RWTH Aachen
Lehrgebiet Theoretische Informatik
Reidl—Rossmanith—Sánchez—Sikdar

SS 2014
Problem Set 2
11.12.2014

Parameterized Algorithms Tutorial

Tutorial Exercise T1

In this tutorial and Homework 1, we want to design an FPT-algorithm for the Odd Cycle Transversal problem. An input to this problem consists of a graph $G = (V, E)$ and an integer parameter k, and the question is whether there exists at most k vertices whose deletion renders the graph bipartite.

An input to the *compression* version of this problem has, in addition, a set $S' \subseteq V$ of size $k+1$ that is a valid solution. That is, if you delete the vertices of S' from G, you get a bipartite graph. The question, in this case, is whether there exists a solution of size k. In what follows, we solve the compression version of the problem.

1. First show that if you can solve the compression version in time $O(f(k) \cdot n^c)$, then the Odd Cycle Transversal problem can be solved in time $O(f(k) \cdot n^{c+1})$.
2. For a bipartite graph $G = (V_1 \uplus V_2, E)$ with partite sets V_1, V_2, show that:
   (a) For $i \in \{1, 2\}$, no walk from V_i to V_i has an odd number of edges.
   (b) No walk from V_1 to V_2 has an even number of edges.

Now we are given a solution S' with $k+1$ vertices and we have to decide whether there exists a solution with k vertices. Suppose such a solution S exists, then there exists a partition $S' = L \uplus R \uplus T$, where $T = S \cap S'$ and L, R are the left and right partite sets of the remaining S' in the resulting bipartite graph. The next step is to show that for each partition of S' into $L \uplus R \uplus T$, we can decide in polynomial time whether there exists a vertex set X of size $k - |T|$ in $G - S'$ such that $G - (T \cup X)$ is bipartite with bipartition V_l and V_r such that $L \subseteq V_l$ and $R \subseteq V_r$.

Let $A \uplus B$ be a bipartition of $G - S'$ and let A_l, B_l be the neighbors of L in A and B, respectively. Similarly let A_r, B_r be the neighbors of R in A and B, respectively. First show that if $X \subseteq V(G) \setminus S'$ is a set of vertices such that $G - (T \cup X)$ is bipartite with bipartition V_l and V_r with $L \subseteq V_l$ and $R \subseteq V_r$, then in $G - (S' \cup X)$ there are no paths between A_l and B_l; B_l and B_r; B_r and A_r; and, A_r and A_l.

Proposed Solution

Any path from A_l to B_l in $G - (S' \cup X)$ has odd length and can be extended to a walk from L to L of odd length in $G - (T \cup X)$, a contradiction! A symmetric argument as to why there are no B_r to A_r paths. Any path from B_l to B_r in $G - (S' \cup X)$ must be of even length and this can be extended to a path from L to R in $G - (T \cup X)$ of even length, again a contradiction. A symmetric argument for A_r to A_l paths.

Homework H1 We now prove the converse of what we showed about the set X. Suppose $S' = L \uplus R \uplus T$ such that $G[L]$ and $G[R]$ are empty graphs (graphs without edges). Let $X \subseteq V(G) \setminus S'$ be such that in $G - (S' \cup X)$ there are no paths between A_l and B_l; B_l to B_r; B_r and A_r; and, A_r and A_l.

1. Show that $G - (T \cup X)$ is bipartite and that there exists a bipartition $V_l \uplus V_r$ such that $L \subseteq V_l$ and $R \subseteq V_r$. Proceed as follows.
   (a) Show that every path from a vertex in L to a vertex in L with inner vertices in $V \setminus (S' \cup X)$ has even length. Also show that every path from a vertex in L to a vertex in R with inner vertices in $V \setminus (S' \cup X)$ has odd length. Conclude that *if* $G - (T \cup X)$ is bipartite with partite sets V_l and V_r then $L \subseteq V_l$ and $R \subseteq V_r$.
   (b) Now show that $G - (T \cup X)$ is bipartite. (Show all cycles have even length.)
2. How quickly can you find X? (Use flows.)
3. Describe the algorithm for the compression version of Odd Cycle Transversal and analyze its running time.

[20 points]

Proposed Solution

To show that $G - (T \cup X)$ is bipartite consider a cycle C in this graph. If no vertex of C is from $L \cup R$ then C must be even, as the graph $G - S'$ is bipartite. Therefore let $v_1, \dots, v_t$ be the vertices of $(L \cup R) \cap V(C)$ in the order in which they appear in C. Also let $v_0 = v_t$. The length of the cycle is then

$$|E(C)| = d_C(v_0, v_1) + d_C(v_1, v_2) + \ldots + d_C(v_{t-1}, v_t).$$

Now the number of terms $d_C(v_i, v_{i+1})$ where $v_i \in L$ and $v_{i+1} \in R$ is equal to the number of terms $d_C(v_j, v_{j+1})$, where $v_j \in R$ and $v_{j+1} \in L$. Each of these terms is odd, but together they sum up to be even. All the other terms are even since they describe either L–L or R–R paths. Hence $|E(C)|$ is even.

To find a set X of size $k - |T|$ in $G - S'$, we construct an auxilliary graph $\widetilde{G}$ by introducing two extra vertices s and t, and connecting s to all vertices of $A_l \cup B_r$; t to all vertices of $B_l \cup A_r$. The property that X is a separator of $A_l \cup B_r$ and $B_l \cup A_r$ helps us find such a set using max flow in time $O(k \cdot |E|)$.

The algorithm for the compression version is clear now. Given a solution S' of size $k + 1$, we consider all possible partitions $S' = L \uplus R \uplus T$ such that $G[L]$ and $G[R]$ are empty and $|T| \le k$. For each such partition, we try to compute a set X of size at most $k - |T|$ in the graph $G' - (S' \cup X)$ such that X is a $(A_l \cup B_r, B_l \cup A_r)$ separator using max flow. If such a set X exists, the given instance is a yes-instance with solution $T \cup X$. If for every partition $L \uplus R \uplus T$, there is no such set X, the given instance is a no-instance.

The running time of the compression version is $O(3^k \cdot k \cdot |E|)$ and the running time of the "entire" algorithm is then $O(3^k \cdot k \cdot |V| \cdot |E|)$.