

Parameterized Algorithms Tutorial

Tutorial Exercise T4

An algorithm for the k -PATH problem using color-coding was discussed in class. This algorithm colored the vertices of the input graph using k colors uniformly and independently at random. The main idea was that if the input graph does in fact have a path on k vertices, the vertices of this path get distinct colors with a probability that is solely a function of k . Moreover one can use dynamic programming and detect such a ‘colorful’ path in time $O(2^{O(k)} \cdot \text{poly}(n))$.

We now consider the following idea of using only two colors to obtain a randomized FPT-algorithm for k -PATH. As before assume that the input graph G has a path \mathcal{P} on k vertices. Color the vertices of the graph using two colors, say, black and white uniformly and independently at random. Show that the probability of obtaining a coloring where \mathcal{P} is split into two roughly equal parts G_1 and G_2 is 2^{-k} . Use this idea and recursion to design an algorithm for k -PATH. Analyze its running time.

Tutorial Exercise T5

In this tutorial and Homework 4, we want to design an FPT-algorithm for the ODD CYCLE TRANSVERSAL problem. An input to this problem consists of a graph $G = (V, E)$ and an integer parameter k , and the question is whether there exists at most k vertices whose deletion renders the graph bipartite.

An input to the *compression* version of this problem has, in addition, a set $S' \subseteq V$ of size $k + 1$ that is a valid solution. That is, if you delete the vertices of S' from G , you get a bipartite graph. The question, in this case, is whether there exists a solution of size k . In what follows, we solve the compression version of the problem.

1. First show that if you can solve the compression version in time $O(f(k) \cdot n^c)$, then the ODD CYCLE TRANSVERSAL problem can be solved in time $O(f(k) \cdot n^{c+1})$.
2. For a bipartite graph $G = (V_1 \cup V_2, E)$ with partite sets V_1, V_2 , show that:
 - (a) For $i \in \{1, 2\}$, no walk from V_i to V_i has an odd number of edges.
 - (b) No walk from V_1 to V_2 has an even number of edges.

Now we are given a solution S' with $k + 1$ vertices and we have to decide whether there exists a solution with k vertices. Suppose such a solution S exists, then there exists a partition $S' = L \cup R \cup T$, where $T = S \cap S'$ and L, R are the left and right partite sets of the remaining S' in the resulting bipartite graph. The next step is to show that for each partition of S' into $L \cup R \cup T$, we can decide in polynomial time whether there exists a vertex set X of size $k - |T|$ in $G - S'$ such that $G - (T \cup X)$ is bipartite with bipartition V_l and V_r such that $L \subseteq V_l$ and $R \subseteq V_r$.

Let $A \cup B$ be a bipartition of $G - S'$ and let A_l, B_l be the neighbors of L in A and B , respectively. Similarly let A_r, B_r be the neighbors of R in A and B , respectively. First show that if $X \subseteq V(G) \setminus S'$ is a set of vertices such that $G - (T \cup X)$ is bipartite with bipartition V_l and V_r with $L \subseteq V_l$ and $R \subseteq V_r$, then in $G - (S' \cup X)$ there are no paths between A_l and B_l ; B_l and B_r ; B_r and A_r ; and, A_r and A_l .

Homework H3

Consider the following problem whose inputs consist of a graph $G = (V, E)$ and an integer parameter k . The question is to decide whether G has k vertex-disjoint 3-cycles. Use the idea of randomly coloring the vertices of G with k colors to enable easy detection of vertex-disjoint triangles. What is the expected running time of your algorithm?

[10 points]

Homework H4 We now prove the converse of what we showed about the set X . Suppose $S' = L \cup R \cup T$ such that $G[L]$ and $G[R]$ are empty graphs (graphs without edges). Let $X \subseteq V(G) \setminus S'$ be such that in $G - (S' \cup X)$ there are no paths between A_l and B_l ; B_l to B_r ; B_r and A_r ; and, A_r and A_l .

1. Show that $G - (T \cup X)$ is bipartite with bipartitions V_l and V_r such that $L \subseteq V_l$ and $R \subseteq V_r$. Proceed as follows.
 - (a) Show that every path from a vertex in L to a vertex in L with inner vertices in $V \setminus (S' \cup X)$ has even length. Also show that every path from a vertex in L to a vertex in R with inner vertices in $V \setminus (S' \cup X)$ has odd length. Conclude that if $G - (T \cup X)$ is bipartite with partite sets V_l and V_r then $L \subseteq V_l$ and $R \subseteq V_r$.
 - (b) Now show that $G - (T \cup X)$ is bipartite.
2. How quickly can you find X ? (Use flows.)
3. Describe the algorithm for the compression version of ODD CYCLE TRANSVERSAL and analyze its running time.

[10 points]