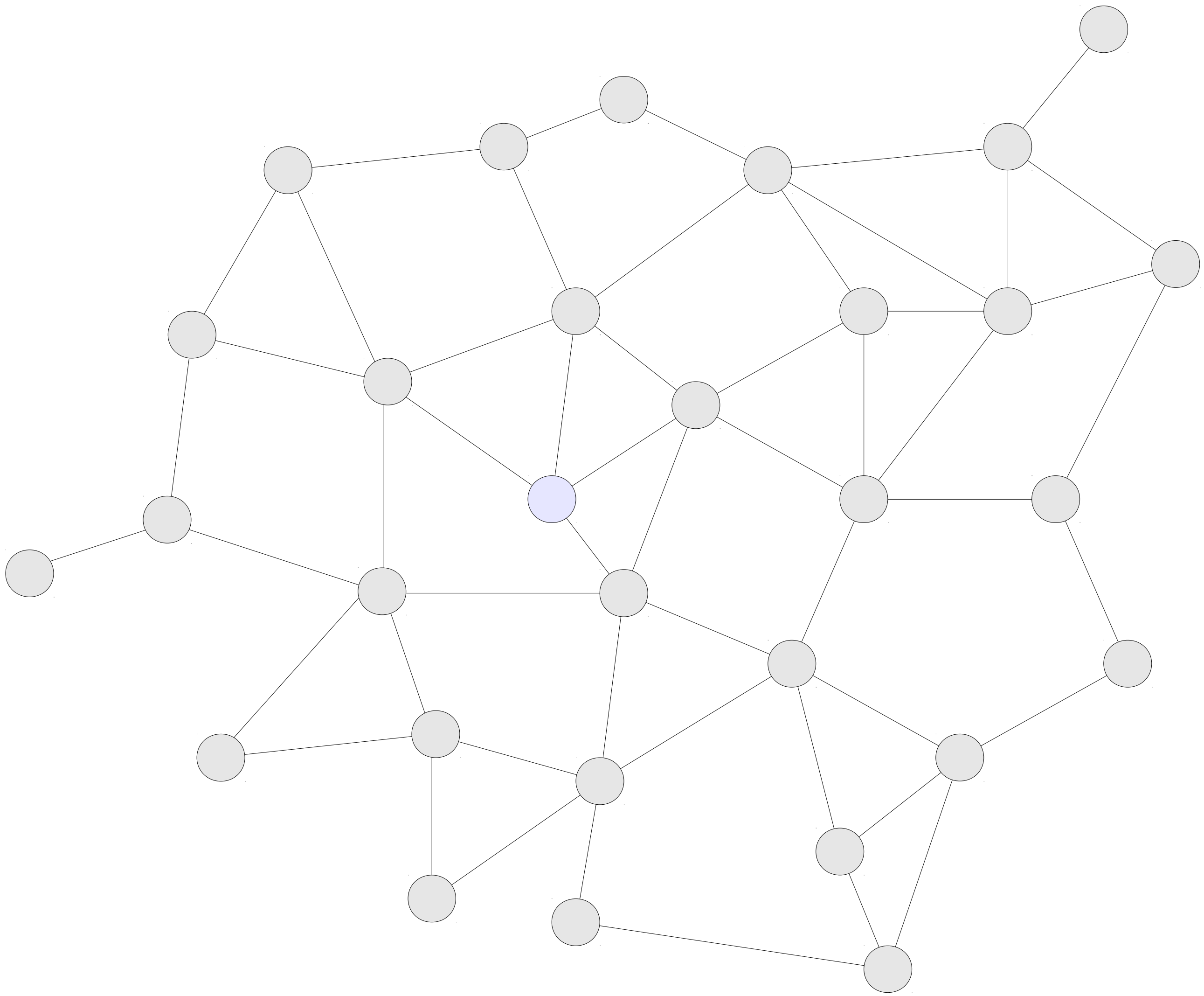
Weight: $0.3 \leq 2/3$

Separator: $4 \leq 2\sqrt{(2)}\sqrt{(16)}$

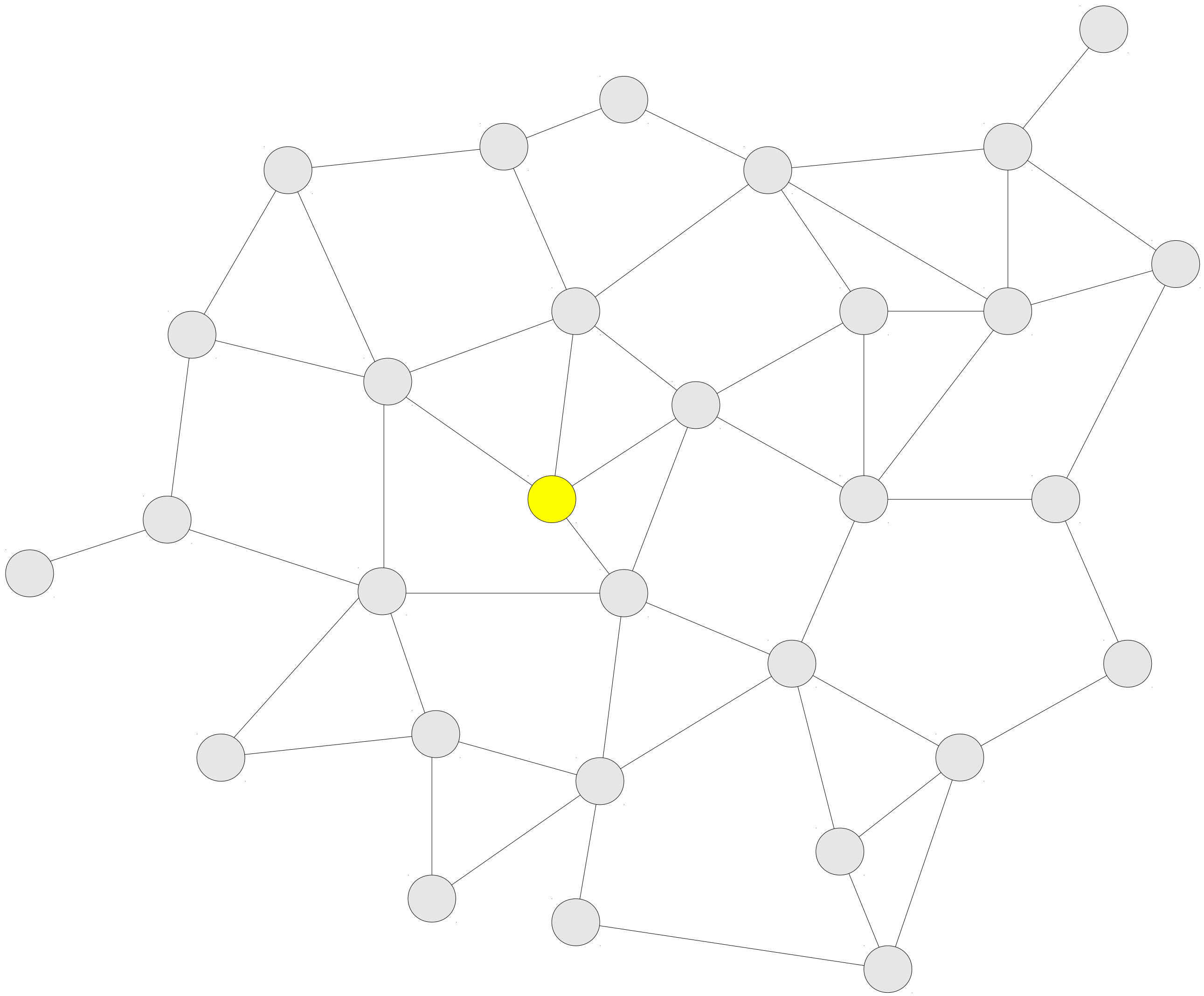
Questions

- **How to find a separator in a graph?**

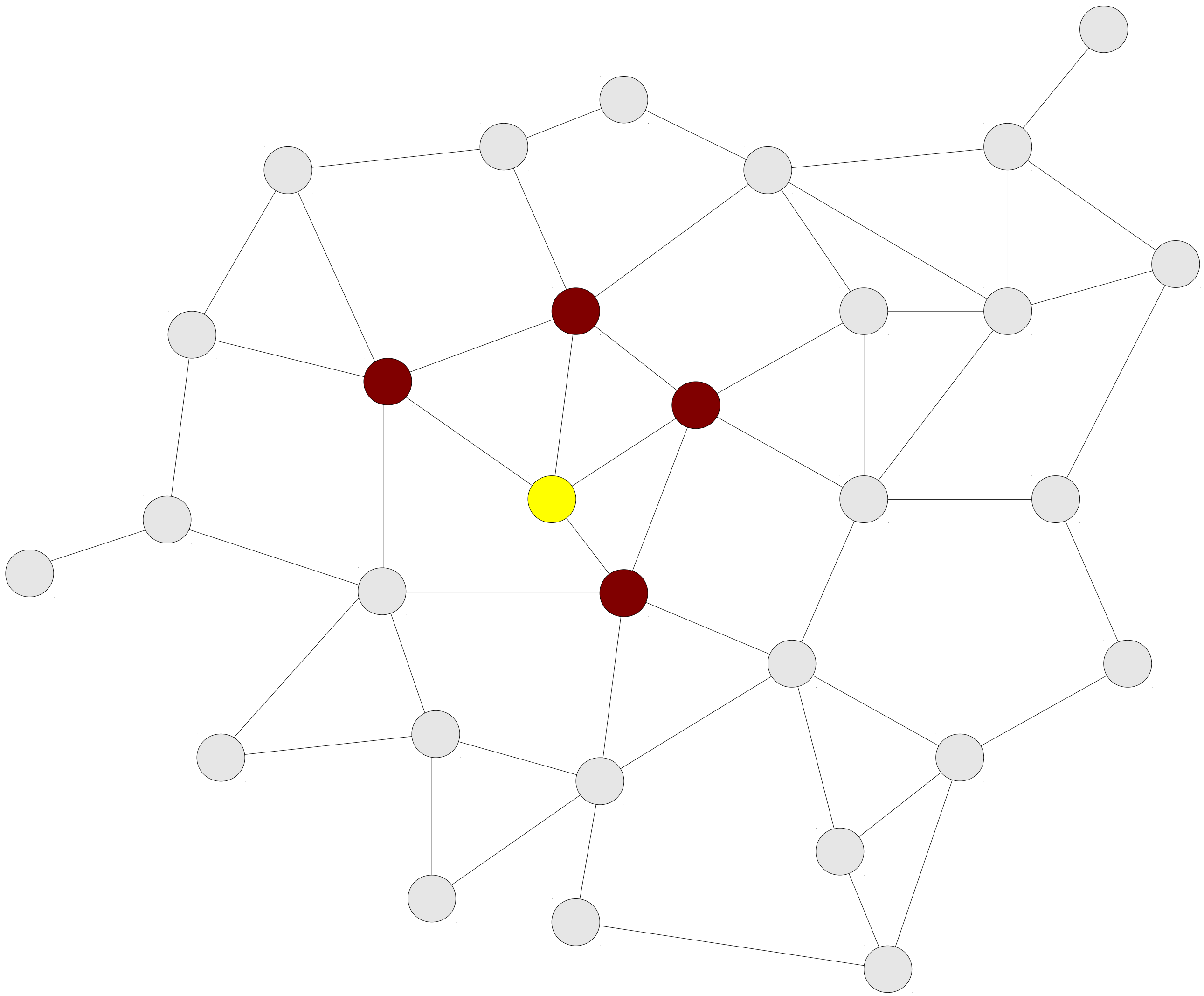
Levels



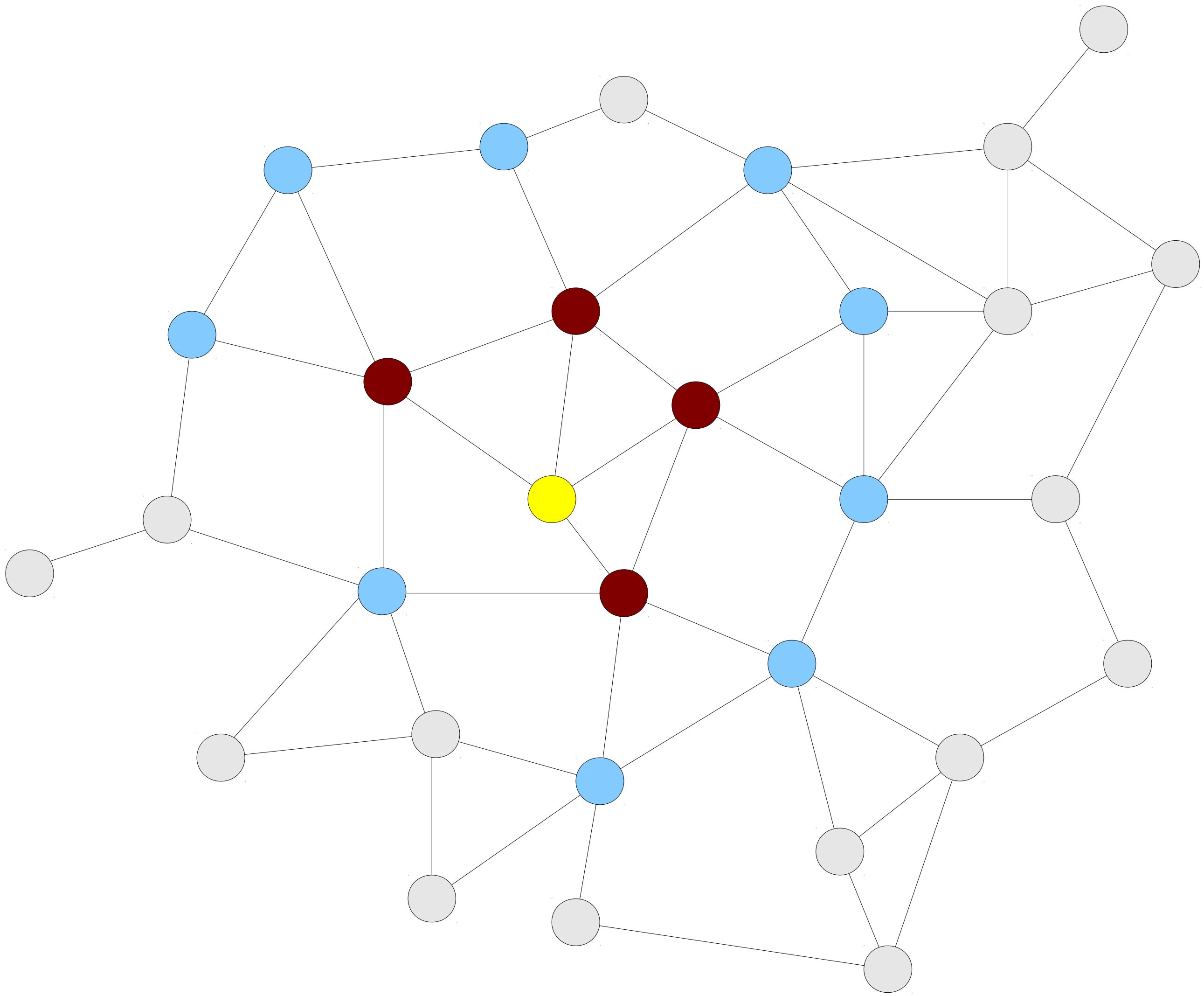
Levels



Levels



Levels



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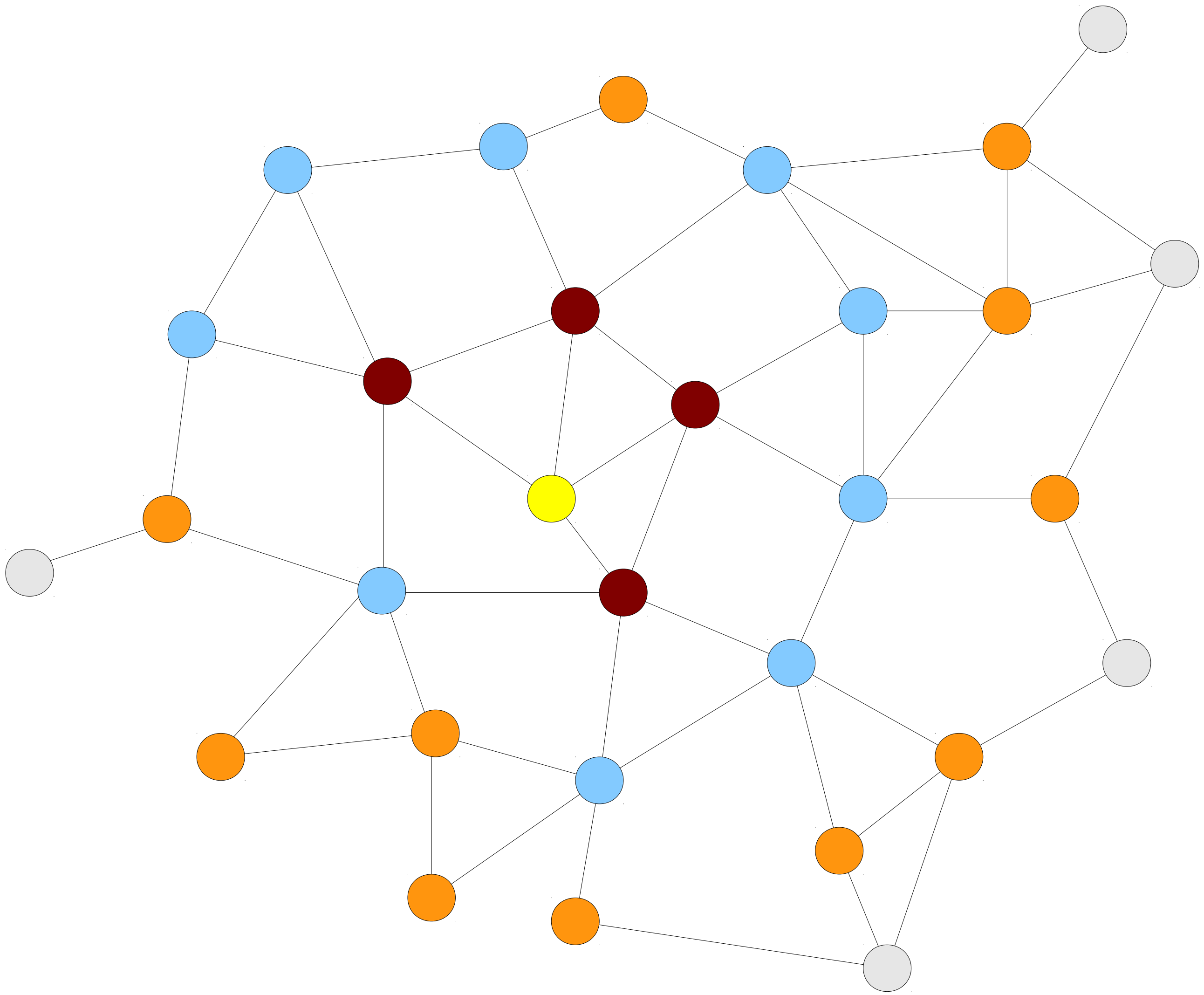

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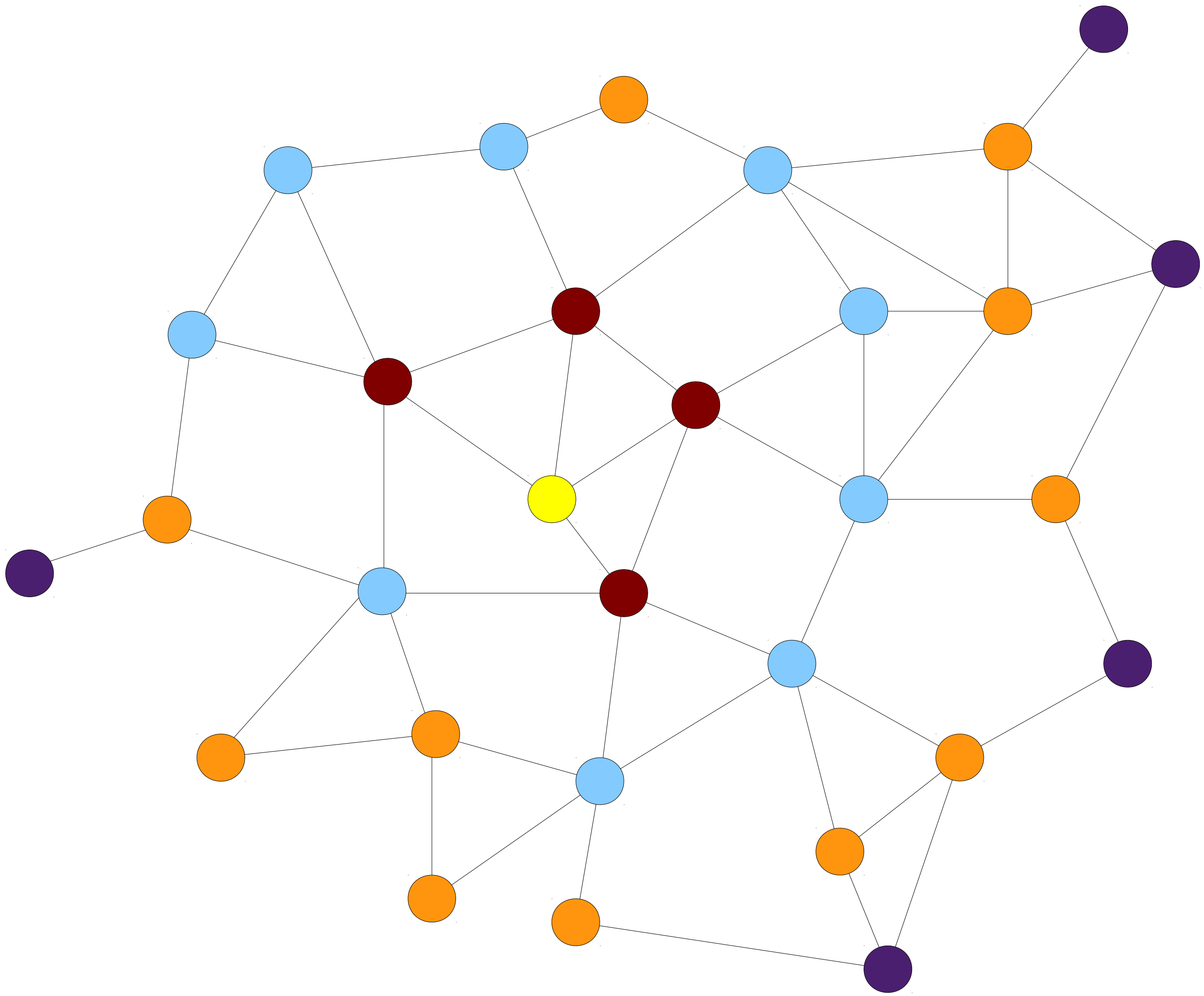
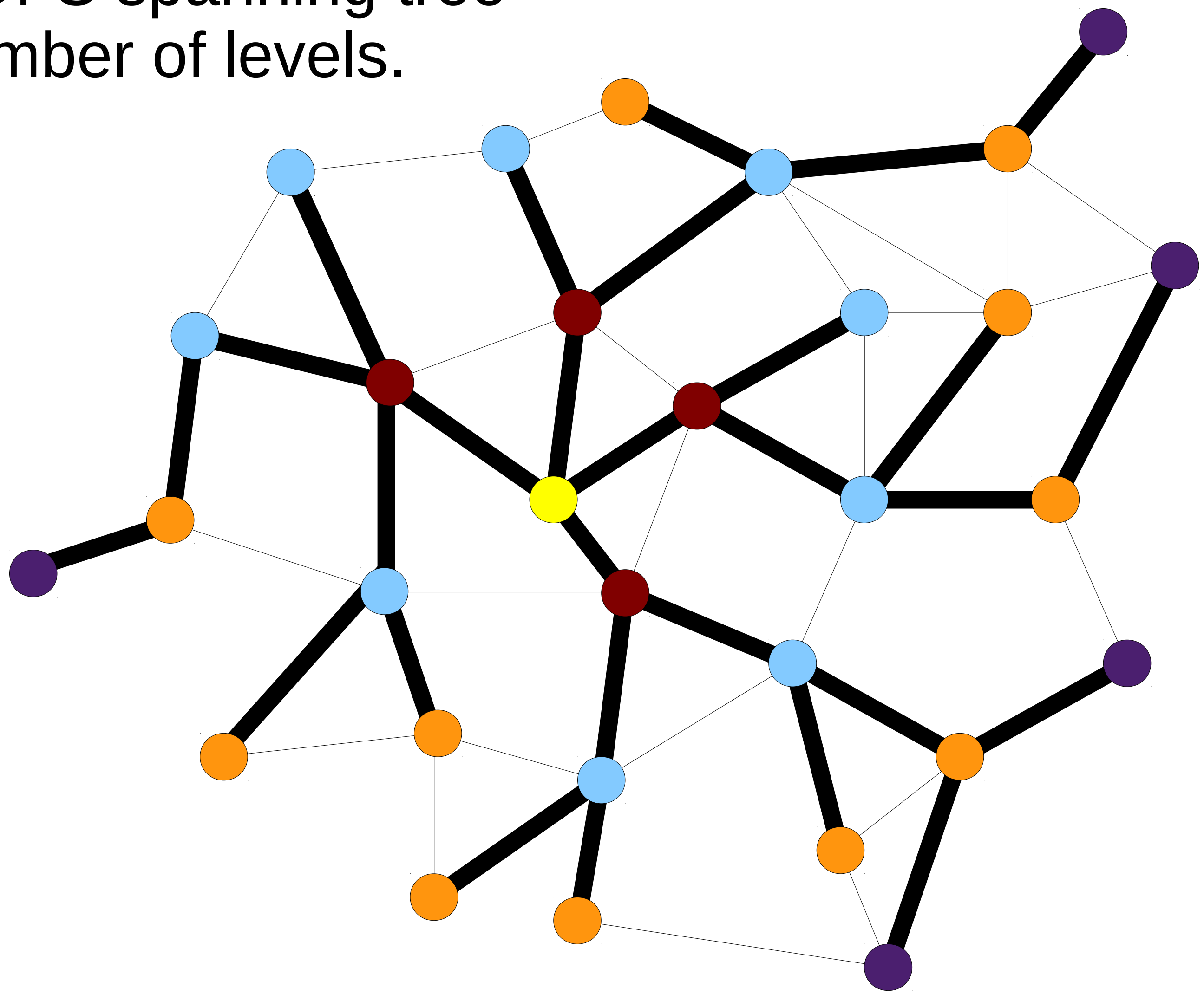
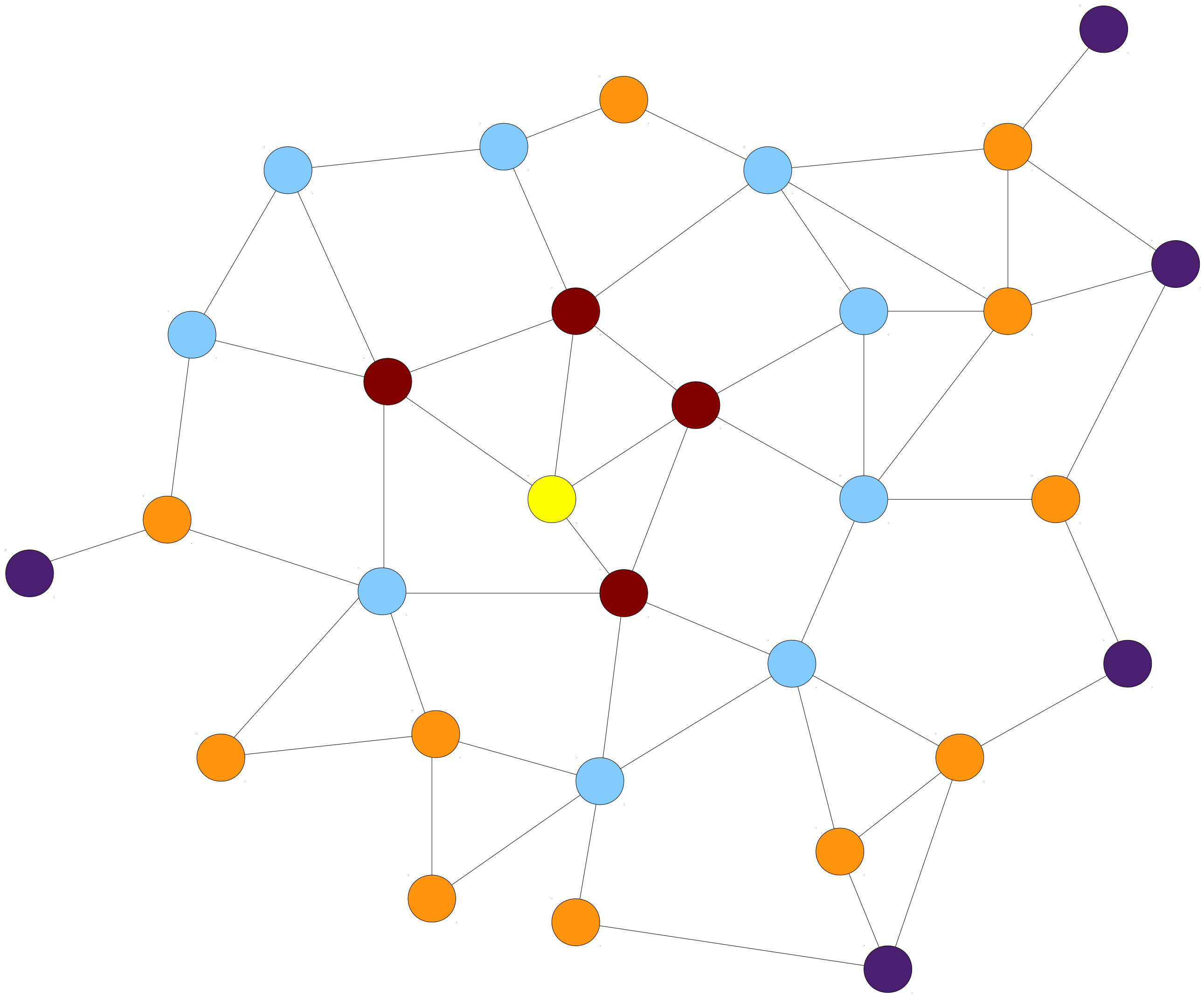


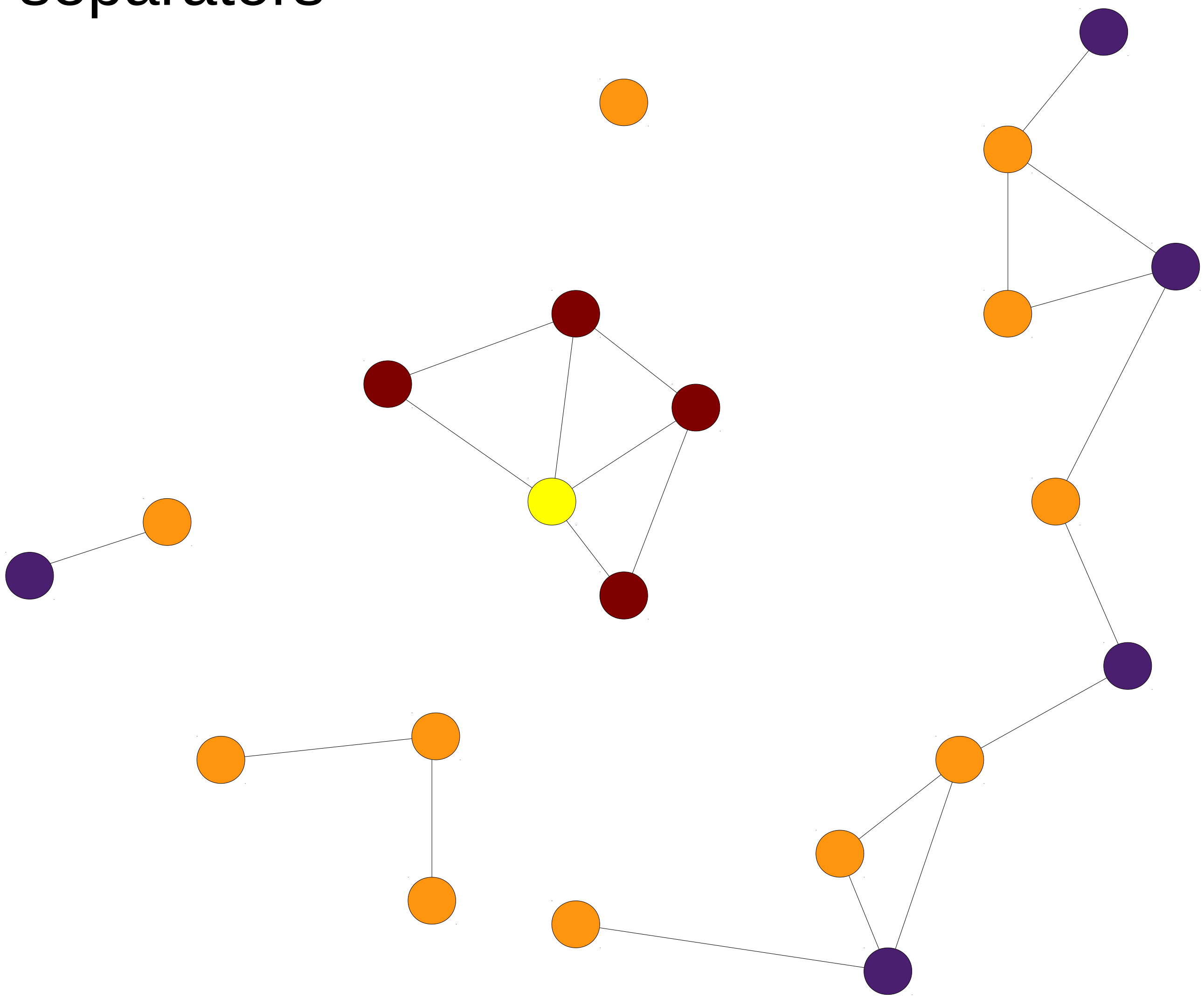
First observation:
Depth of DFS spanning tree equals number of levels.



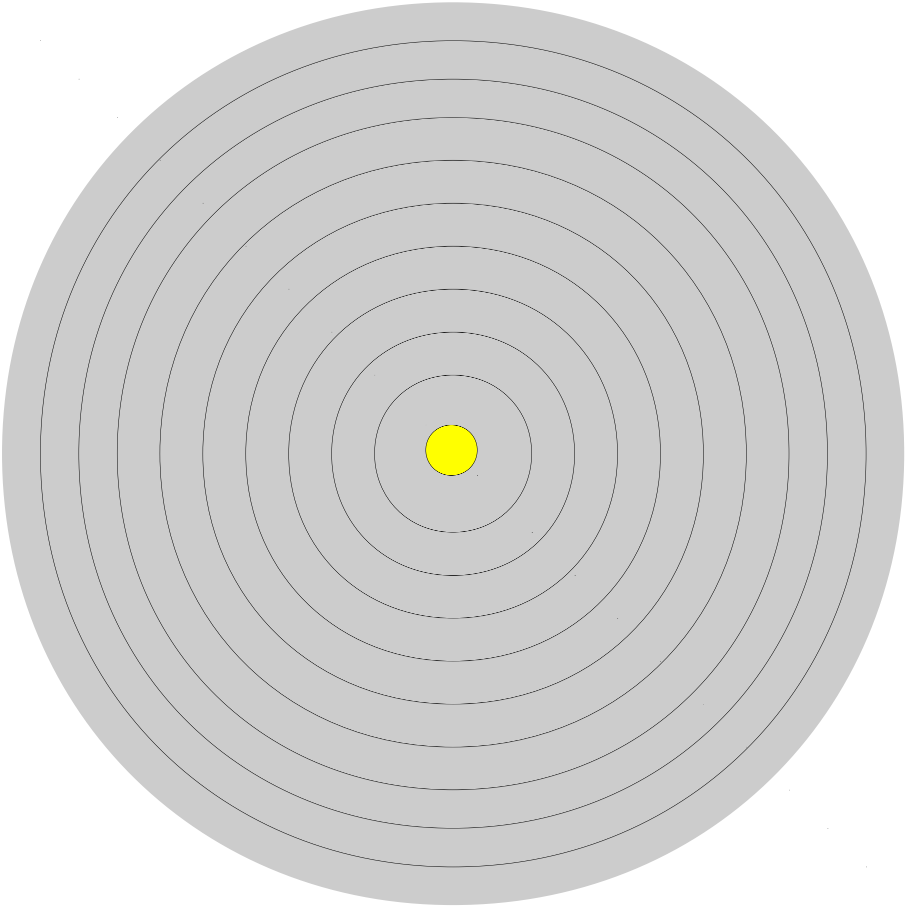
Levels



Second observation:
Levels are separators



Levels



Levels

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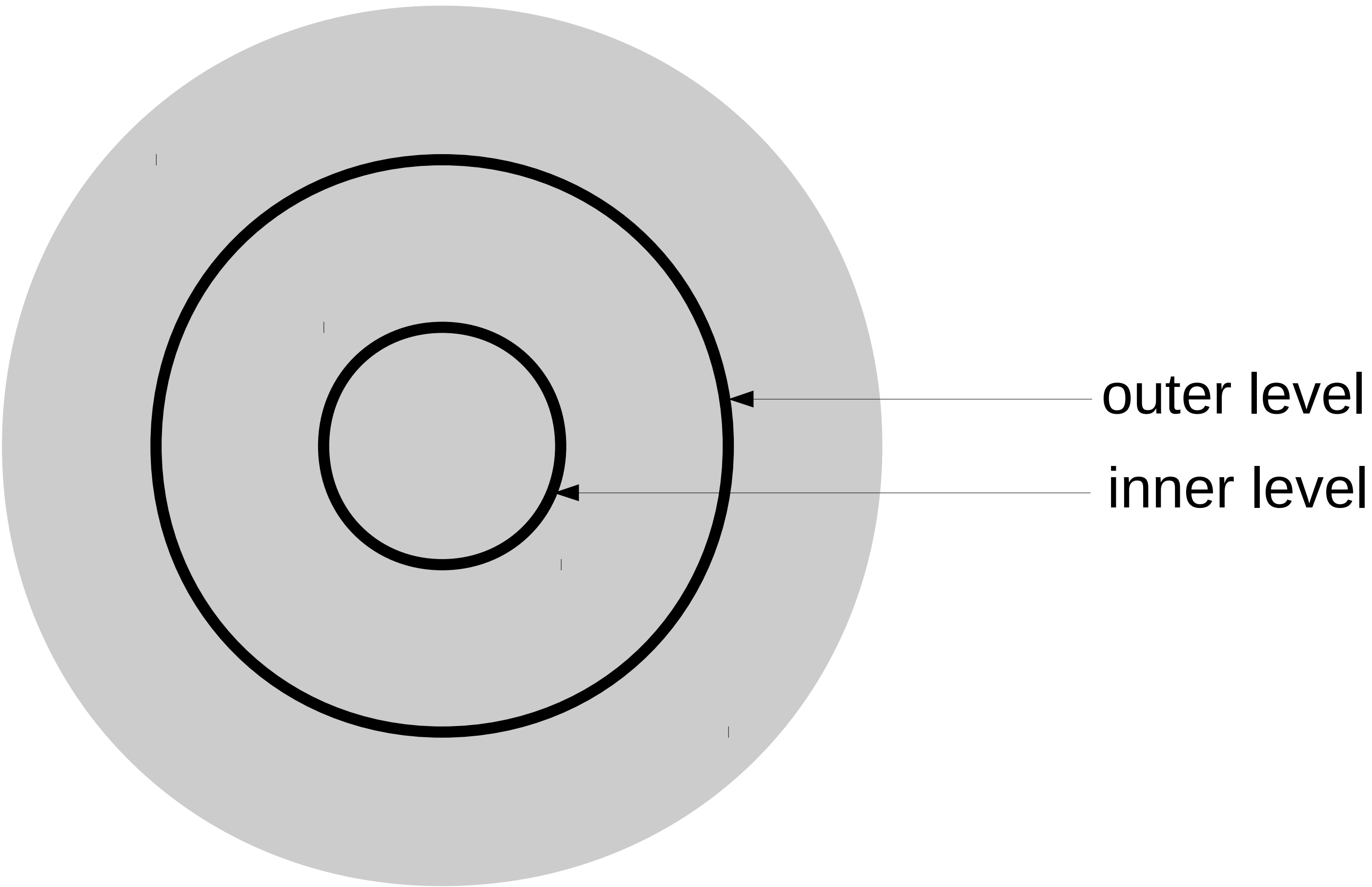

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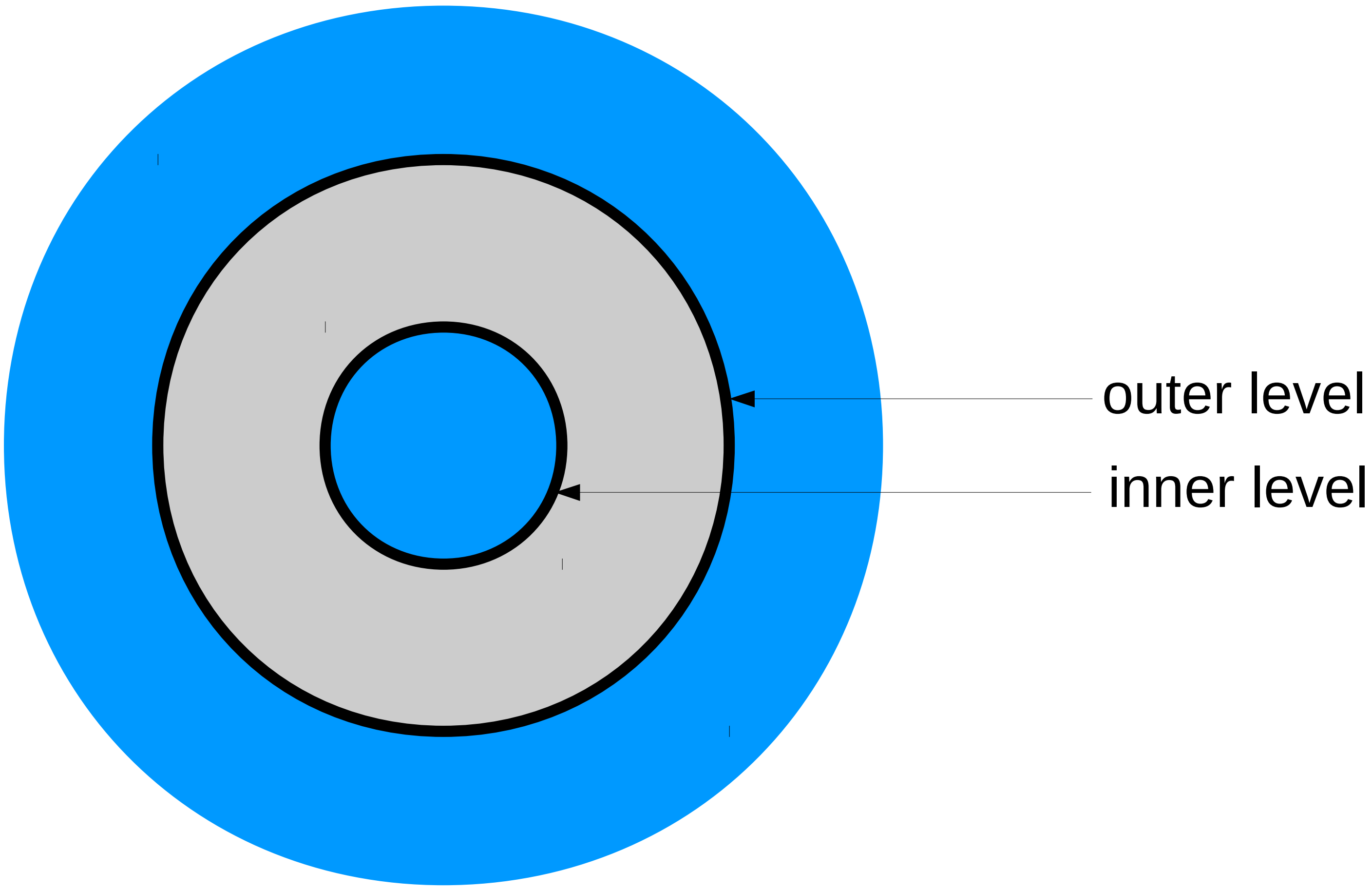
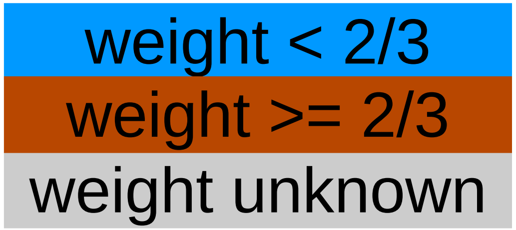
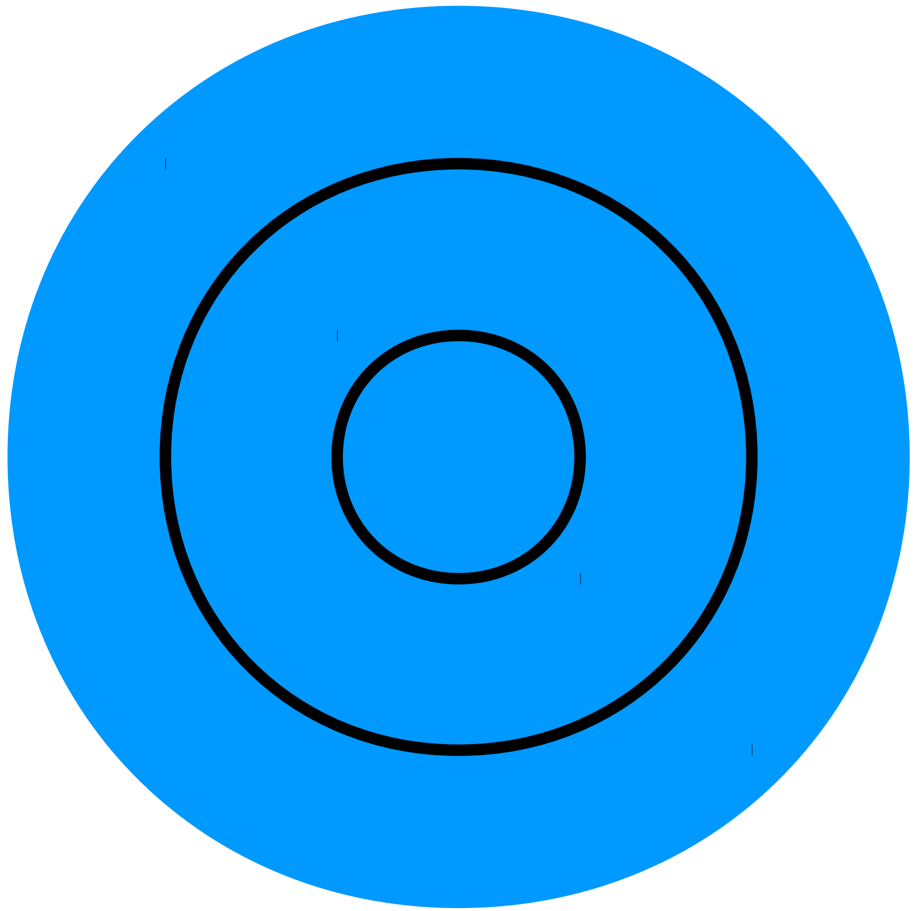




First Case:

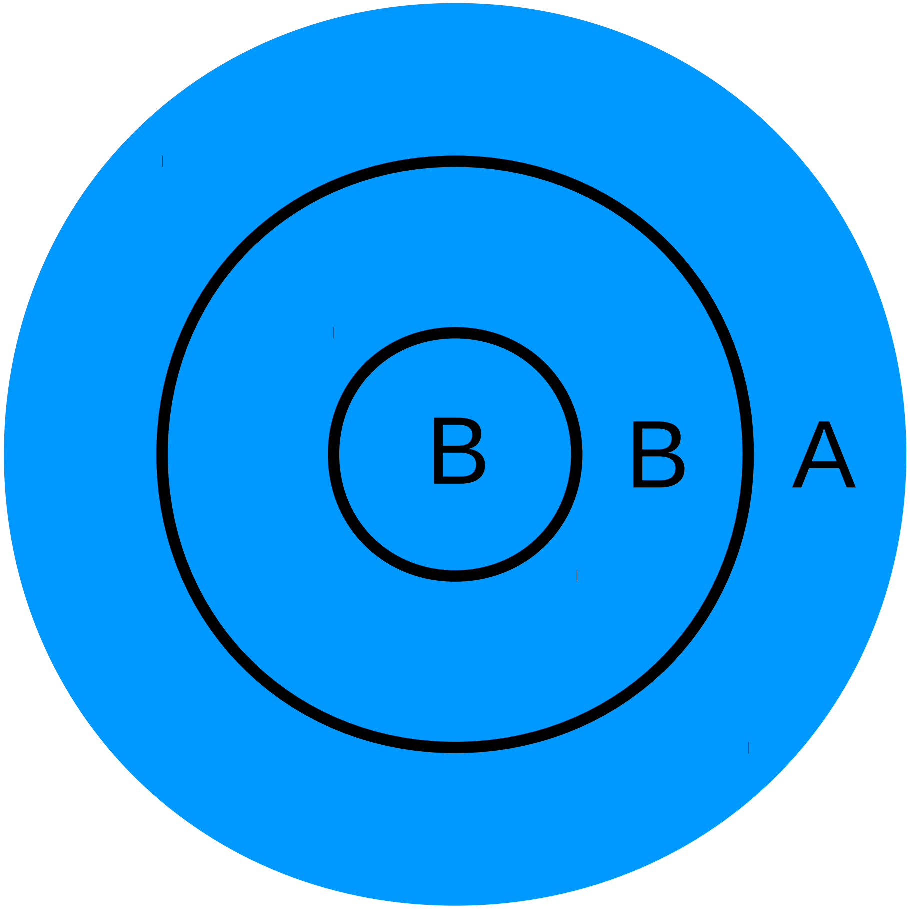


weight < 2/3

weight >= 2/3

weight unknown

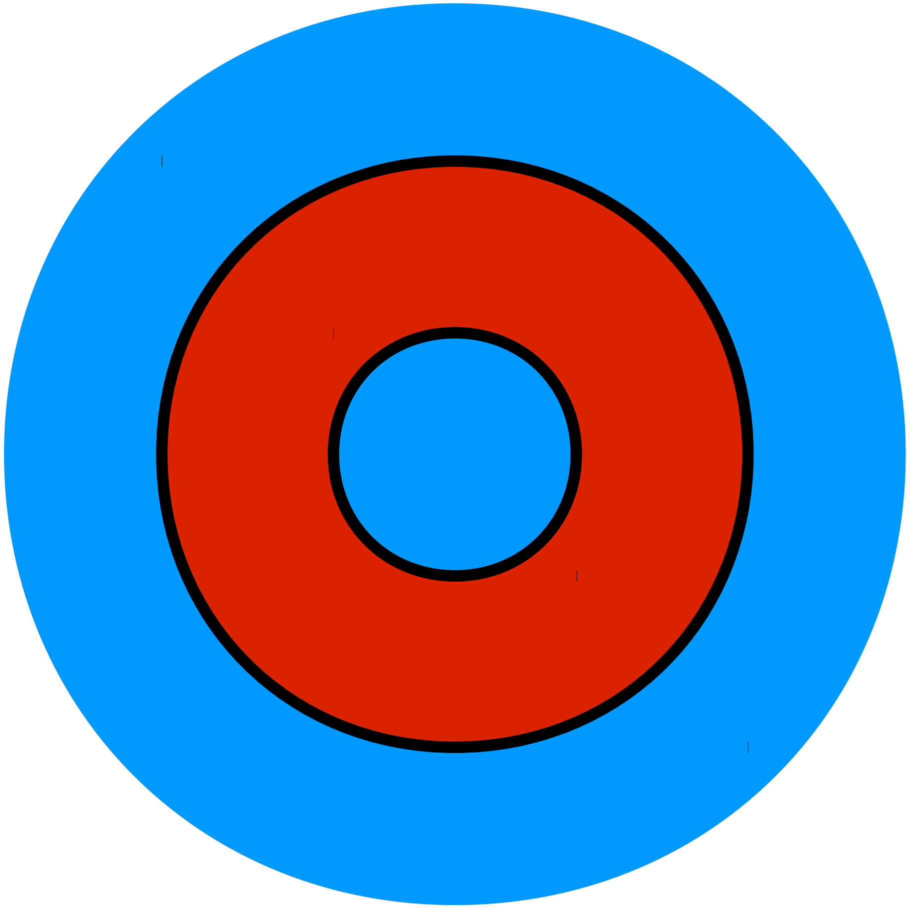
First Case:



- **Choose:**
  - **Set A**: heavier set
  - **Set B**: two lighter sets
  - **Separator**: nodes on inner and outer level

- Weight of A and B is smaller than $\frac{2}{3}$

Second Case:



weight < 2/3

weight >= 2/3

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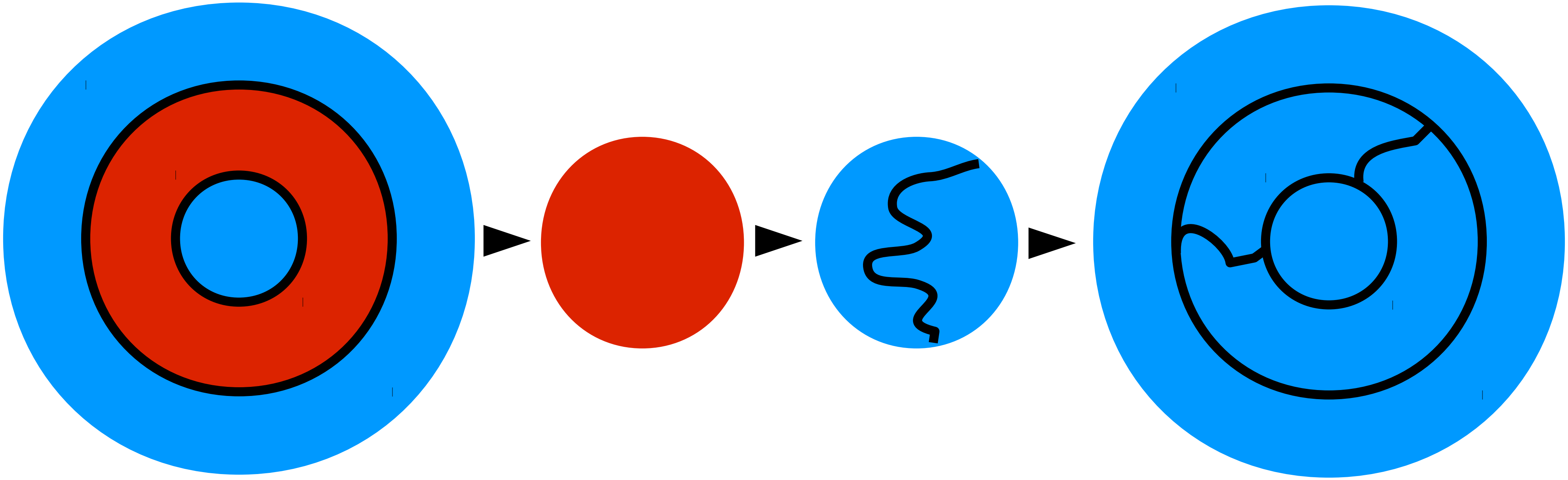
Second Case:



weight < 2/3

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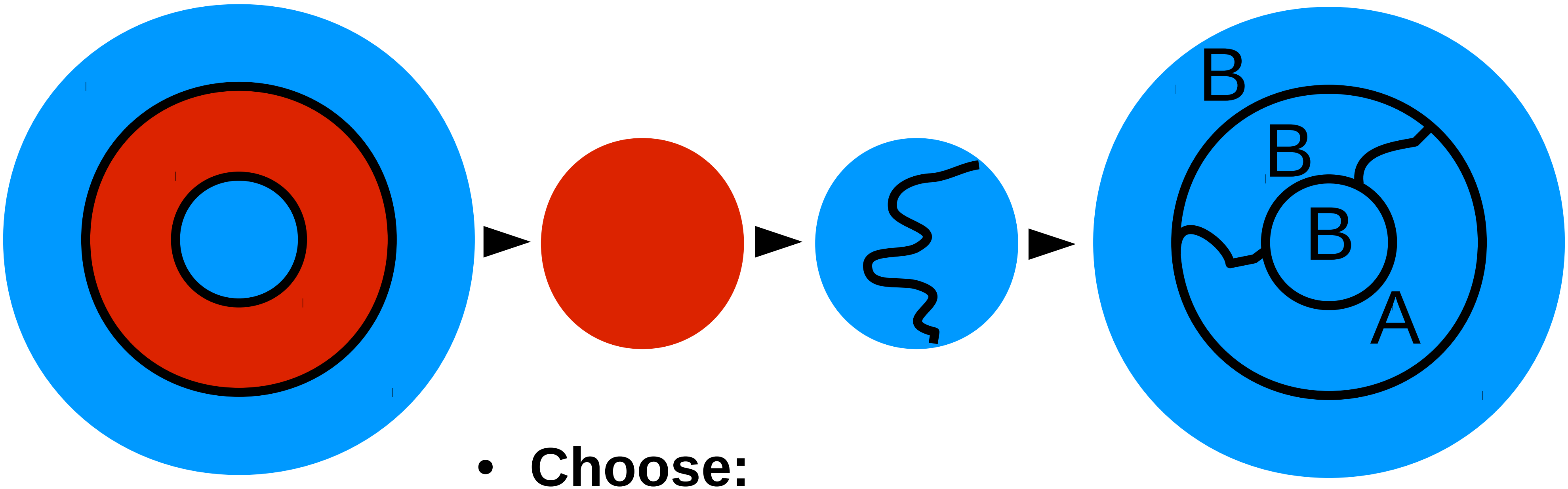
Second Case:

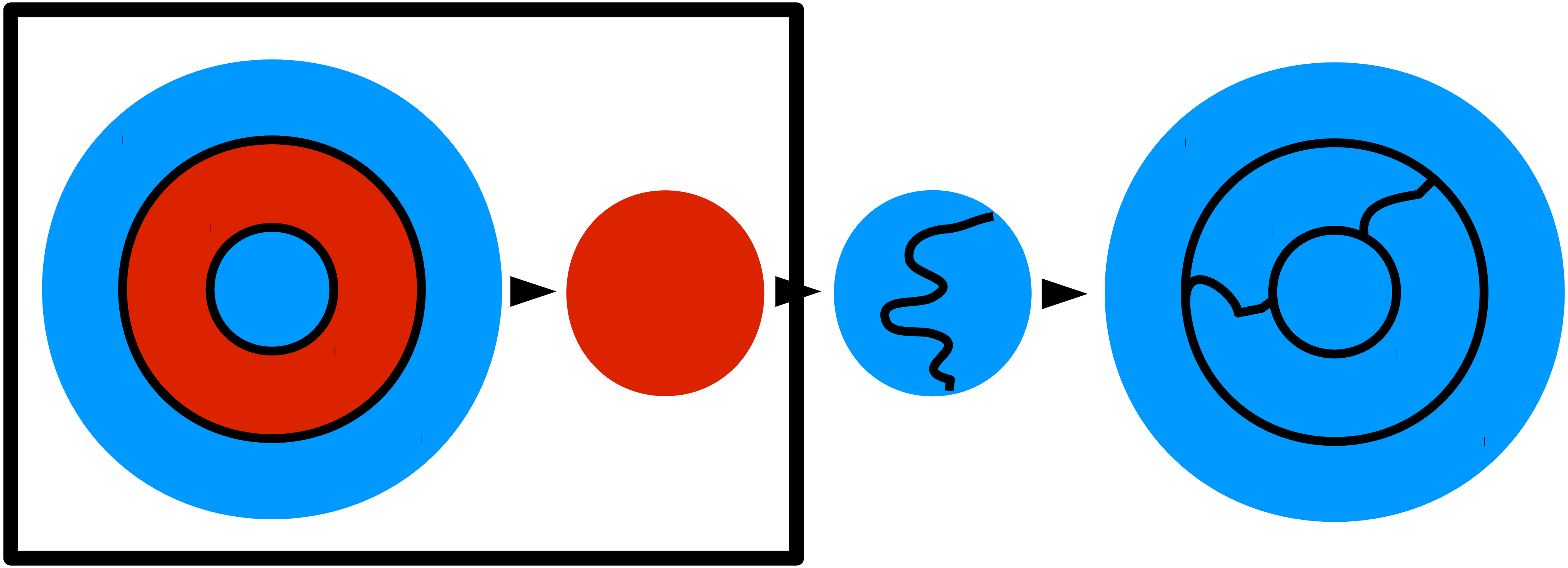


- **Choose:**
  - **Set A**: heavier half of middle set
  - **Set B**: remaining sets
  - **Separator**: nodes on inner and outer level and separator of middle set

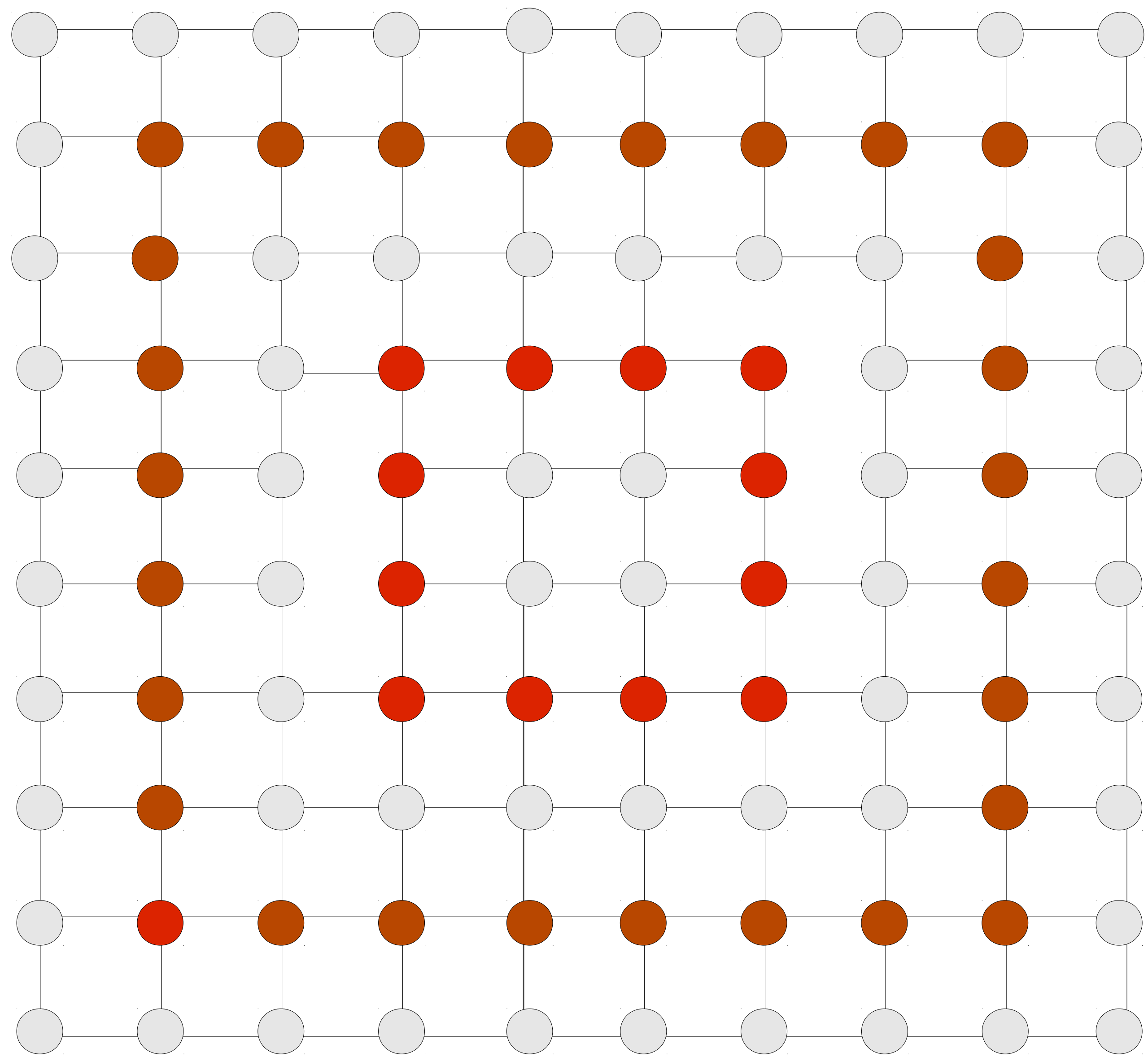
weight < 2/3
weight >= 2/3
weight unknown

- Weight of A and B is smaller than $\frac{2}{3}$

- **How can we separate the middle set?**
  - Extract new graph
  - Find separator
  - Integrate in old graph



- **Idea**
  - Remove outer set and separator
  - Merge inner set and separator



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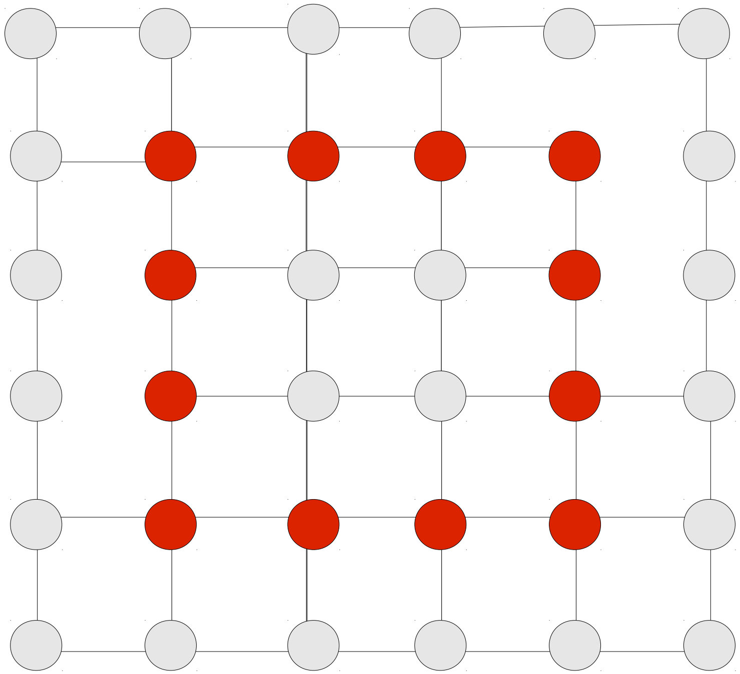

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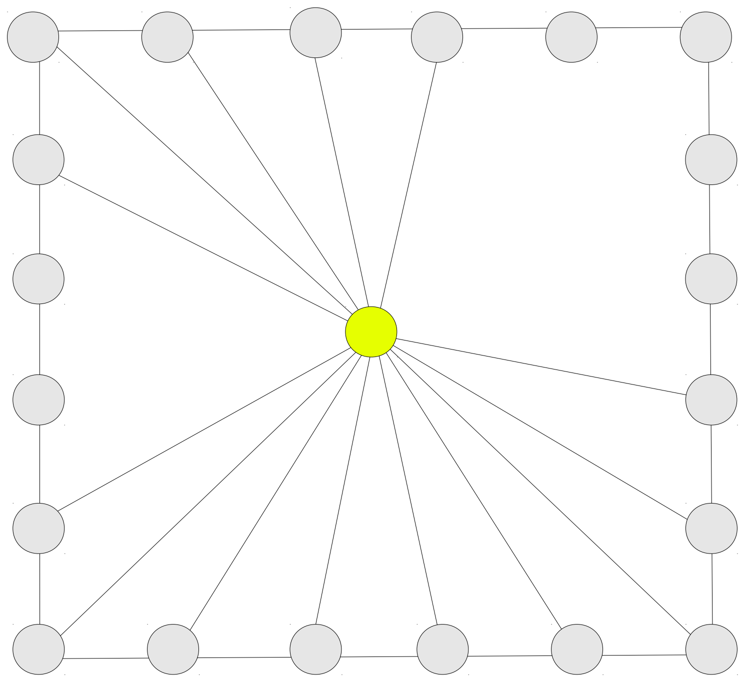

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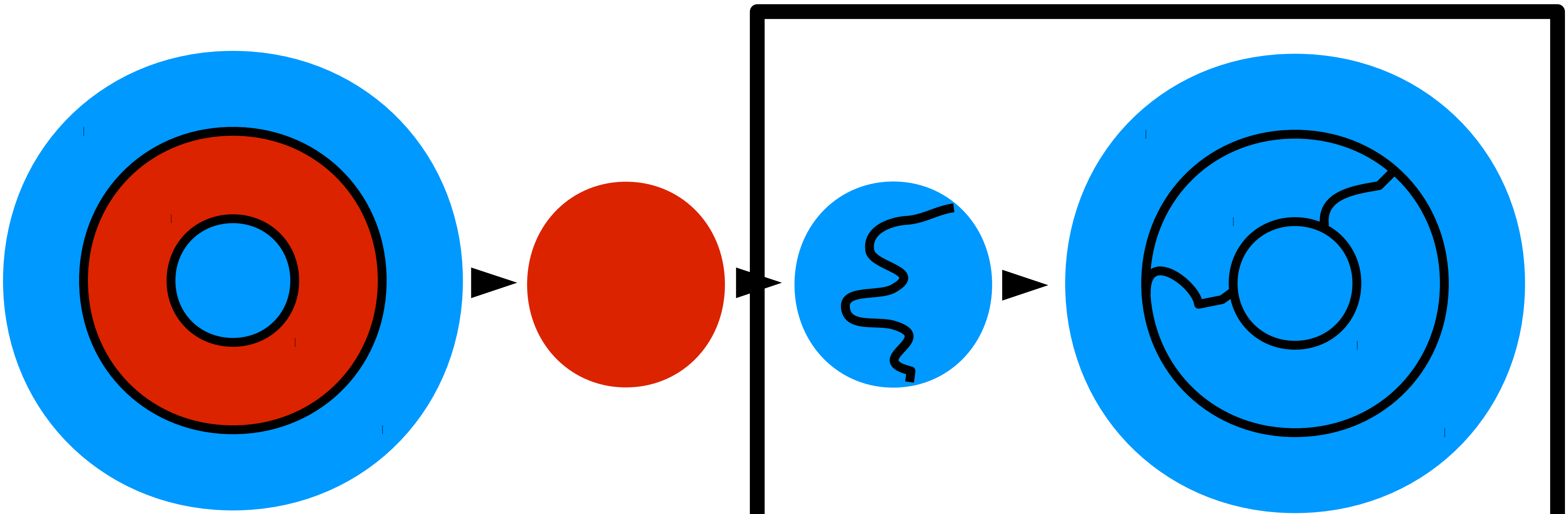
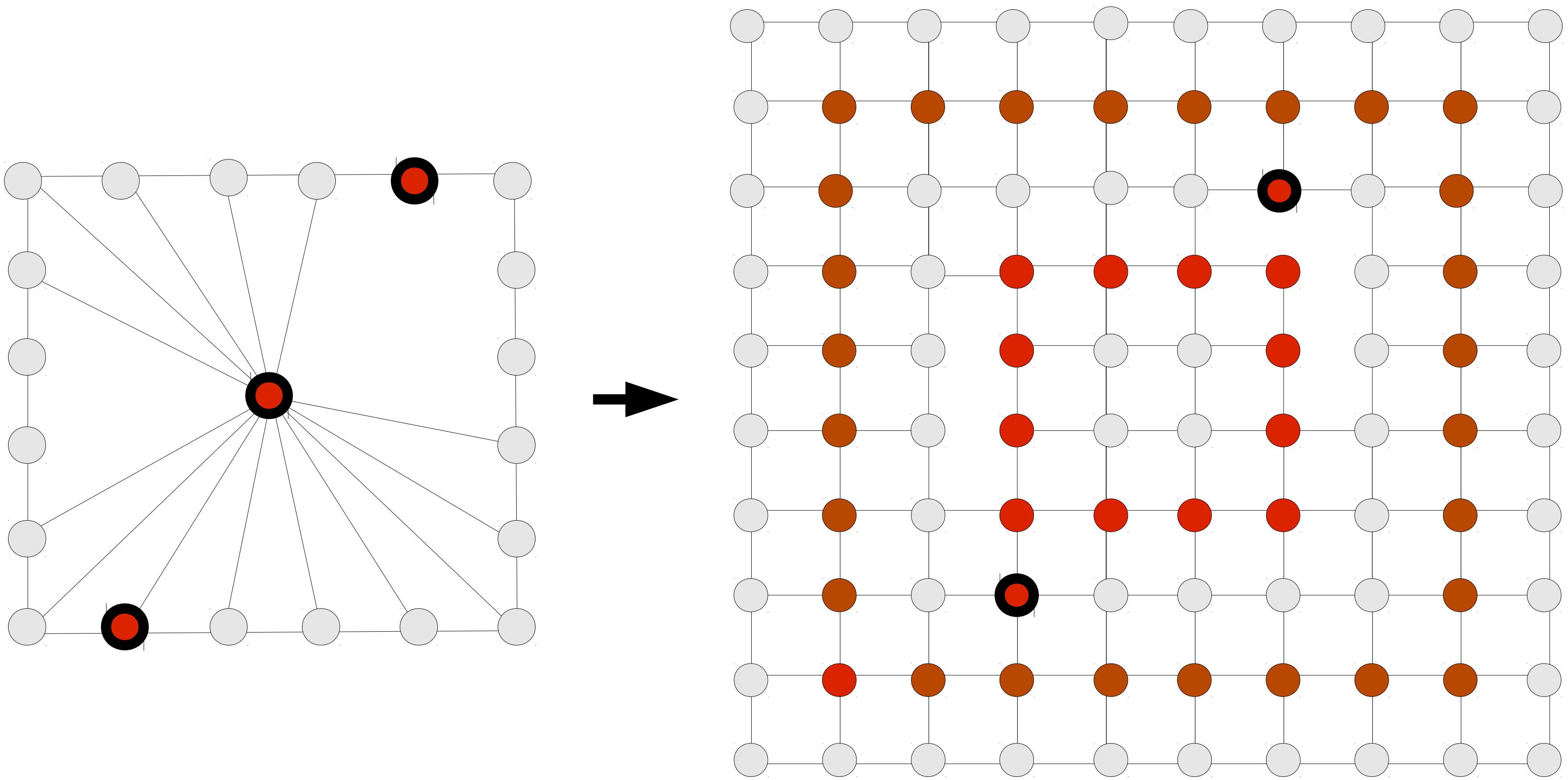
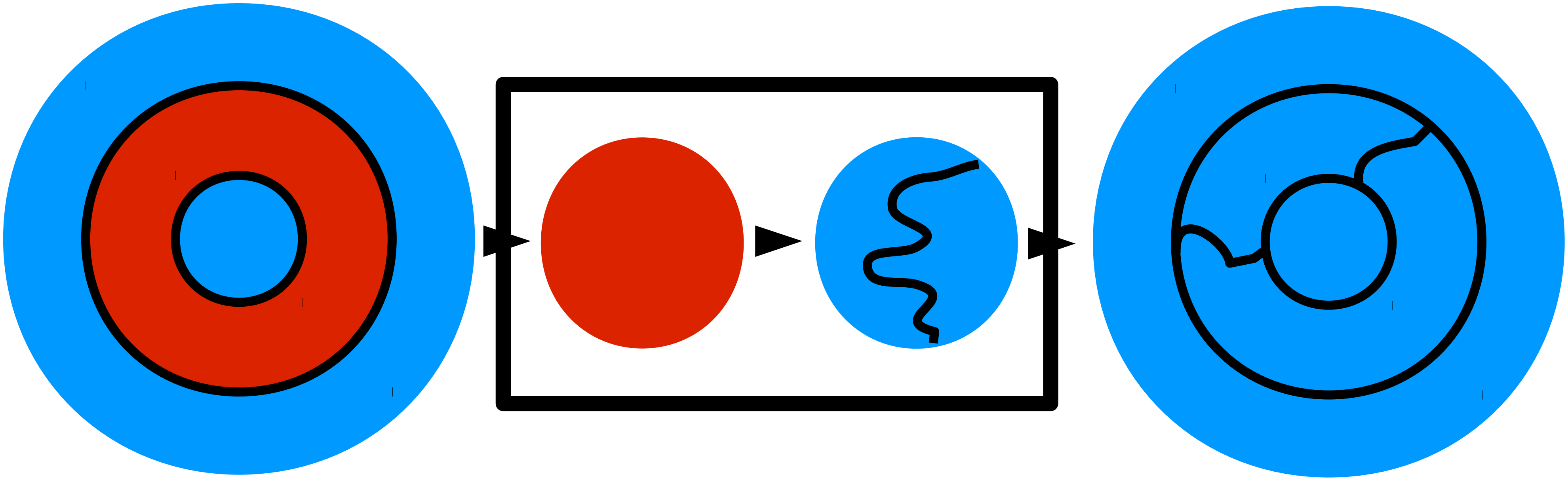


Separating the middle set via spanning tree



- **How can we separate the middle set?**
  - Extract new graph
  - <u>Find separator</u>
  - Integrate in old graph



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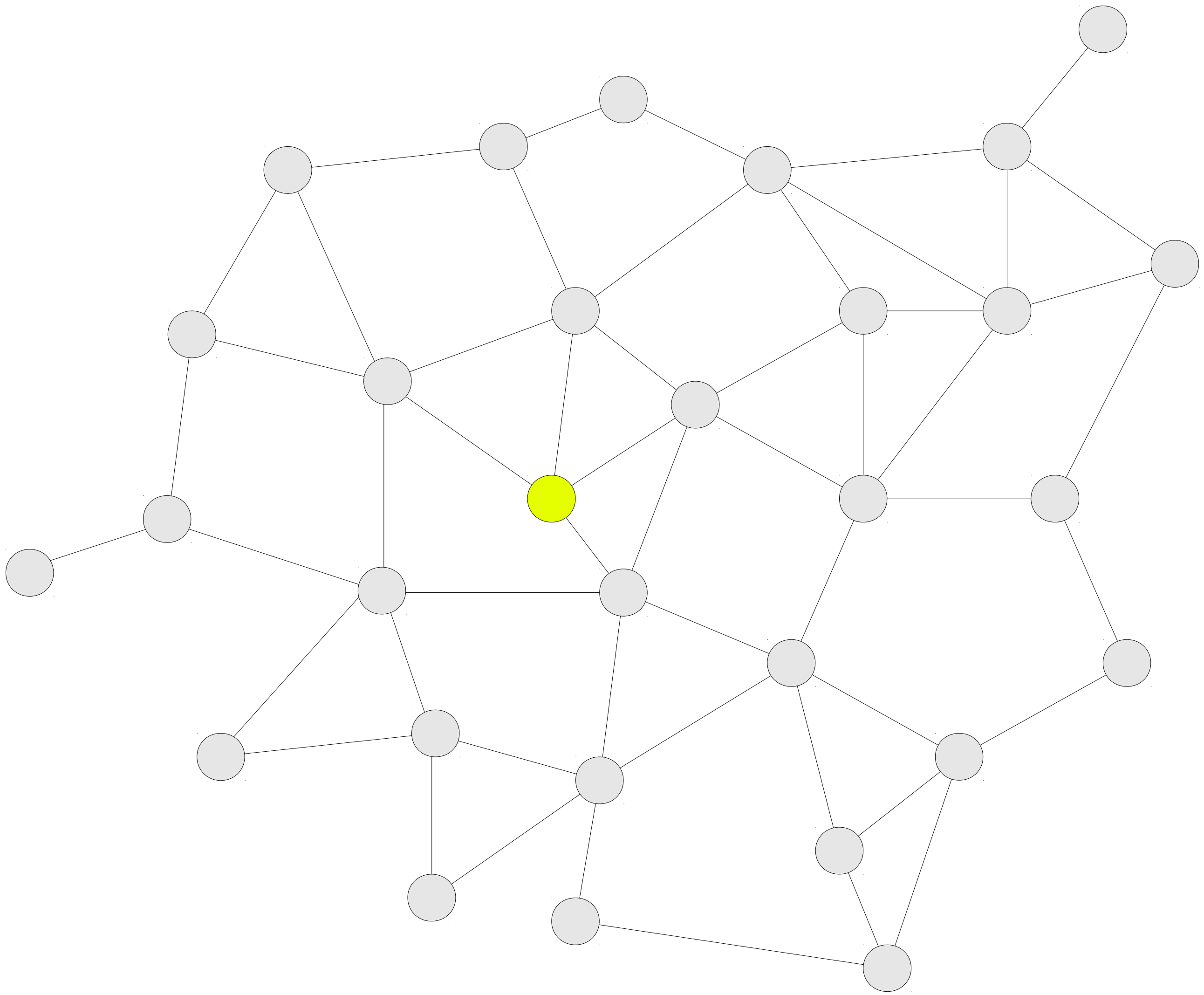
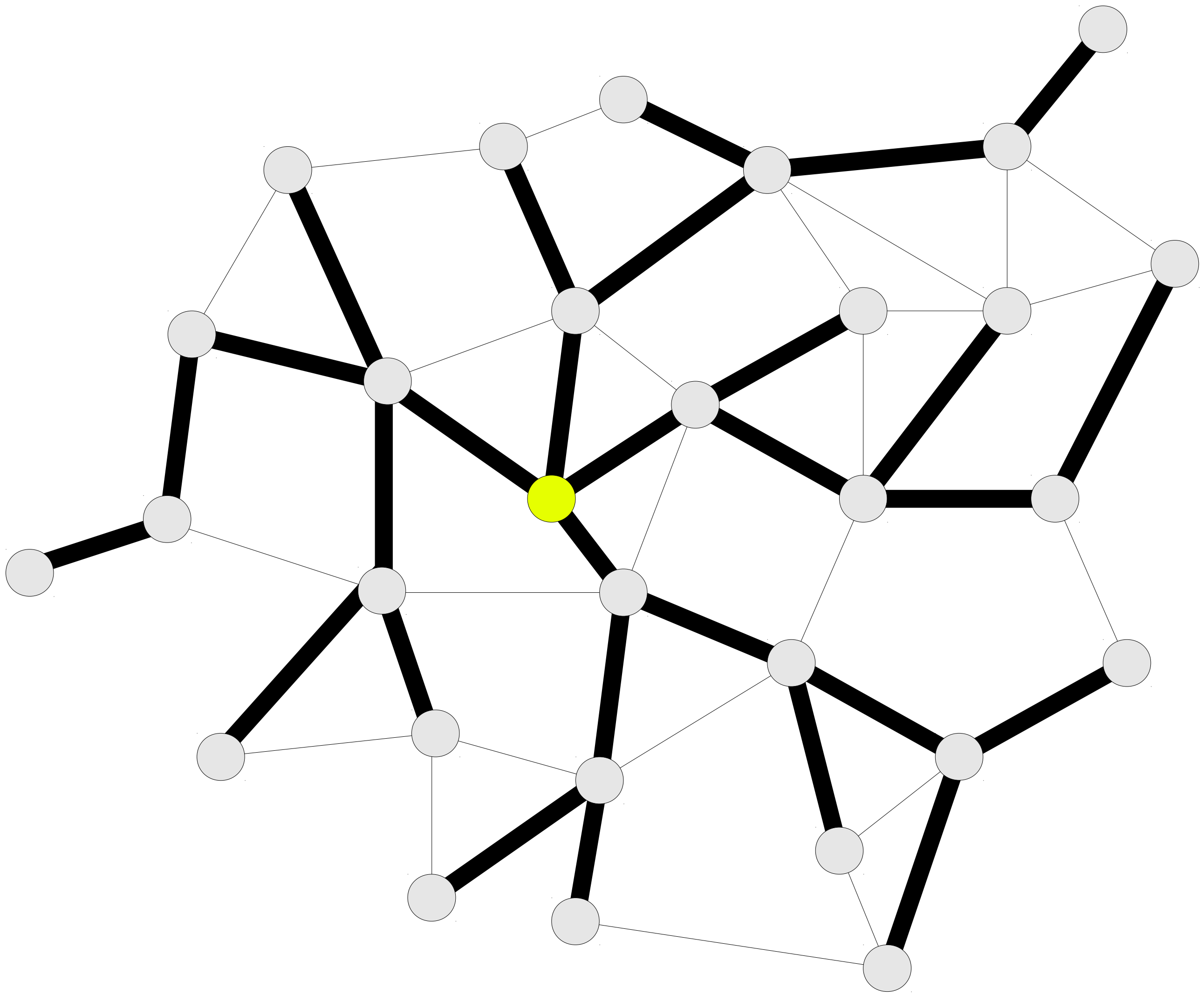
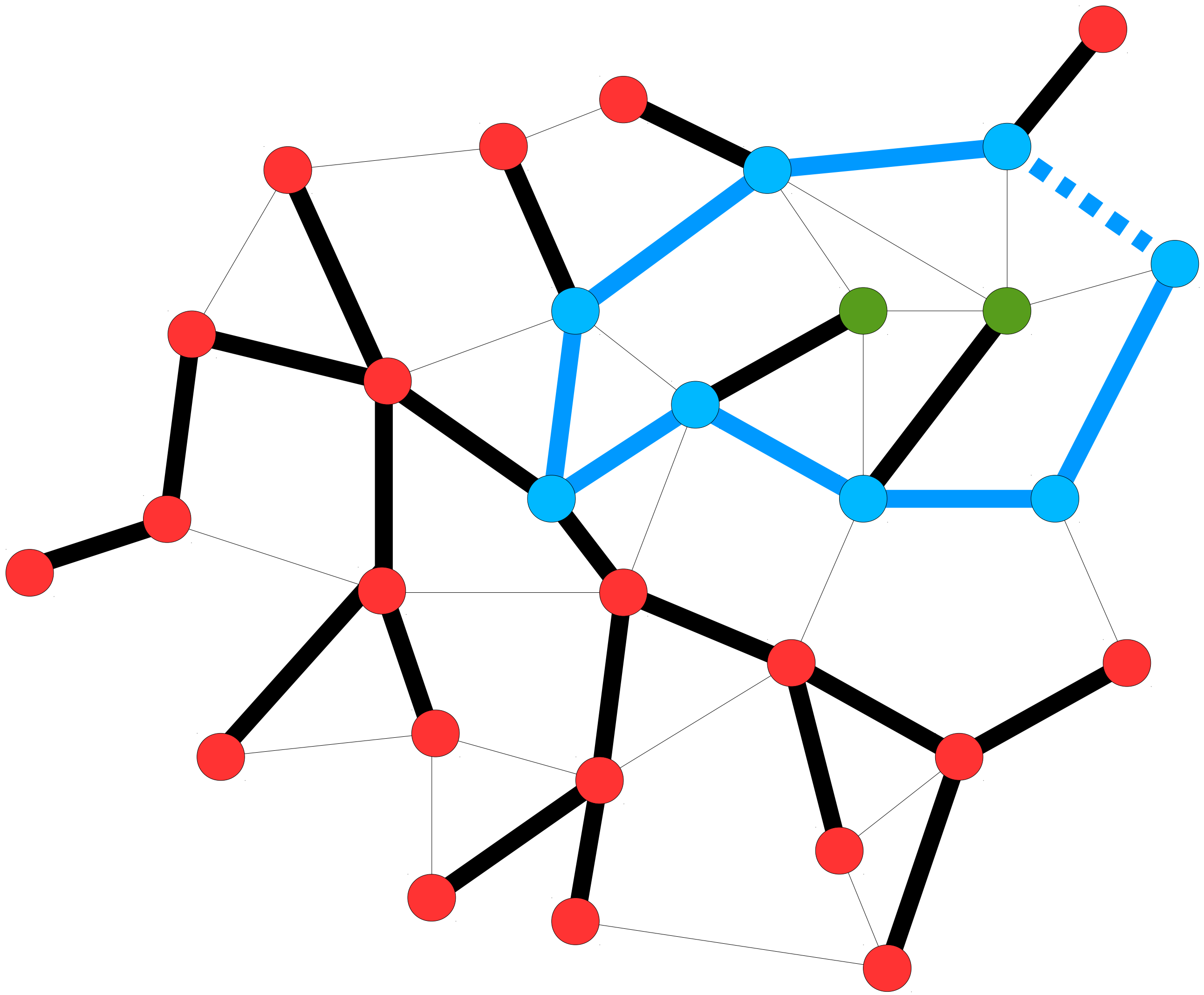


- **Calculate**: Planar representation and DFS spanning tree



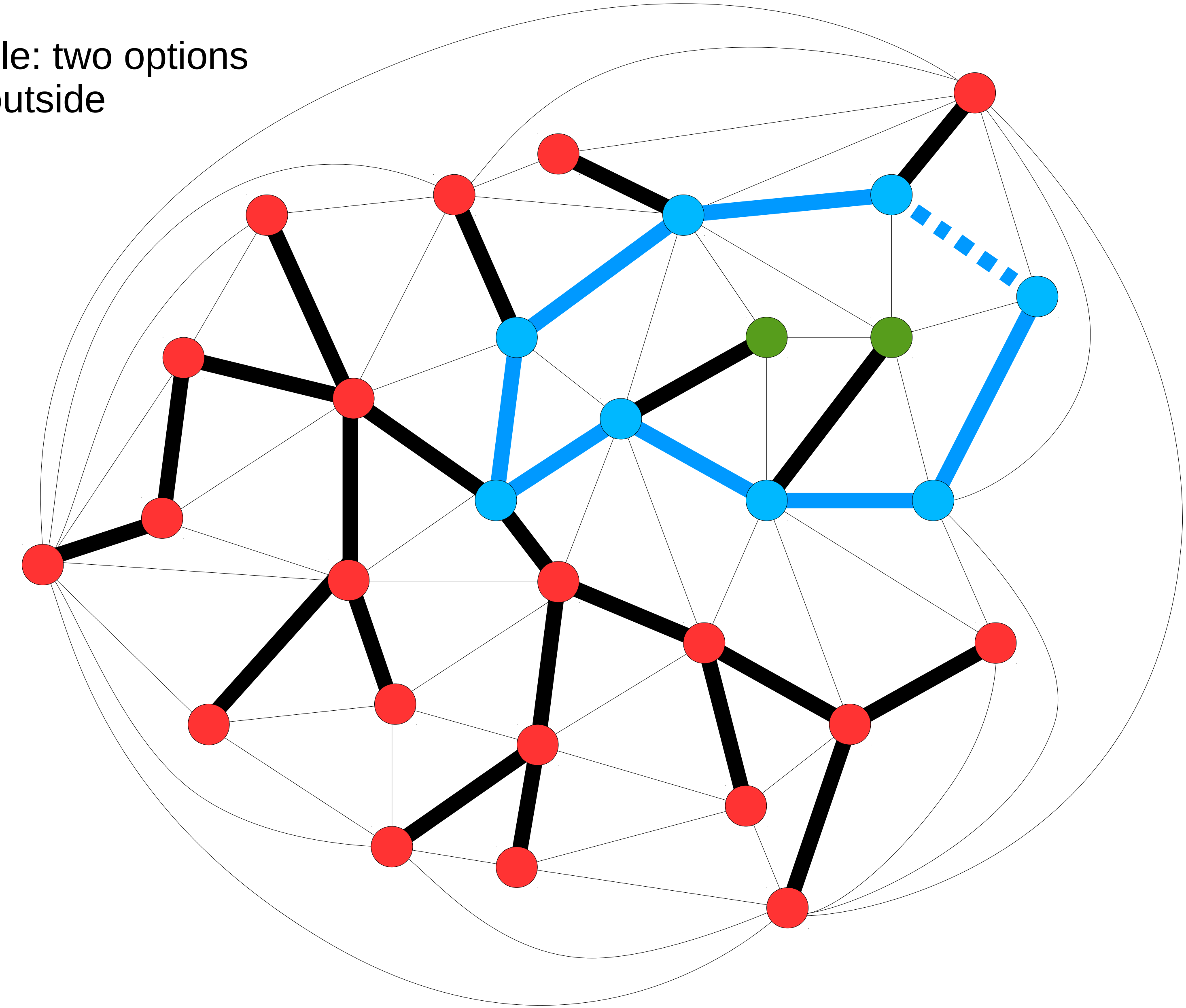
Separating the middle set via spanning tree

- Pick random edge to create a circle



Separating the middle set via spanning tree

Triangle: two options
Why outside



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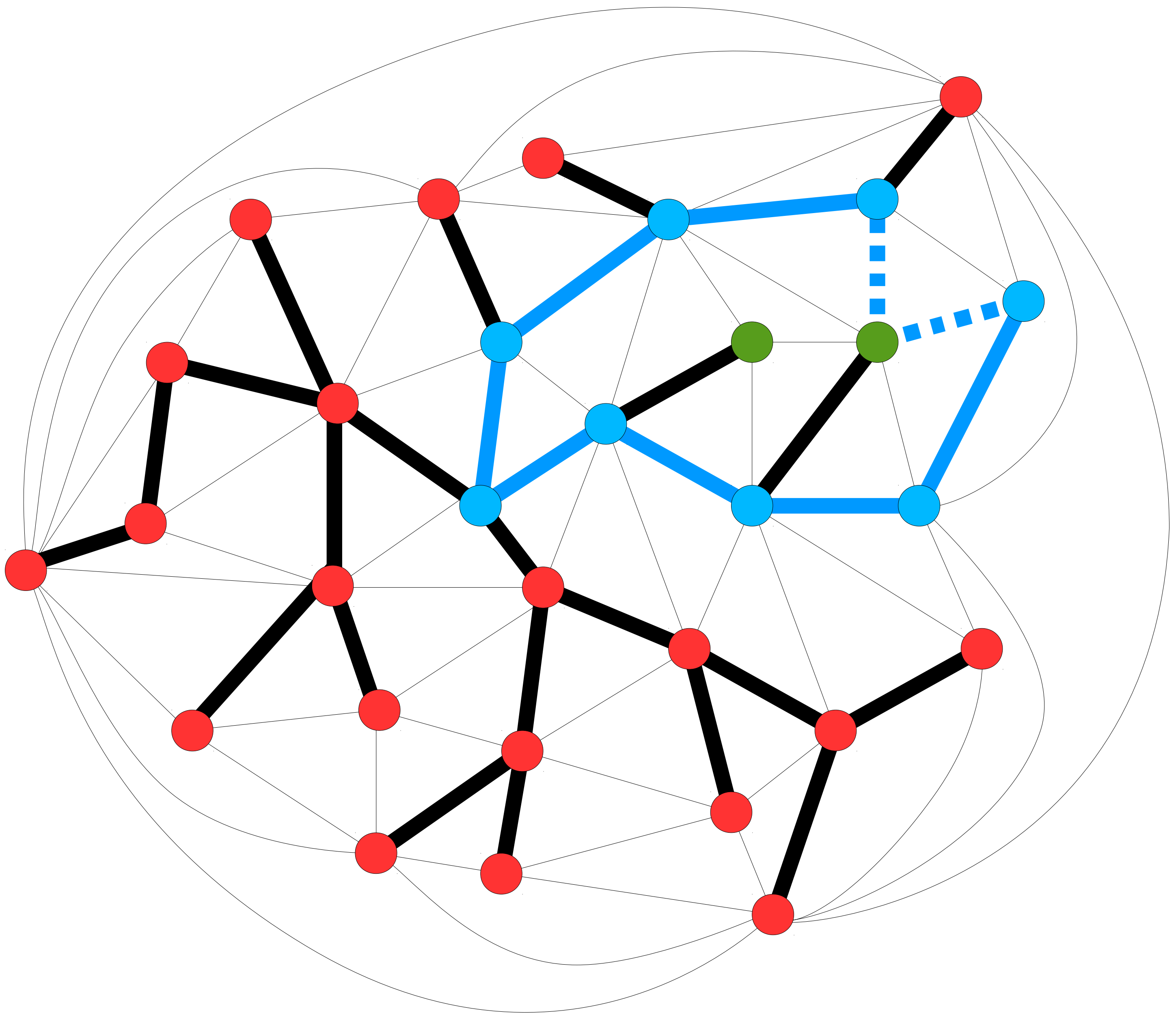

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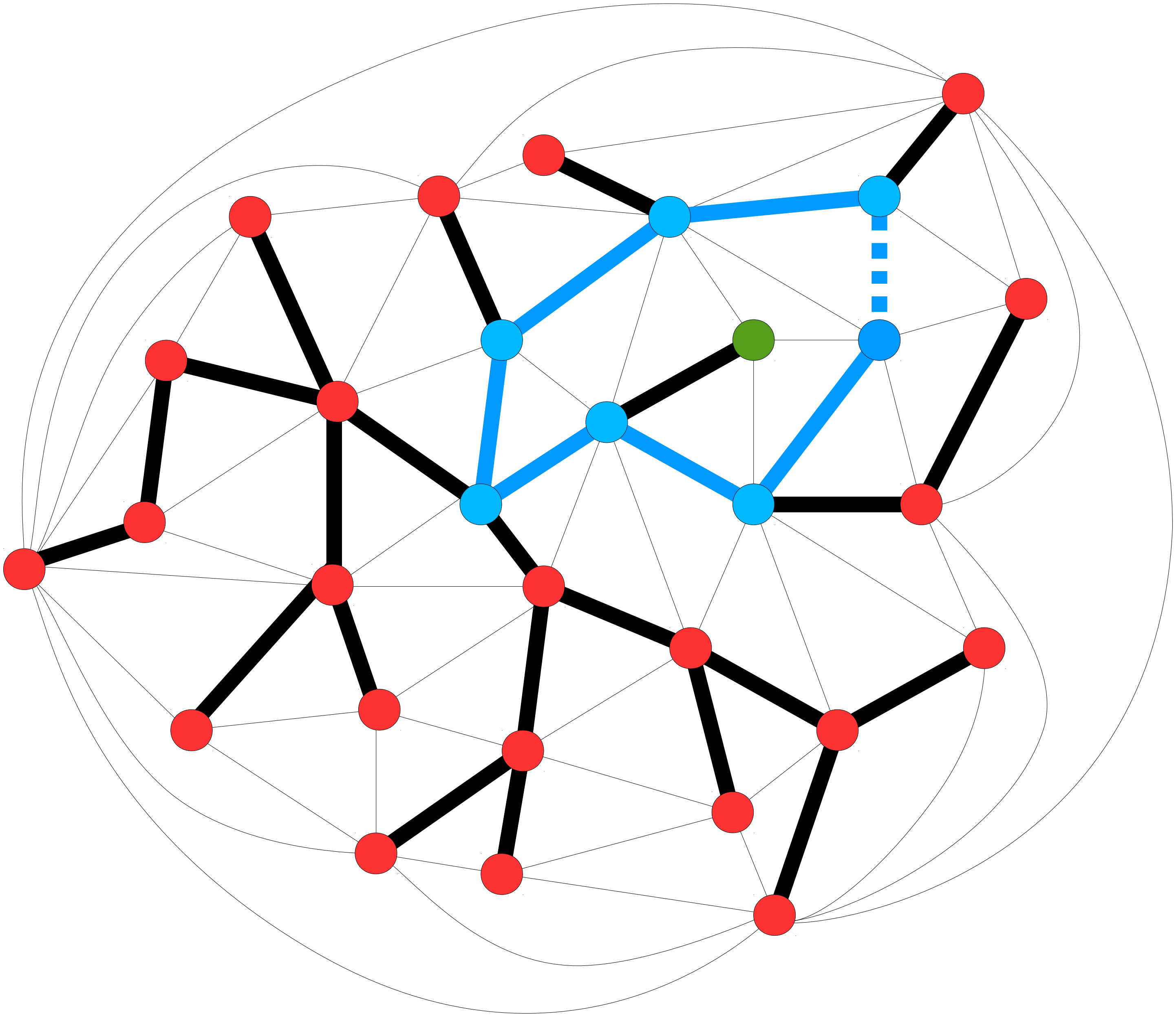
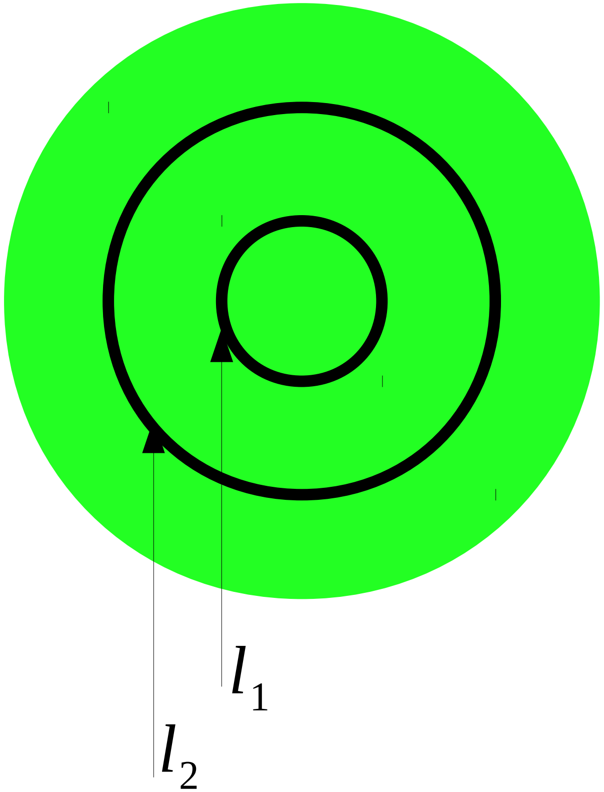
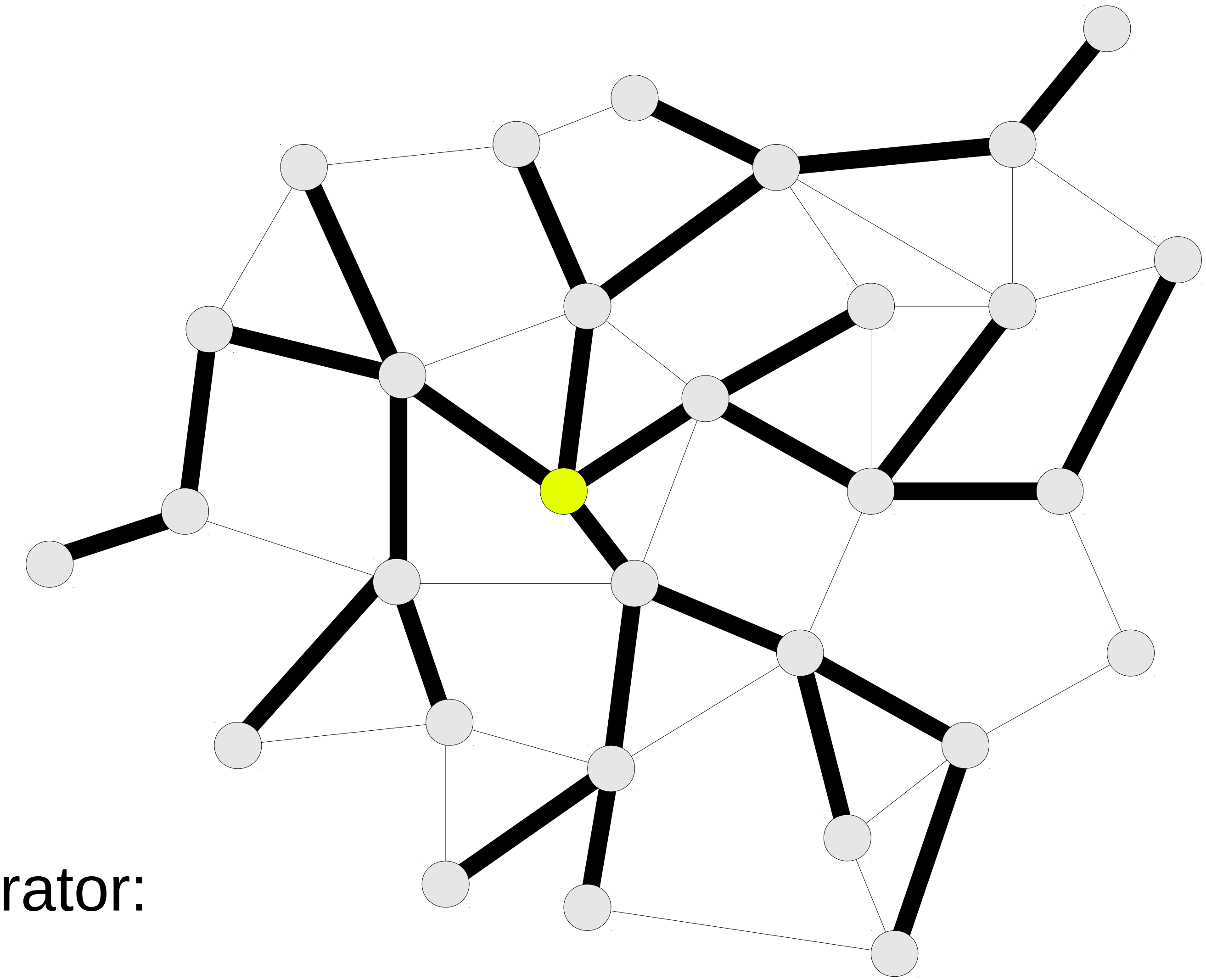


Size of the separator

- **What is the size of the separator?**

Size of middle separator

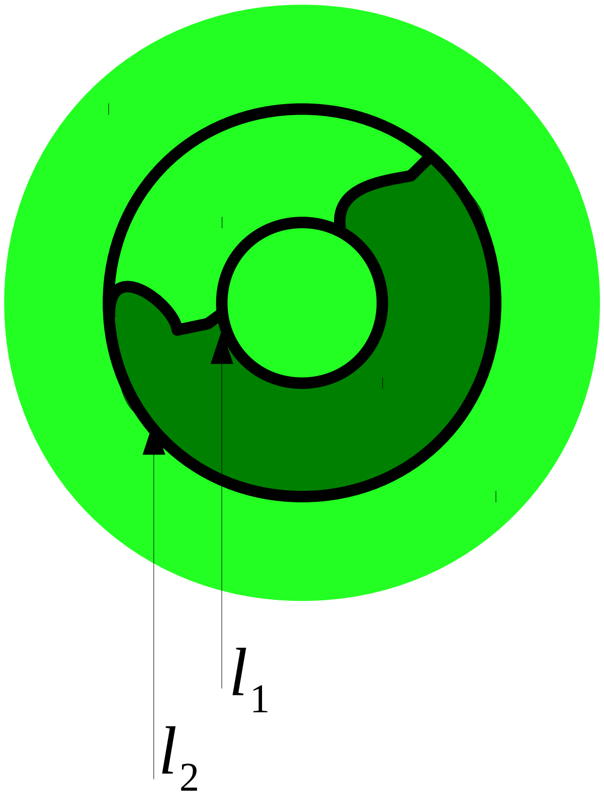




- Radius of tree:
  $r = l_2 - l_1 - 1$

- Size of middle separator:
  $|S_{middle}| \leq 2(l_2 - l_1 - 1)$

Size of the whole separator



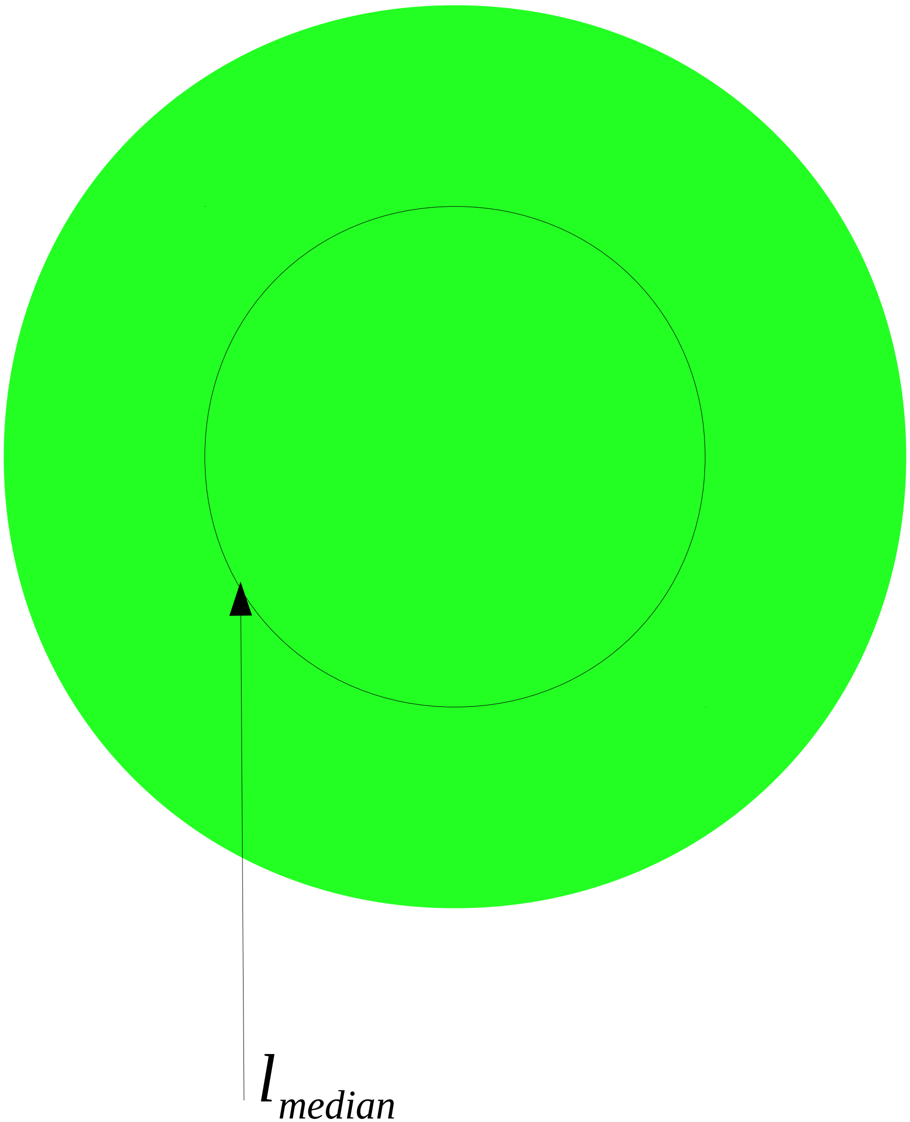
- Separator is linear in number of nodes on both levels plus the middle separator:

$$|S| \leq N(l_1) + N(l_2) + |S_{middle}|$$

$$|S| \leq N(l_1) + N(l_2) + 2(l_2 - l_1 - 1)$$

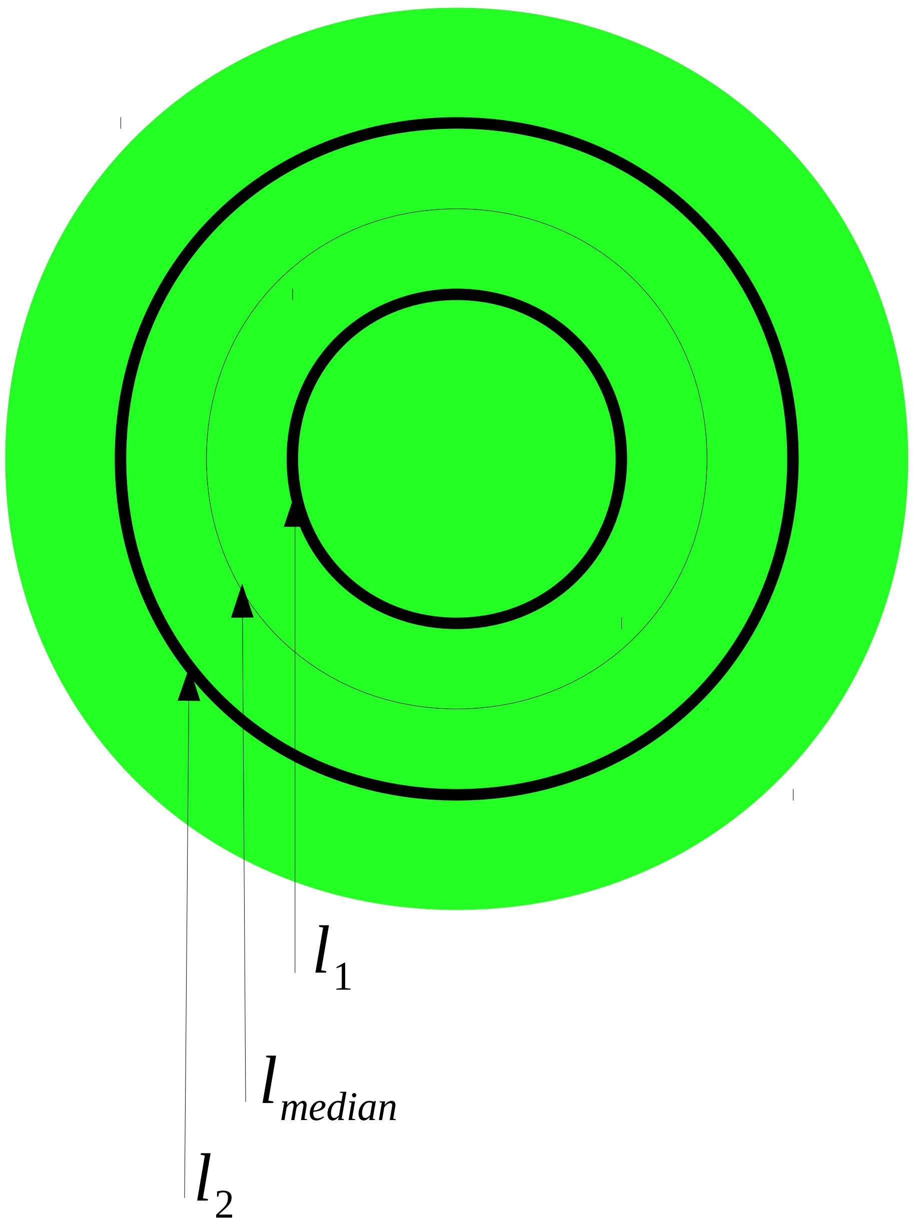
- How to choose l_1 and l_2 so that size of separator is in $O(\sqrt{(n)})$?

Size of the whole separator



- Find median level l_{median}, so that inner and outer set have weight of less than ½
- k := number of nodes on this level

Size of the whole separator



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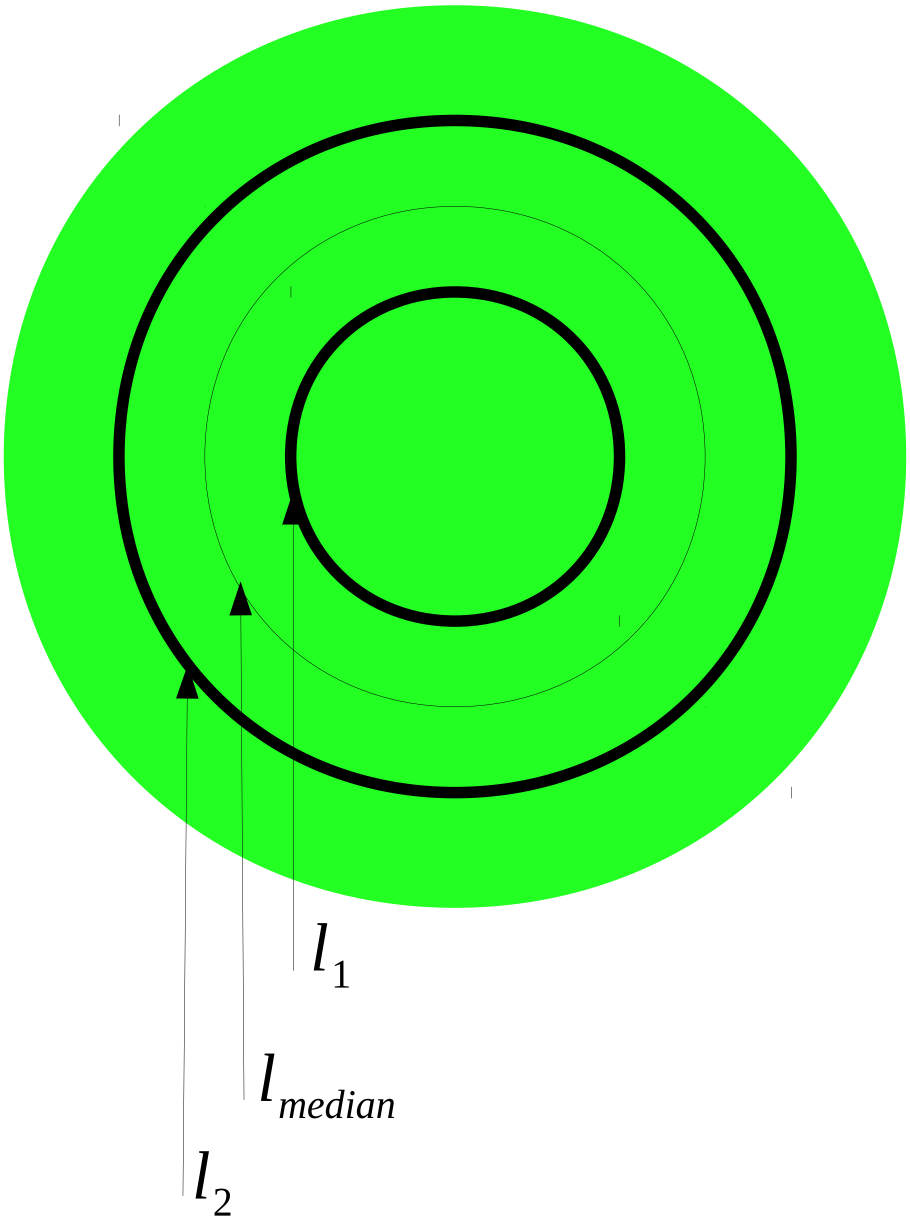

- Find median level l_{median}, so that inner and outer set have weight of less than ½
- k := number of nodes on this level
- Choose l_1, with:

  $N(l_1)+2(l_{median}-l_1)<2\sqrt{(k)}$
- Choose l_2, with:

  $N(l_2)+2(l_2-l_{median}-1)<2\sqrt{(n-k)}$

Size of the whole separator

$$|S| \leq N(l_1) + N(l_2) + 2(l_2 - l_1 - 1)$$

$$\left| \begin{array}{l} N(l_1) + 2(l_{median} - l_1) < 2\sqrt{(k)} \\ N(l_2) + 2(l_2 - l_{median} - 1) < 2\sqrt{(n-k)} \end{array} \right.$$

$$|S| \leq 2\sqrt{(k)} + 2\sqrt{(n-k)}$$

$$|S| \leq 2\sqrt{(n/2)} + 2\sqrt{(n/2)}$$

$$|S| \leq 2\sqrt{(2)}\sqrt{(n)}$$

Sources:

- Lipton, Richard J.; Tarjan, Robert E. (1979), "A separator theorem for planar graphs", SIAM Journal on Applied Mathematics 36 (2): 177–189