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WS 2010/11
Tutorial 11
11.01.2011

Tutorial Exact Algorithms

The aim of this tutorial is to provide practice using the principle of Inclusion-Exclusion.

Exercise T22 Determine the number of positive integers n where $1 \leq n \leq 100$ and n is *not* divisible by 2, 3, or 5.

Solution The objects in this case are the integers $1, \ldots, 100$ and the properties are as follows:

$$\begin{aligned} A_1 &= \{1 \leq n \leq 100\colon 2 \text{ divides } n\}. \\ A_2 &= \{1 \leq n \leq 100\colon 3 \text{ divides } n\}. \\ A_3 &= \{1 \leq n \leq 100\colon 5 \text{ divides } n\}. \end{aligned}$$

We are required to find $|\overline{A_1 \cup A_2 \cup A_3}|$. By the Principle of Inclusion-Exclusion, we know that

$$\begin{aligned} |\overline{A_1 \cup A_2 \cup A_3}| &= |\mathcal{U}| - (|A_1| + |A_2| + |A_3|) + |A_1 \cap A_2| \\ &\quad + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

Now $|A_1| = \lfloor 100/2 \rfloor = 50$, $|A_2| = \lfloor 100/3 \rfloor = 33$, and $|A_3| = \lfloor 100/5 \rfloor = 20$;

- $|A_1 \cap A_2| = \lfloor 100/6 \rfloor = 16$;
- $|A_1 \cap A_3| = \lfloor 100/10 \rfloor = 10$;
- $|A_2 \cap A_3| = \lfloor 100/15 \rfloor = 6$;
- $|A_1 \cap A_2 \cap A_3| = \lfloor 100/30 \rfloor = 3$.

Hence,

$$\begin{aligned} |\overline{A_1 \cup A_2 \cup A_3}| &= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3 \\ &= 26. \end{aligned}$$

Exercise T23 Count the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_r = n,$$

where $r, n \geq 1$? Answer the same question under the extra restriction that $x_i < m$ for all $1 \leq i \leq r$.

Solution The number of nonnegative integer solutions is just the number of binary strings with $r-1$ ones and n zeros, which is

$$\binom{n+r-1}{n}.$$

We use Inclusion-Exclusion to count the number of solutions when $x_i < m$. The objects in this setting are r-tuples $(x_1, \ldots, x_r)$ such that $x_i \geq 0$ for all i and $\sum_{i=1}^r x_i = n$. An object satisfies property A_j, for $1 \leq j \leq r$, if $x_j \geq m$. By symmetry, $|A_i| = |A_j|$ for all $1 \leq i \leq j \leq r$ and in fact for all $X, Y \subseteq [r]$ with $|X| = |Y|$, we have that

$$\left|\bigcap_{i \in X} A_i\right| = \left|\bigcap_{j \in X} A_j\right|.$$

We need to estimate $|\bar{A}_1 \cap \cdots \cap \bar{A}_r|$, which by Inclusion-Exclusion is given by:

$$|\bar{A}_1 \cap \cdots \cap \bar{A}_r| = \sum_{X \subseteq [r]} (-1)^{|X|} \left|\bigcap_{j \in X} A_j\right|.$$

Now for $X \subseteq [r]$ with $|X| = j$, the term $|\bigcap_{j \in X} A_j|$ is simply the number of nonnegative integral solutions to the equation

$$x_1 + \cdots + x_r = n - jm,$$

which is given by

$$\binom{n - jm + r - 1}{n - jm}.$$

Hence the number of solutions where $x_i < m$ for all i is

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{n - jm + r - 1}{n - jm}.$$

Exercise T24 THE EULER ϕ-FUNCTION. For $n \in \mathbf{Z}^+$ and $n \geq 2$, let $\phi(n)$ denote the number of positive integers that are relatively prime to n, that is,

$$\phi(n) := \left|\{1 \leq m \leq n \colon \gcd(n, m) = 1\}\right|.$$

For example, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, and $\phi(6) = 2$. Show that

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right),$$

where $p_1, \ldots, p_r$ are the primes in the prime factor decomposition of n.

Solution Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factor decomposition of n. The objects in this setting are the numbers $1, \ldots, n$ and property A_j for $1 \leq j \leq r$ is defined as follows:

$$A_j := \{1 \leq q \leq n \colon p_j \text{ divides } q\}.$$

Then $|\bar{A}_1 \cap \cdots \cap \bar{A}_r|$ counts the number of positive integers that are relatively prime to n. By the principle of Inclusion-Exclusion,

$$\begin{aligned} |\bar{A}_1 \cap \cdots \cap \bar{A}_r| &= \sum_{X \subseteq [r]} (-1)^{|X|} \left|\bigcap_{j \in X} A_j\right| \\ &= \sum_{X \subseteq [r]} (-1)^{r} \left\lfloor \frac{n}{\prod_{i \in X} p_i} \right\rfloor. \end{aligned}$$

Dividing the numerator and denominator by $p_1 p_2 \dots p_r$, we obtain

$$\frac{n}{p_1 p_2 \dots p_r} \left[p_1 p_2 \dots p_r - (p_2 \dots p_r + p_1 p_3 \dots p_r + \cdots p_1 \dots p_{r-1}) + \cdots \right],$$

which evaluates to

$$\frac{n}{p_1 p_2 \dots p_r} (p_1 - 1)(p_2 - 1) \cdots (p_r - 1) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right).$$

Homework Assignment H23 (10 Points) Determine the number of positive integers n, $1 \leq n \leq 2000$, that are

1. not divisible by 2, 3, or 5.
2. not divisible by 2, 3, 5, or 7.
3. not divisible by 2, 3, or 5, but are divisible by 7.

Homework Assignment H24 (10 Points) Determine how many integer solutions there are to $x_1 + x_2 + x_3 + x_4 = 19$, if

1. $0 \leq x_i$ for all i.
2. $0 \leq x_i < 8$ for all i.
3. $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 6$, $3 \leq x_3 \leq 7$, $3 \leq x_4 \leq 8$.

Homework Assignment H25 (10 Bonus Points) Let $n \in \mathbf{Z}^+$. Determine

- $\phi(2^n)$.
- $\phi(2^n p)$, where p is an odd prime.