RWTH Aachen | **SS 2014**
Lehrgebiet Theoretische Informatik | Problem Set 5
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Parameterized Algorithms Tutorial

Tutorial Exercise T1

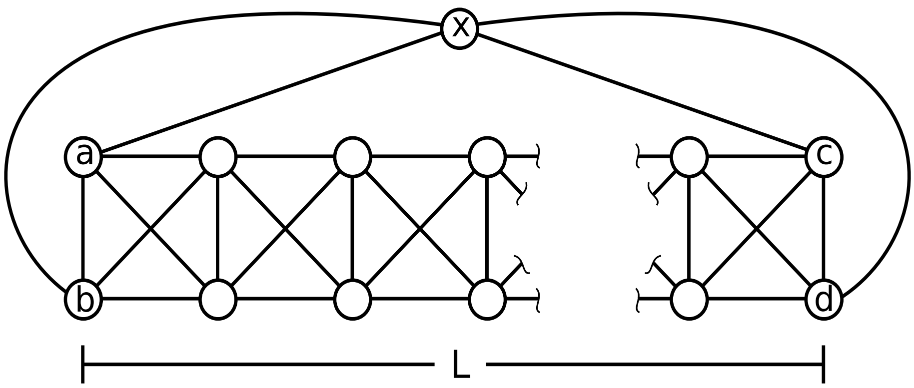
The 4-Coloring problem is the following: given a graph $G = (V, E)$, decide whether there is an assignment of colors to the vertices of G such that adjacent vertices receive distinct colors and the total number of colors used is at most four. This problem is known to be NP-complete.

Show that the property of 4-colorability is hereditary. Does this property admit a finite forbidden set? If not, construct an infinite family $\mathcal{F}$ of non-four-colorable graphs such that

- no two distinct graphs in $\mathcal{F}$ are induced subgraphs of one another;
- for all $G \in \mathcal{F}$, every proper subgraph of G admits a four-coloring.

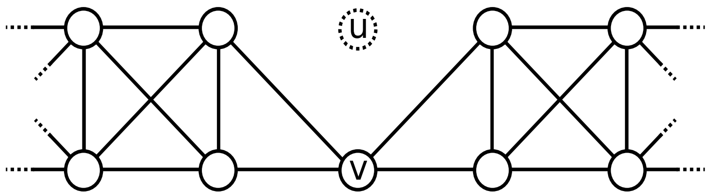
Proposed Solution

Consider the family $\mathcal{F}$ depicted by the following image:



Note that we include only such graphs in $\mathcal{F}$ for which L is even. Clearly, no pair of graphs $G_1, G_2 \in \mathcal{F}$ exists with $G_1 \subsetneq G_2$. We claim that no $G \in \mathcal{F}$ is four-colorable. Assume a, b are assigned the colors 1 and 2, then every second pair of vertices has these colors, too. It follows that every other pair of vertices (starting with the ones right after a, b) has colors 3 and 4, including the pair c, d. Note that this coloring is unique up to permutations of the colors! ut then x is adjacent to four differently-colored vertices and therefore G cannot be colored with four colors.

What is left to show is that if we remove *any* vertex from a graph $G \in \mathcal{F}$, the remaining graph is four-colorable (this excludes the possiblity that there might be a finite number of forbidden subgraphs which are contained in all the graphs of $\mathcal{F}$). This is obviously true if we remove a,b,c,d or x, so let us look at the other vertices. Say we remove a vertex u. The remaining graph then looks like this:



Assume that u, v had the colors $1, 2$ in a coloring of the original graph, which means that the pairs left and right of u, v, had the colors $3, 4$. Without loss of generatily, let v have the color 1 in that coloring. Then we can color $G - u$ by using the coloring of G, but for all vertices on the right of u, v we exchange color 2 and 3. One can easily verify that this is also a valid coloring. But now, x is adjacent to only three colors, namely $1, 2$ and 4 and we can assign color 3 to it.

It follows that $\mathcal{F}$ has the above properties and therefore the property of being four-colorable does not have a finite forbidden set.

Tutorial Exercise T2

The goal of this exercise is to consider problems of the following general type: Let Π be a fixed hereditary property. Given as input a graph G and an integer k, decide whether G has an *induced* subgraph with at least k vertices that satisfy property Π. Note that this is the *dual* version of the problem dealt with in class. Example: Planarity is a hereditary property and the corresponding problem would be to decide whether an input graph has a planar subgraph with at least k vertices.

In this exercise, we will deal with hereditary properties that contain *all* independent sets and *all* cliques. Since this involves some Ramsey-theoretic arguments, that's where we begin:

1. Show that among six people, there are three any two of whom are friends or there are three no two of whom know each other.

2. Show that any sufficiently large graph has either a large independent set or a large clique. More formally, let $R(s, t)$ be the minimum n such that any graph on n vertices has either an independent set of size s or a clique of size t. Use induction on $s + t$ to show that $R(s, t)$ is bounded.

   (a) What are $R(1, t)$, $R(s, 1)$?

   (b) For any $s, t > 1$, show that $R(s, t) \leq R(s - 1, t) + R(s, t - 1)$.

   (c) Show that:
   $$R(s, t) \leq \binom{s + t - 2}{s - 1}.$$
   satisfies the inequality obtained in the last step.

3. Let Π be a hereditary property that contains all cliques and all independent sets. Show that Π-Induced Subgraph problem is in FPT.

Solution

1. Let the six people be A, B, C, D, E, F. Start with A. Among the five people $\{B, C, D, E, F\}$, A either knows three or does not know three. We consider the case when A *knows* three people, as the other case is symmetric. Without loss of generality, assume that A knows B, C, D. If either

   (a) B and C know each other, or

   (b) C and D know each other, or

   (c) B and D know each other,

   we already obtain a triple of people, each of whom knows the others (Draw a picture!). Therefore suppose that this is not the case. Then B, C and D form a triple none of whom knows the others.

2. The proof consists of inducting on $s+t$. Observe that $R(1,t) = R(s,1) = 1$, so that in particular $R(s,t)$ is bounded by $\binom{s+t-2}{s-1}$ when $s+t \leq 2$. The induction hypothesis is: assume that $R(s,t)$ is bounded from above by $\binom{s+t-2}{s-1}$ for all values of s and t with $s+t \leq k$, where $k \geq 2$. We simultaneously prove the inequality $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

   Let $n = R(s-1,t) + R(s,t-1)$ and let G be any graph on n vertices. Fix an arbitrary vertex v of G and consider the two vertex sets S and T, where S is the set of neighbors of v and T is the set of non-neighbors of v. Then $|S| + |T| + 1 = n$ and one of the two inequalities is true:

   (a) $|S| \geq R(s,t-1)$, or

   (b) $|T| \geq R(s-1,t)$.

   For if both are false, we would have:

   (a) $|S| + 1 \leq R(s,t-1)$, and

   (b) $|T| + 1 \leq R(s-1,t)$,

   which in turn implies that $|S|+|T|+2 \leq R(s,t-1)+R(s-1,t)$, a contradiction! If $|S| \geq R(s,t-1)$, apply induction hypothesis on the graph induced by S to conclude the existence of either an s-independent set or a $(t-1)$-clique; the latter can be combined with v to obtain a t-clique. Hence, G has either an s-independent set or a t-clique. If, on the other hand, $|T| \geq R(s-1,t)$, apply the induction hypothesis to the graph induced by T to obtain either an $(s-1)$-independent set or a t-clique. Add v to the independent set to obtain an s-independent set. This would then show that G has either an s-independent set or a t-clique. We must therefore have that $R(s,t) \leq R(s-1,t) + R(s,t-1)$. Again apply induction hypothesis to obtain:

   $$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \\ &\leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} \\ &= \binom{s+t-2}{s-1}. \end{aligned}$$

   This complete the proof.

3. Let Π be a hereditary property that contains all independent sets and all cliques; and let (G, k) be an instance of the Π-Induced Subgraph problem. If $|V(G)| \geq R(k, k)$, then G has either a k-independent set or a k-clique and hence (G, k) is a yes-instance. Else, $|V(G)| < R(k, k) \leq (2k)^k$. We can now use brute-force to check whether G indeed has a k-vertex induced subgraph satisfying Π.

Homework H1

Let k be a constant and consider the class of graphs that have a vertex cover of size *exactly* k. Does this class define a hereditary property? What can you say about the class of graphs that have a vertex cover of size *at most* k? In case you believe that the property is hereditary, how large is the forbidden set?

[10 Points]

Proposed Solution

The property asking for a vertex cover of *exactly* size k is not hereditary: especially all graphs of size *smaller* than k do not have this property. As these occur as subgraphs of graphs which have it, the property is note hereditary.

The property asking for a vertex cover of size *at most* k is hereditary by the following argument: say G is a graph with a vertex cover S of size k. Then for every subgraph $G' \leq G$, the set $S \cap V(G')$ is a vertex cover.

We will now prove that a finite set of forbidden subgraph suffices to characterize this property. Consider the set $\mathcal{F}$ of all graphs which have a minumum vertex cover of size $k+1$. For any such graph G, let S be a vertex cover of that size. As we saw with the Buzz kernel, any vertex of degree at least $k+2$ must be inside S. We will use this fact to prove that a set of graphs of *bounded size* suffices as a forbidden set, which implies that this set is finite.

First, take all graphs $G \in \mathcal{F}$ where all vertices in S have a degree bounded by $k+1$. Call this set $\mathcal{F}_{small}$. We can diretly see that this set is finite as it contains graphs of size at most $O(k^2)$.

Now, consider graphs $G \in \mathcal{F}$ where a set of vertices $S' \subseteq S$ have degree larger than $k+1$. We want to argue that if the degree of a vertex $v \in S'$ is *very* high, we can find a subgraph of G which also has S as a minumum vertex cover—which of course means that G does not need to be included in the forbidden set.

Suppose there exists a vertex $v \in V(G) \setminus S$ which is not adjacent to any vertex of $S \setminus S'$ such that in $G - v$, all vertices of S' *still* have degree $> k+1$. Then clearly, S is still a vertex cover of $G - v$ and no smaller vertex cover can exist: the vertices of S' must be included in it anyway, as their degree is still large, and the vertices of $S \setminus S'$ are not affected by the removeal of v (any smaller cover of $G - v$ would imply a smaller cover for G which contradicts our initial assumption).

Now, we build our set $\mathcal{F}_{large}$ as follows: include any graph from $\mathcal{F} \setminus \mathcal{F}_{small}$ which does *not* have such a vertex.

Homework H2

Let Π be a hereditary property that *excludes* some clique and some independent set. Show that Π-Induced Subgraph is FPT.

Proposed Solution

Let the size of the smallest independent set and smallest clique excluded by Π be s and t, respectively. Then no graph with $R(s,t)$ vertices or more satisfies Π. Since s and t are constants, this implies that Π is finite and hence the Induced Subgraph problem is *constant* time solvable.