RWTH Aachen **WS 2018**
Lehrgebiet Theoretische Informatik Problem Set 12
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Parameterized Algorithms Tutorial

In the lecture characterized graph classes with bounded expansion using centered colorings. A *centered coloring* is a coloring of a graph such that every connected subgraph has a color that is used exactly once. A graph class $\mathcal{G}$ has *bounded expansion* if for every $r \in \mathbb{N}$ there exists $f(r)$ such that for every $G \in \mathcal{G}$ there exists a coloring with at most $f(r)$ colors where every subgraph with at most r colors has a centered coloring.

Tutorial Exercise T1

The tree-depth of a graph G may be defined as the minimum height of a forest F with the property that every edge of G connects a pair of nodes that have an ancestor-descendant relationship to each other in F. Show that a graph has a centered coloring with at most d colors if and only if it has treedepth at most d.

Proposed Solution

Assume a graph G has treedepth at most d. We consider a forest F with $V(F) = V(G)$ and depth at most d. We color vertices on level i with color i. Since edges of G are only between ancestors and descendants, the resulting coloring is a centered coloring.

Assume a graph G has a centered coloring with at most d colors. We will construct a forest F of depth at most d. We remove the vertex with the unique color of each component and put them as the roots of F. The resulting components have again each a vertex with a unique color, which we put into the next level of F. After d steps we obtain a forest F of depth at most d such that edges in G are only between ancestors and descendants. Therefore, G has treedepth at most d.

Tutorial Exercise T2

We consider robber-cops games with the additional constraint that once a cop is placed at a certain location it cannot leave the location. Show that a graph G has treedepth at most d if and only if d cops can catch a robber in G.

Proposed Solution Assume G has treedepth at most d. This gives us a centered coloring with at most d colors. We observe in which component the robber is and place a cop on the unique color of that component. We repeat at most d times until we find the robber.

Assume d cops can catch the robber in a graph G. We fix a component of G. Assuming the robber is in this component one cop would be placed on a vertex v. We color this vertex with color 1. We repeat the same proceture for every component of G. Then we remove all vertices with color 1. We repeat the procedure d times. Since d cops can always catch the robber the remaining graph is empty. We obtain a centered coloring with d colors which implies that the treedepth is at most d.

Let G be a graph which is colored and H be a subgraph with r colors. It is sufficient to show that the following two statements are identical.

- H has a centered coloring.
- H has treedepth at most d.

Remark

One can obtain an alternative definition of bounded expansion:
$\mathcal{G}$ has bounded expansion if for every $r \in \mathbb{N}$ there exists $f(r)$ such that for every $G \in \mathcal{G}$ there exists a coloring with at most $f(r)$ colors where every subgraph with at most r colors has treedepth at most r.

Homework H1

Show that the treewidth of a graph is smaller than the treedepth of a graph.

Proposed Solution This follows from both robber-cop characterizations. The treewidth characterization gives the cops more freedom, therefore less cops are required.

Homework H2

Show that a path of length 2^d has treedepth at most $O(d)$.

Proposed Solution We use the robber-cop definition. The placement of every cop halves the length of the remaining path.

Homework H3

Show that a graph with treedepth d may contain paths of length at most $2^{O(d)}$.

Proposed Solution Proof via induction. Introducing the root node of a treedepth decomposition to a graph may at most double the maximum path length.