RWTH Aachen **SS 2013**
Lehrgebiet Theoretische Informatik Problem Set 4
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Parameterized Algorithms Tutorial

Tutorial Exercise T10 Let G be a graph and let $S \subseteq V(G)$ be some vertex subset. Show that the following properties are MSO-expressible:

- S is a vertex cover of G
- S is an independent set of G
- G is a connected graph
- S induces a cycle in G
- G has a Hamiltonian path
- S induces an even cycle in G

Proposed Solution

- Vertex cover: $vc(S) = \forall x \forall y(\neg xEy \vee x \in S \vee y \in S)$
- Independent set: $is(S) = \forall x \forall y(\neg xEy \vee x \notin S \vee y \notin S)$
- Connected: Let us introduce a slightly more general formula.

  $$con(S) = \forall A \forall B((A \subseteq S \wedge B \subseteq S) \rightarrow \exists a \exists b(a \in A \wedge b \in B \wedge aEb))$$

  Where $con(S)$ means that $G[S]$ is connected ($con(V)$ is the formula we need for this part of the exercise)
- Cycle:

  $$\begin{aligned} cycle(S) = con(S) \wedge \forall x \exists a \exists p \forall y((x \in S \wedge y \in S) \\ \rightarrow a \neq p \wedge a \in S \wedge p \in S \\ \wedge (xEy \rightarrow y = a \vee y = p)) \end{aligned}$$

  If we would leave out the connectedness-condition, our formula would also be satisfied by a collection of disjoint cycles.
- Hamiltonian path:

  $$\begin{aligned} hampath =& \exists F \subseteq E \exists s \exists t\, path(s,t,V,F) \\ path(s,t,S,F) =& \forall x((x \in S \wedge x \neq s \wedge x \neq t) \\ & \rightarrow \exists a \exists p \forall y(a \in S \wedge p \in S \wedge a \neq p \wedge (xFy \rightarrow y = a \vee y = p))) \end{aligned}$$
- Even cycle:

  $$\begin{aligned} evencycle(S) =& cycle(S) \wedge bipartite(S) \\ bipartite(S) =& \exists A \exists B \forall a \forall b((a \in S \wedge b \in S \wedge aEb) \\ & \rightarrow (a \in A \wedge b \in B) \vee (b \in A \wedge a \in B)) \end{aligned}$$

Tutorial Exercise T11

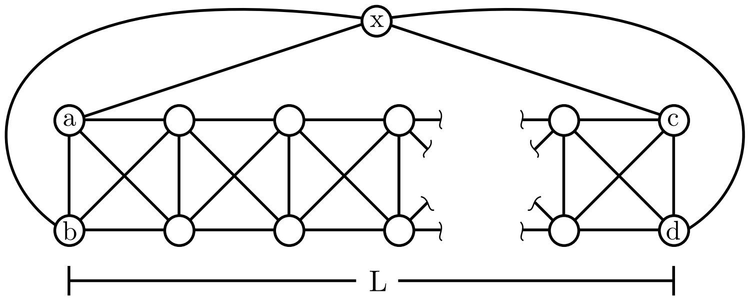
The 4-COLORING problem is the following: given a graph $G = (V, E)$, decide whether there is an assignment of colors to the vertices of G such that adjacent vertices receive distinct colors and the total number of colors used is at most four. This problem is known to be NP-complete.

Show that the property of 4-colorability is hereditary. Does this property admit a finite forbidden set? If not, construct an infinite family $\mathcal{F}$ of non-four-colorable graphs such that

- no two distinct graphs in $\mathcal{F}$ are induced subgraphs of one another;
- for all $G \in \mathcal{F}$, every proper subgraph of G admits a four-coloring.

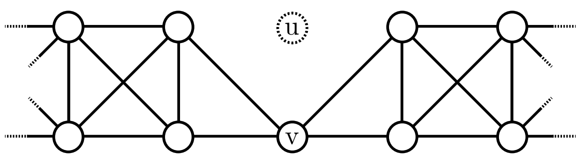
Proposed Solution

Consider the family $\mathcal{F}$ depicted by the following image:



Note that we include only such graphs in $\mathcal{F}$ for which L is even. Clearly, no pair of graphs $G_1, G_2 \in \mathcal{F}$ exists with $G_1 \lneq G_2$. We claim that no $G \in \mathcal{F}$ is four-colorable. Assume a, b are assigned the colors 1 and 2, then every second pair of vertices has these colors, too. It follows that every other pair of vertices (starting with the ones right after a, b) has colors 3 and 4, including the pair c, d. Note that this coloring is unique up to permutations of the colors! ut then x is adjacent to four differently-colored vertices and therefore G cannot be colored with four colors.

What is left to show is that if we remove *any* vertex from a graph $G \in \mathcal{F}$, the remaining graph is four-colorable (this excludes the possiblity that there might be a finite number of forbidden subgraphs which are contained in all the graphs of $\mathcal{F}$). This is obviously true if we remove a,b,c,d or x, so let us look at the other vertices. Say we remove a vertex u. The remaining graph then looks like this:



Assume that u, v had the colors $1, 2$ in a coloring of the original graph, which means that the pairs left and right of u, v, had the colors $3, 4$. Without loss of generatily, let v have the color 1 in that coloring. Then we can color $G - u$ by using the coloring of G, but for all vertices on the right of u, v we exchange color 2 and 3. One can easily verify that this is also a valid coloring. But now, x is adjacent to only three colors, namely $1, 2$ and 4 and we can assign color 3 to it.

It follows that $\mathcal{F}$ has the above properties and therefore the property of being four-colorable does not have a finite forbidden set.

Homework H7

Let G be a graph and $S \subseteq V(G)$ some vertex subset. Show that the following properties are MSO-expressible:

- S is a dominating set of G
- S induces a path in G
- S induces an even path in G
- S induces an odd cycle in G
- G is 3-colorable

[10 points]

Homework H8

Show that planarity is a hereditary property. Is the forbidden set finite or infinite? If your answer is "finite" then construct the forbidden set; if your answer is "infinite", then construct an infinite family $\mathcal{F}$ of non-planar graphs such that

- for all $G \in \mathcal{F}$, all proper subgraphs of G are planar;
- for all distinct $G_1, G_2 \in \mathcal{F}$, we have that G_1 is not an induced subgraph of G_2.

[10 points]