RWTH Aachen University **WS 2010/11**
Lehrgebiet Theoretische Informatik Tutorial 14
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Tutorial Exact Algorithms

The hints for the exercises and assignments are to found in the slides for the lectures.

Exercise T29 Properties and uses of the Convolution Product. Let U be a set of n elements and let f, g, h be functions from 2^U to $\mathbf{Z}$. Recall that the convolution of f and g, denoted by $f * g$, is a function from 2^U to $\mathbf{Z}$ defined as follows. For all $S \subseteq U$,

$$(f * g)(S) = \sum_{\substack{X,Y \subseteq S \\ X \cap Y = \emptyset}} f(X)g(Y).$$

Show that the convolution product is commutative and associative, that is,

$$\begin{aligned} f * g &= g * f. \\ (f * g) * h &= f * (g * h). \end{aligned}$$

Given a graph $G = (V, E)$, define $f\colon 2^V \to \{0, 1\}$ as follows. For $S \subseteq V$, we let $f(S) = 1$ iff S is an independent set in G.

1. Show that $(f * f)(V)$ represents the number of two colorings of the graph G.
2. What does the iterated k-fold convolution $(f * \cdots * f)(V)$ (k times) represent? How would you compute the chromatic number of a graph G, if you were given an algorithm that computes the convolution $f * g$?
3. If $f(S)$ is defined to be 1 iff S is a dominating set of G, then what does the iterated k-fold convolution $(f * \cdots * f)(V)$ (k times) represent?

Exercise T30 Fast Ranked Zeta Transform. In class we saw how to compute the zeta and Möbius transforms of a function $f\colon 2^U \to \mathbf{Z}$ in time $O^*(2^{|U|} \cdot \log m)$ where m is a positive integer such that for all $S \subseteq U$, it holds that $|f(S)| \leq m$. The objective of this exercise is to show that the ranked version of the zeta transform can also be computed within this time. As usual, we assume that $f(S)$ for all $S \subseteq U = [n]$ can be computed in time $O^*(2^n \cdot \log m)$ using space $O^*(2^n \cdot \log m)$. The steps of the fast algorithm for the ranked zeta transform are as follows:

1. First compute the values of $f(S)$ for all $S \subseteq U$ and store them in a table of size $O^*(2^n \cdot \log m)$.
2. For $0 \leq i \leq n$, define $\zeta_k^{(i)}(f)$ as follows: for all $S \subseteq [n]$,

   $$\zeta_k^{(i)}(f)(S) = \sum_{\substack{X \subseteq S; |X| = k \\ S \setminus [i] \subseteq X}} f(X).$$

   What do $\zeta_k^{(0)}(f)(S)$ and $\zeta_k^{(n)}(f)(S)$ represent?

3. If $i \in S$, show that

$$\zeta_k^{(i)}(f)(S) = \zeta_k^{(i-1)}(f)(S \setminus i) + \zeta_k^{(i-1)}(f)(S).$$

4. Now write down a recurrence for $\zeta_k^{(i)}(f)(S)$ in terms of $\zeta_k^{(i-1)}(f)(S\setminus i)$ and $\zeta_k^{(i-1)}(f)(S)$.

5. What is the time and space taken to compute $\zeta_k^{(i)}(f)(S)$ for all i, k, and S?

Homework Assignment H30 (10 Points) COUNTING k-PARTITIONS. Let U be a set of n elements and let $\mathcal{F}$ be a family of subsets of U. A k-partition of the set system $(U, \mathcal{F})$ is a collection $S_1, \ldots, S_k$ of k sets from the family $\mathcal{F}$ such that $S_i \cap S_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^k S_i = U$. Show that the number of k-partitions of a set system can be written as the iterated k-fold convolution of an appropriate function. This will show that an algorithm for computing the convolution of two functions may be used to count the number of k-partitions of a set system.

Homework Assignment H31 (10 Points) THE MÖBIUS AND ZETA TRANSFORM. In class we saw that the Möbius transform of the zeta transform of a function $f\colon 2^U \to \mathbf{Z}$ is the function f itself. That is, for all $S \subseteq U$

$$\mu(\zeta(f))(S) = \sum_{X \subseteq S} (-1)^{|S\setminus X|} \zeta(f)(X) = f(S).$$

Show that the converse also holds, that is, the zeta transform of the Möbius transform of f is the function f itself.

Homework Assignment H32 (10 Bonus Points) FAST RANKED MÖBIUS TRANSFORM. Given a function $f\colon 2^{[n]} \to \mathbf{Z}$, show that the ranked Möbius transform of f can be computed in time $O^*(2^n \cdot \log m)$ time, where $m \in \mathbf{N}$ is such that for all $S \subseteq [n]$, it holds that $|f(S)| \leq m$. As usual, assume that for all $S \subseteq [n]$ one can compute $f(S)$ in time $O^*(2^n \cdot \log m)$ and space $O^*(2^n \cdot \log m)$.