RWTH Aachen University **WS 2010/11**
Lehrgebiet Theoretische Informatik Tutorial 12
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Tutorial Exact Algorithms

Exercise T25 An $n \times n$ matrix is *stochastic* if all its entries are nonnegative and every row sums up to 1. An $n \times n$ matrix is *doubly stochastic* if its entries are nonnegative and each row and each column sums up to 1.

1. Let M be the incidence matrix of a regular bipartite graph of degree $r \geq 1$. Show that $\frac{1}{r}M$ is doubly stochastic.
2. The Permanent Inequality states that for any doubly stochastic matrix A

$$\text{perm}(A) \geq \text{perm}\left(\frac{1}{n}J_n\right) \geq \frac{n!}{n^n},$$

where J_n is the all-ones $n \times n$ matrix. This inequality was conjectured by van der Waerden in 1926 and proved independently by Egorichev and Falikman in 1980. Use this inequality to show that a regular bipartite graph of degree r has at least $(r/e)^n$ perfect matchings.

Exercise T26 The Hamiltonian Cycle Problem. Given a graph $G = (V, E)$ decide whether G has a simple cycle $(u_1, u_2, \ldots, u_{|V|}, u_1)$ containing all vertices. Formulate this problem as an Inclusion-Exclusion formula and show that it can be solved in time $O^*(2^n)$ and polynomial space.

Solution Fix a vertex $s \in V(G)$ and assume that $V = \{s, u_1, \ldots, u_n\}$. In order to formulate this problem as an IE-formula, we need to decide what the objects are and what properties they must satisfy. An object in this context is a closed walk that starts and ends at s and is of length $n+1$. An object satisfies property A_v for $v \in \{u_1, \ldots, u_n\}$ if it contains vertex v. Clearly an object that satisfies all properties $A_{u_1}, \ldots, A_{u_n}$ is a Hamiltonian cycle of G. Now

$$|A_{u_1} \cap \cdots \cap A_{u_n}| = \sum_{X \subseteq V \setminus s} (-1)^{|X|} \left| \bigcap_{i \in X} \bar{A}_{u_i} \right|.$$

The term $\left|\bigcap_{i \in X} \bar{A}_{u_i}\right|$ represents the number of closed walks of length $n+1$ that start and end at s and avoid the vertices in X. This can be easily computed by finding the (s, s)th entry in the $(n+1)$st power of the adjacency matrix of $G \setminus X$. The time taken to compute this for a fixed subset X is $O(n^{\omega+1})$ and the space required is $O(n^2)$. An alternative way of computing the terms $\left|\bigcap_{i \in X} \bar{A}_{u_i}\right|$ is by dynamic programming. For $X \subseteq \{u_1, \ldots, u_n\}$ and $u \in V \setminus X$ define $P_X(u, k)$ to be the number of walks from s to u of length k that avoid all the vertices in X. Then

$$\left| \bigcap_{i \in X} \bar{A}_{u_i} \right| = \sum_{u \in N(s) \setminus X} P_X(u, n).$$

Also $P_X(u,0) = 1$ if $u = s$ and 0 otherwise. For $k \geq 1$,

$$P_X(u,k) = \sum_{v \in N(u) \setminus X} P_X(v, k-1).$$

For a fixed $X \subseteq \{u_1, \ldots, u_n\}$, the time taken to compute $\left|\bigcap_{i \in X} \bar{A}_{u_i}\right|$ is $O(mn)$ and the space required is $O(n^2)$. Thus the total time taken to evaluate $|A_{u_1} \cap \cdots \cap A_{u_n}|$ is $O^*(2^n)$ and the space required is polynomial.

Homework Assignment H26 (10 Points) Given an $n \times n$ matrix $A = [a_{ij}]$ show that

$$\text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i=1}^{n} \sum_{j \notin S} A[i,j].$$

Hint. One can view the elements of A as variables and the permanent of A as a multinomial polynomial. One way of proving the above is by determining the coefficient of these terms on either side of the purported identity. For example, for a fixed permutation $\pi\colon [n] \to [n]$, the coefficient of the term $a_{1,\pi(1)}, \ldots, a_{n,\pi(n)}$ is a 1 on either side. Now determine the coefficient of terms that involve variables from k columns. The coefficient of such a term is 0 on the left-hand side. What is the coefficient on the right-hand side? Conclude that the permanent of an $n \times n$ matrix can be computed in time $O^*(2^n)$ and space polynomial in n.

Homework Assignment H27 (10 Points) The Travelling Salesperson Problem (TSP). Given a graph $G = (V,E)$ where $V = \{s, u_1, \ldots, u_n\}$ and a weight function $w\colon E \to \{1, \ldots, c\}$, determine the weight of the smallest tour that starts and ends at s. Use Inclusion-Exclusion to solve TSP in time $O^*(2^n c)$. What is the space requirement of your algorithm? Comment on whether it is a polynomial space algorithm.