RWTH Aachen
Lehrgebiet Theoretische Informatik
Reidl—Rossmanith—Sánchez—Sikdar

SS 2013
Problem Set 5
04.06.2013

Parameterized Algorithms Tutorial

Tutorial Exercise T12

Let $k > 0$ be a constant and consider the class of graphs that have a vertex cover of size *exactly* k. Does this class define a hereditary property? What can you say about the class of graphs that have a vertex cover of size *at most* k? In case you believe that the property is hereditary, how large is the forbidden set?

Proposed Solution

The property asking for a vertex cover of *exactly* size k is not hereditary. Any graph with a vertex cover of size exactly $k > 0$ has an independent set on $n - k$ vertices and the graph induced by the independent set has a vertex cover of size zero.

The set $\mathcal{F}$ of graphs with a vertex cover of size *at most* k is hereditary since if S is a vertex cover of $G \in \mathcal{F}$ of size at most k, then for any subgraph $H \subseteq G$, $S \cap V(H)$ is a vertex cover of H.

Let $\mathcal{F}$ be a set of graphs that do not have a vertex cover of size k or less. Moreover suppose that $\mathcal{F}$ is minimal in the sense that

1. No two members of $\mathcal{F}$ are induced subgraphs of each other.
2. Every proper subgraph of any member of $\mathcal{F}$ has a vertex cover of k or less.

We show that such a set $\mathcal{F}$ has size $2^{O(k^4)}$. First note that no member of $\mathcal{F}$ has a vertex cover of size $k + 2$ or more. For if G is such a graph and if S is a vertex cover of G of size $k + 2$, then deleting any vertex of $x \in S$ along with all its degree-one neighbors in $V(G) \setminus S$ creates an induced subgraph with vertex cover number $k+1$, contradicting the second requirement.

Therefore let $G \in \mathcal{F}$ with vertex cover number $k + 1$ and let $S = \{v_1, \dots, v_{k+1}\}$ be an optimal vertex cover of G. If every vertex of S has degree at most $k + 1$, then clearly $|V(G)| \leq k+1+(k+1)^2$. Hence consider the case when the first r vertices of S, $v_1, \dots, v_r$, have degree at least $k + 2$. Let $S' = \{v_1, \dots, v_r\}$. Consider a vertex $x \in V(G) \setminus S$ which is incident to vertices in S' only (that is, it has no neighbors in S). We claim that the graph $G - x$ has a vertex cover number of $k + 1$. Suppose $G - x$ has a vertex cover of size k, then every vertex of S' would be in this vertex cover since in $G - x$, every vertex of S' has degree at least $k + 1$. But this means that any optimal vertex cover of $G - x$ is also a vertex cover of G (of size k), contradicting the hypothesis that G has a vertex cover number of $k + 1$. Since we assumed that $\mathcal{F}$ is minimal in the sense that any proper subgraph of a member has a vertex cover of size k or less, this means that such a graph G cannot be in $\mathcal{F}$. Therefore, if $x \in V(G) \setminus S$ is a neighbor of some vertex in S', it must be a neighbor of some vertex in $S \setminus S'$. But since every vertex of $S \setminus S'$ has degree at most $k + 1$, the total number of such neighbors is at most $(k + 1)^2$. The total number of vertices of G is therefore $k + 1 + (k + 1)^2$. Therefore every member of $\mathcal{F}$ has at most $O(k^2)$ vertices and since k is a constant, this means that $|\mathcal{F}|$ is finite.

Tutorial Exercise T13
The Nemhauser–Trotter-Theorem states that given an undirected graph $G = (V, E)$, one can in polynomial time to find two disjoint vertex sets $C_0 \uplus V_0 \subseteq V$ such that

1. If $D \subseteq V_0$ is a vertex cover of $G[V_0]$, then $D \cup C_0$ is a vertex cover of G.
2. There is an optimal vertex cover S of G with $C_0 \subseteq S$.
3. Every vertex cover of $G[V_0]$ has size at least $|V_0|/2$.

Show that:

1. If S' is an optimal vertex cover of $G[V_0]$, then $S' \cup C_0$ is an optimal vertex cover of G.
2. If G has a vertex cover of size k, then $|V_0| + |C_0| \leq 2k$.

Proposed Solution
Let S_0 be an optimal vertex cover of $G[V_0]$. By condidtion 1, we know that $S' = S_0 \cup C_0$ is a vertex cover of G—it remains to show that is also optimal for G. Assume the contrary and let thus S'' be a vertex of G with $|S''| < |S'|$. By condition 2, we can assume that $C_0 \subseteq S''$ holds. Similary, $C_0 \subseteq S'$ holds by construction of S'. It follows that

$$
\begin{aligned}
& |S''| < |S'| \\
\Rightarrow \quad & |S''| - |C_0| < |S'| - |C_0| \\
\Rightarrow \quad & |S'' \setminus C_0| < |S' \setminus C_0| = S_0
\end{aligned}
$$

and therefore $S'' \cap V_0$ is smaller than S_0. But this means we found a smaller vertex cover for $G[V_0]$, which contradicts our initial assumption. This proves statement 1.
For statement 2, assume S is a vertex cover of G with $|S| = k$. Again we assume that $C_0 \subseteq S$. By condition 3 we know that $|S \cap V_0| \geq |V_0|/2$. Putting this together we obtain

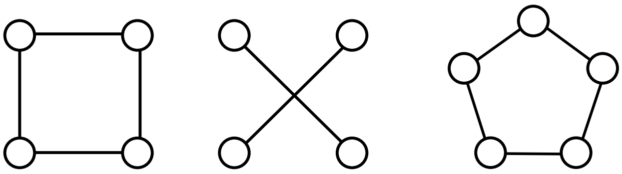
$$
\begin{aligned}
|V_0| + |C_0| &= |V_0| + |S \cap C_0| \leq 2|S \cap V_0| + |S \cap C_0| \\
&= 2|S \cap V_0| + (|S| - |S \cap V_0|) \\
&= |S \cap V_0| + |S| \leq 2k
\end{aligned}
$$

Tutorial Exercise T14
In the Planar Independent Set problem, you are given a planar graph $G = (V, E)$ and an integer parameter k. The question is to decide whether G has an independent set of size at least k. Show that this problem admits a linear kernel.

Proposed Solution
Using the Four-Color Theorem, we know that any independent set of a planar graph is of size at least $n/4$. Therefore if $k \leq n/4$, the instance is a yes-instance and we return with a trivial instance. If $n/4 < k$ then $n < 4k$ and we have a linear kernel.

Homework H9

A graph $G = (V, E)$ is a *divorce graph* if its vertex set can be partitioned into sets $X \uplus Y$ such that $G[X]$ is a complete graph and $G[Y]$ has no edges (there can be edges between the sets X and Y). Show that a graph is a divorce graph if and only if it does not contain the following two graphs as induced subgraphs:



[10 points]

Homework H10

Consider the following problem. You are given a set U of n elements; a family $\mathcal{F}$ of subsetes of U such that each subset has at most three elements; and an integer k. The problem is to decide whether there exists a set $S \subseteq U$ of size at most k such that $S \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. That is, is there a set of size at most k that "hits" all the members of $\mathcal{F}$? Show that this problem admits a kernel of size $O(k^3)$. Proceed as follows:

1. Show that we may reduce to an equivalent instance where every pair of distinct elements occurs together in at most k sets. That is, it is no loss of generality to assume that for each distinct pair x and y, there are at most k sets in $\mathcal{F}$ where they occur together.

2. Next show that we may then reduce to an equivalent instance where every element of U occurs in at most k^2 sets of $\mathcal{F}$.

3. Using the above two properties conclude that every yes-instance of the problem may be reduced to an equivalent instance with $|\mathcal{F}| \leq k^3$ and $|U| \leq 3k^3$.

[10 points]