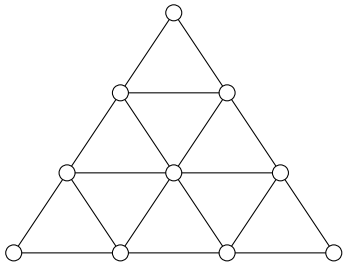
RWTH Aachen
Lehrgebiet Theoretische Informatik
Dreier—Kuinke—Rossmanith

WS 2018
Problem Set 4
12.11.2018

Parameterized Algorithms Tutorial

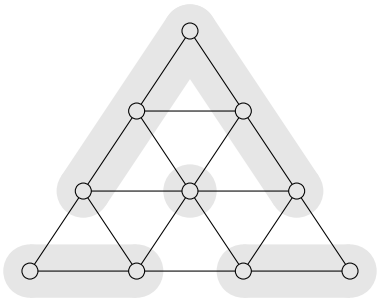
Tutorial Exercise T1

Determine the treewidth of the graph on the right.



Solution

One can give a treewidth decomposition of width three so the treewidth is at most three. The following bramble of order four shows that the treewidth is at least three. Therefore, the treewidth is three.



Tutorial Exercise T2

Recall that a tree-decomposition of a graph $G = (V, E)$ is a pair $\langle T, \mathcal{X} = \{X_i \mid i \in V(T)\}\rangle$, where T is a tree whose vertices are called *nodes* and $\mathcal{X}$ is a collection of subsets of $V(G)$ called *bags* such that the following hold.

1. $\bigcup_{i \in V(T)} X_i = V(G)$.
2. For each edge $\{x, y\} \in E(G)$, there exists $i \in V(T)$ such that $x, y \in X_i$.
3. For all $i, j, k \in V(T)$, if X_j is in the path between X_i and X_k in the tree T, then $X_i \cap X_k \subseteq X_j$.

Show that the last condition can be replaced by the equivalent condition: For each vertex $u \in V(G)$, the set of bags that contain u is a subtree of T.

Solution

We will show the two directions of "Bag condition $\Leftrightarrow$ Subtree condition"

($\Rightarrow$) Assume for some $v \in V(G)$, the bags containing v would *not* form a subtree in $\langle T, \mathcal{X} \rangle$. Then, because of Condition 1, we would could pick two bags X_i, X_j containing v such that some bag X_l inbetween X_i and X_j does *not* contain v. This already implies $X_l \not\supseteq X_i \cap X_j$.

($\Leftarrow$) Assume there exist three bags X_i, X_j, X_l such that l lies on the path between i and j in T but $X_l \not\supseteq X_i \cap X_j$. Pick some vertex v from $(X_i \cap X_j) \setminus X_l$. But then the set of bags containing v cannot be connected and thus not form a tree.

Tutorial Exercise T3

A tree-decomposition $\langle T, \mathcal{X} = \{X_i \mid i \in V(T)\} \rangle$ of a graph $G = (V, E)$ is *nice* if it is rooted at some node and has only four types of nodes.

1. *Leaf nodes* i, the leaves of the decomposition, with $|X_i| = 1$.
2. *Introduce nodes* i that have exactly one child j such that $X_i = X_j \cup x$, for some vertex $x \in V(G)$.
3. *Forget nodes* i that have exactly one child j such that $X_i = X_j \setminus x$, for some vertex $x \in V(G)$.
4. *Join nodes* i that have exactly two children j and k such that $X_i = X_j = X_k$.

Given a tree decomposition $\langle T', \mathcal{X}' \rangle$ of G of width w, construct a nice tree-decomposition $\langle T, \mathcal{X} \rangle$ of G in polynomial time of width w such that $|V(T)| = O(w \cdot |V(T')|)$.

Solution

If a bag X_i is a leaf with $|X_i| > 1$ we can add a bag $X_i \setminus \{v\}$ for some $v \in X_i$ to it without changing the width of the decomposition. If we do so repeatedly, at some point each leaf has size 1. The number of bags added through this process is polyomial in the size of the treedecomposition.

Consider a bag X_i with child X_j such that $|X_i \cap X_j| > 1$. We add a new bag X_l between i and j with $X_l = X_i \cap X_j$. This neither invalidates the tree decomposition nor does it increase its width. After this process, we know that each parent-child pair X_i, X_j either satisfiyes $X_i \subseteq X_j$ or $X_j \subseteq X_i$. Wlog, assume $X_i \subseteq X_j$. If now $|X_i \cap X_j| > 1$, we introduce a bag X_l inbetween i and j with $X_l = (X_i \cap X_j) + v$ for some $v \in X_j \setminus X_i$. Repeating this process exhausively will leave us with a tree decomposition where the intersection of two adjacent bags has size exactly one, which either constitues a Forget- or an Introduce.

Finally, we need to take care of the join bags. Consider a join bag X_i with children $X_{j_1}, \ldots, X_{j_r}$. We replace X_i by a binary tree of copies of X_i with r leafs and connect each leaf to one of the children $X_{j_1}, \ldots, X_{j_r}$. Again, this neither invalidates the tree decomposition nor does it increase the width.

The polynomial running time of this procedure should be obvious.

Tutorial Exercise T4

The notion of treewidth can be defined in several ways. One way to frame the definition of treewidth is by using the following game called the *cops-and-robber* game. The game consists of a set of cops trying to catch a robber. The robber lives in the the graph and can move with infinite speed along the edges of the graph. He cannot, however, move through a vertex should a cop be guarding it. The cops move about in helicopters, the point being that they are not constrained to move along the edges of the graph, but they have finite speed. The game proceeds as follows. Initially, the robber occupies some vertex of the graph. The cops announce their positions (a set of vertices) and move towards them with finite speed. Seeing their positions, the robber announces his position (a vertex) and moves to that vertex instantaneously. Not all cops need land on vertices at once and not all cops need change positions, that is, if a cop occupies a vertex, it may continue occupying that vertex in the next move of the game. The cops *catch* the robber when one of them lands on a vertex occupied by him.

Show that if a graph has treewidth k then $k+1$ cops can always catch the robber in a cops-and-robber game on the graph. The converse also holds and this is not so easy to show. In the exercises, we will assume this equivalent formulation of treewidth.

Solution

Assume a nice tree decomposition. Then the $k+1$ cops can catch the robber as follows: first k of them occupy the vertices of the root bag. As shown in T3, this separates the subgraphs induced by the non-occupied bags. When the robber announces his position, the cops can close in on him by moving in the direction of that subgraph. This is cleary possible if the current bag is a non-join bag by moving a single cop (Introduce: move a cop to the new vertex, Forget: remove a cop). If the bag is a join bag, then, in a nice tree decomposition, all children contain the exact same vertices—thus this reduced to the previous case.

As in each step the component in which the robber resides shrinks by one vertex, the cops always win in a finite number of steps.

Homework H1

Use the properties of a tree-decomposition to show that if a graph G contains a clique C, then every tree-decomposition $\langle T, \mathcal{X} \rangle$ of G has a bag X such that $V(C) \subseteq X$.

Solution

For $|C| = 1$ and $|C| = 2$ this holds per definition. Let us prove the rest by induction. Assume the above holds for cliques of size r. Consider a clique $C \subseteq V(G)$ with $|C| = r+1$. If the above does not hold, then there exists a tree decomposition $\langle T, \mathcal{X} \rangle$ of G such that $C \not\subseteq X$ for each $X \in \mathcal{X}$. Pick three distinct vertices $u, v, w \in C$. By assumption, the cliques $C - u$,$C - v$ and $C - w$ must occur in at least one bag. Let us denote these bags by X_u,X_v and X_w, i.e. $C - a \subseteq X_a$ for $a \in \{u, v, w\}$.

Now, there must exist a bag $Y \in \mathcal{X}$ from which every of the three bags can be reached via a path that does not touch the other bag (this includes the case that Y is one of the three bags). But then we have the following:

- $C \setminus \{u, v, w\} \subseteq Y$ as these vertices are in the common intersection of all three bags

- $u \in Y$ because Y lies between X_v and X_w which contain u
- $v \in Y$ because Y lies between X_u and X_w which contain v
- $w \in Y$ because Y lies between X_u and X_v which contain w

It immediately follows that therefore, $C \subseteq Y$ contrary to our assumption.

Homework H2

Let $\langle T, \mathcal{X} \rangle$ be a tree-decomposition of a graph G. Suppose that the subtrees obtained by deleting a node t from T are $T_1, \ldots, T_r$ and for $1 \leq i \leq r$, let G_i be the graph induced by the vertices of G that are in the bags of T_i. Then the subgraphs $G_1 - X_t, \ldots, G_r - X_t$ have no vertices in common and there are no edges between them.

Solution

First, note that if $\langle T, \mathcal{X} \rangle$ is a valid tree decomposition for a graph G then $\langle T', \mathcal{X}' \rangle$ with $T' \leq T$ and $\mathcal{X}' = \{X_i \in \mathcal{X} \mid i \in T'\}$ is a valid tree decomposition for $G' := G[\bigcup_{X' \in \mathcal{X}'} X']$. Otherwise, whichever condition is violated in $\langle T', \mathcal{X}' \rangle$ will also be violated in $\langle T, \mathcal{X} \rangle$.
Furthermore, if $\langle T, \mathcal{X} \rangle$ is a valid tree decomposition for G then it is also a valid tree decomposition for $G - v$ for some $v \in G$ (we can simply remove v from every bag).
Now, suppose the above was false, i.e. there are at least two subgraph $H_i = G_i - X_t$,$H_j = G_j$,$1 \leq i, j \leq r$ with $H_1 \neq H_2$ that either share a vertex or an edge.

- *Vertex case*: Let $v \in H_i, H_j$ be a common vertex. By construction of H_1, H_2 it follows that $v \notin X_t$. As $\langle T_i, \mathcal{X}_i \rangle$ with $\mathcal{X}_i := \{X_l \setminus X_t \mid l \in T_i\}$ is valid tree decomposition for H_i, it must contain v in some bag. But the same holds for H_j with a similar defined tree decomposition $\langle T_j, \mathcal{X}_j \rangle$ and it follows that the then there exist two bags in subtrees of X_t that contain a vertex that is *not* in X_t. This contradicts our assumption of a valid tree decomposition $\langle T, \mathcal{X} \rangle$.
- *Edge case*: Analogous.

Homework H3

Let $\langle T, \mathcal{X} \rangle$ be a tree-decomposition of a graph G. Suppose that the subtrees obtained by deleting an edge $\{i, j\} \in E(T)$ are T_i, T_j and let G_i, G_j be the subgraphs induced by the vertices in the bags of T_i and T_j, respectively. Show that deleting the set $X_i \cap X_j$ from $V(G)$ disconnects G into two subgraphs $G'_i := G_i - (X_i \cap X_j)$ and $G'_j := G_j - (X_i \cap X_j)$; that is, they do not share vertices and there is no edge with one end in each of them.

Solution

Analogous to T3. First note that if we start with a valid tree-decomposition T of G, and then by deleting the vertex set $X_i \cap X_j$ from each bag of the decomposition, we obtain valid tree-decompositions T_i and T_j of G'_i and G'_j, respectively. Therefore, if G'_i and G'_j shared an edge $\{u, v\}$, then both these vertices would appear in some bag B_1 of T_i and a bag B_2 of T_j. Since T_i and T_j were obtained from T, by Property 3 of a tree-decomposition, every bag on the path from B_1 to B_2 contains u, v. But this means that $\{u, v\} \subseteq X_i \cap X_j$, a contradiction, since we assumed that T_i and T_j were obtained by deleting $X_i \cap X_j$. This shows that G'_i and G'_j cannot share an edge. A similar argument shows that they cannot share vertices.