RWTH Aachen **WS 2011/12**
Lehrgebiet Theoretische Informatik Problem Set 9
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Parameterized Algorithms Tutorial

In this tutorial we will revisit the basic technique of proving that certain problems that are fixed-parameter tractable do not admit polynomial kernels. In this context, recall that a *distillation algorithm* for a decision problem $L \subseteq \Sigma^*$ is an algorithm that

- recieves as input a sequence $(x_1, \ldots, x_t)$, with $x_i \in \Sigma^*$ for each $1 \leq i \leq t$;
- uses time polynomial in $\sum_{i=1}^{t} |x_i|$;
- outputs a string $y \in \Sigma^*$ of length polynomial in $\max_{1 \leq i \leq t} |x_i|$ such that $y \in L$ iff $x_i \in L$ for some $1 \leq i \leq t$.

It is known that it is implausible for NP-complete problems to admit distillation algorithms (the exact complexity-theoretic collapse is not that relevant for us).
Also recall that a *composition algorithm* for a parameterized problem $L \subseteq \Sigma^* \times \mathbf{N}$ is an algorithm that

- receives as input a sequence $((x_1, k), \ldots, (x_t, k))$, with $(x_i, k) \in \Sigma^* \times \mathbf{N}$ for each $1 \leq i \leq t$;
- uses time polynomial in $\sum_{i=1}^{t} |x_i| + k$;
- outputs $(y, k') \in \Sigma^* \times \mathbf{N}$ such that k' is polynomial in k and $(y, k') \in L$ iff $(x_i, k) \in L$ for some $1 \leq i \leq t$.

Then the basic result that we use to prove the non-existence of polynomial kernels is: *A compositional parameterized problem whose unparameterized version is NP-complete does not admit a polynomial kernel (modulo some complexity-theoretic hypothesis).*

Tutorial Exercise T23

Show that the problems k-Path and k-Cycle are compositional.

Tutorial Exercise T24

Consider the w-Independent Set problem: given a graph G, a tree-decomposition $\mathcal{T}$ of G of width w and an integer k, decide whether G has an independent set of size at least k. This problem is fixed-parameter tractable wrt w as parameter. Our goal is to show that this problem does not admit a polynomial kernel. To do this, first show that the problem w-Independent Set Refinement, defined as follows, does not admit a polynomial kernel. Given a graph G, a tree-decomposition $\mathcal{T}$ of G of width w, and an independent set I of G, decide whether G has an independent set of size at least $|I| + 1$, where w is the parameter. Use this result to show that w-Independent Set does not admit a polynomial kernel.

Tutorial Exercise T25

We generalize an observation from the last exercise. Suppose that A and B are parameterized problems such that the unparameterized version of A is NP-complete and that of B is in NP. Also suppose that there is a polynomial-time algorithm $\mathcal{A}$ that takes an instance (x, k) of A and, in time polynomial in $|x| + k$, outputs an instance (y, k') of B such that k' is polynomial in k and $(x, k) \in A$ iff $(y, k') \in B$. Show that if B admits a polynomial kernel then so does A. What can we say if B is NP-complete and compositional? The algorithm $\mathcal{A}$ is called a polynomial-time parameter-preserving reduction.

Homework H19

Show that the following problems are compositional.

1. w-Clique: An input consists of a graph G, a tree-decomposition of G of width w, and an integer k. The problem is to decide whether G contains a clique of size k. The parameter is the width w.

2. k-Leaf Out-tree: An out-tree is a directed graph with a distinguished root vertex such that the underlying graph is a tree and all arcs are oriented away from the root. This problem is defined as follows: given a directed graph D and an integer parameter k, does D have as a subdigraph an out-tree with at least k leaves.

Homework H20

Show that the problem w-Dominating Set does not admit a polynomial kernel. This problem is defined similar to w-Independent Set: given a graph G, a tree-decomposition of G of width w, an integer k, decide whether G has a dominating set of size k. The parameter is the width w. [Hint: Use the same strategy as for w-Independent Set.]

Homework H21

Show that the following problem does not admit a polynomial kernel. k-Disjoint Paths: given a graph G and an integer parameter k, decide whether G has k vertex disjoint paths, each of length k. [Hint: Give a polynomial-time parameter-preserving reduction from k-Path.]