Parameterized Algorithms WS 2021
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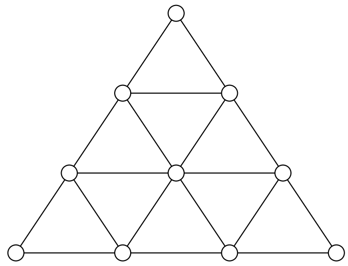
Exercise Sheet 03

Task T8

Determine the treewidth of the graph on the right.



Task T9

The notion of treewidth can be defined in several ways. One way to frame the definition of treewidth is by using the following game called the *cops-and-robber* game. The game consists of a set of cops trying to catch a robber. The robber lives in the the graph and can move with infinite speed along the edges of the graph. He cannot, however, move through a vertex should a cop be guarding it. The cops move about in helicopters, the point being that they are not constrained to move along the edges of the graph, but they have finite speed. The game proceeds as follows. Initially, the robber occupies some vertex of the graph. The cops announce their positions (a set of vertices) and move towards them with finite speed. Seeing their positions, the robber announces his position (a vertex) and moves to that vertex instantaneously. Not all cops need land on vertices at once and not all cops need change positions, that is, if a cop occupies a vertex, it may continue occupying that vertex in the next move of the game. The cops *catch* the robber when one of them lands on a vertex occupied by him.

Show that if a graph has treewidth k then $k+1$ cops can always catch the robber in a cops-and-robber game on the graph. The converse also holds and this is not so easy to show. In the exercises, we will assume this equivalent formulation of treewidth.

Task T10

A tree-decomposition $\langle T, \mathcal{X} = \{X_i \mid i \in V(T)\}\rangle$ of a graph $G = (V, E)$ is *nice* if it is rooted at some node and has only four types of nodes.

1. *Leaf nodes* i, the leaves of the decomposition, with $|X_i| = 1$.
2. *Introduce nodes* i that have exactly one child j such that $X_i = X_j \cup x$, for some vertex $x \in V(G)$.
3. *Forget nodes* i that have exactly one child j such that $X_i = X_j \setminus x$, for some vertex $x \in V(G)$.
4. *Join nodes* i that have exactly two children j and k such that $X_i = X_j = X_k$.

Given a tree decomposition $\langle T', \mathcal{X}'\rangle$ of G of width w, construct a nice tree-decomposition $\langle T, \mathcal{X}\rangle$ of G in polynomial time of width w such that $|V(T)| = O(w \cdot |V(T')|)$.

Task T11

Let G be a graph and let $S \subseteq V(G)$ be some vertex subset. Show that the following properties are MSO-expressible:

- S is a vertex cover of G
- S is an independent set of G
- G is a connected graph
- S induces a cycle in G
- G has a Hamiltonian path
- S induces an even cycle in G

Task H6 (5 credits)

Let $\langle T, \mathcal{X} \rangle$ be a tree-decomposition of a graph G. Suppose that the subtrees obtained by deleting an edge $\{i, j\} \in E(T)$ are T_i, T_j and let G_i, G_j be the subgraphs induced by the vertices in the bags of T_i and T_j, respectively. Show that deleting the set $X_i \cap X_j$ from $V(G)$ disconnects G into two subgraphs $G_i' := G_i - (X_i \cap X_j)$ and $G_j' := G_j - (X_i \cap X_j)$; that is, they do not share vertices and there is no edge with one end in each of them.

Task H7 (10 credits)

Let G be a graph and let $S \subseteq V(G)$ be some vertex subset. Show that the following properties are MSO-expressible:

- S is a dominating set of G
- S is a distance-2 dominating set of G
- P is a longest path in G
- S is a Steiner tree in G