RWTH Aachen University **WS 2010/11**
Lehrgebiet Theoretische Informatik Tutorial 4
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Tutorial Exact Algorithms

Exercise T6 (again)

The problem MaxCut is defined as follows:

Input:	A graph $G = (V, E)$.
Feasible solutions:	Any bipartition $V = V_1 \cup V_2$.
Goal:	Maximize the number of edges that are *cut* by the bipartition.

Design an algorithm or this problem with a running time of $O^*(\tau(3,3)^m)$, where as usual $m = |E|$.

Solution

As with in the lecture, we for each node branch on the two cases whether it belongs to V_1 or V_2. For the analysis, we let $F := \{ \{u, v\} \in E \mid \{u, v\} \subseteq V \setminus (V_1 \cup V_2) \}$, i.e., F contains only those edges whose endpoints are not "colored" yet. For a vertex v, we let $d(v) = |\{ \{u, v\} \in F \}$ be the number of edges in F incident to v.

By reduction rules, we can remove such vertices with $d(v) = 0$ and $d(v) = 1$. Furthermore, if $d(v) = 2$ for all $v \in V$, the instance can be solved in polynomial time (see lecture notes for details). Hence, we can always branch on vertices with $d(v) \geq 3$, and once we add v to V_1 or V_2, the size of F decreases by at least 3 in each step.

Analyzed in $|F| \leq |E|$, we therefore get a running time of $O^*(\tau(3,3,3)^{|F|}) = O^*(\tau(3,3,3)^{|E|}$.

Exercise T7

Let $F = \{C_1, \ldots, C_m\}$ be a formula in 3-CNF with clauses $C_1, \ldots, C_m$. Define for a variable x the "maxdegree" of x as $d(x) := \max\{ |C| \mid x \in C \}$

Now consider the following simple branching algorithm for the 3SAT problem:

Given is a formula F in 3-CNF. If there is x that occurs only positively or negatively, or a clause $\{x\}$ or $\{\bar{x}\}$, choose the value for x accordingly. Otherwise choose x with maximal $d(x)$ and branch on the two possible assignments $x := 0$ and $x := 1$.

What is the running time in the form $O^*(c^n)$ when analyzed in the number n of variables?

Now consider a new measure. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ and define

$$\Phi(F) := \sum_{x \in vars(F)} \alpha_{d(x)}.$$

- Construct the branching vectors for $\Phi(F)$ for all possible cases.
- What are good choices for $\alpha_1, \alpha_2, \alpha_3$?
- What running time to we obtain?

Solution

A "simple" analysis yields a running time of only $O^*(2^n)$, since we basically branch into two subinstances once for each node.

Let x be a variable with $d(x) = 2$ as in the algorithm. Consider w.l.o.g. the following cases — all remaining cases are similar (for example, with interchanged roles of x and $\bar{x}$) or are easily seen to be better. Note that each variable contributes at least a value of α_2 to $\Phi(F)$ after the reduction rules have been applied.

- $\delta(x) = 2$ and there are clauses $\{x, l_1\}$ and $\{\bar{x}, l_2\}$.

  In the branch $x = 0$, we gain α_2 for x and at least α_2 for l_1, since only l_1 is left to satisfy the first clause. In the branch $x = 1$, we analogously gain α_2 each for x and l_2.

  This yields a branching vector of at least+$(2\alpha_2, 2\alpha_2)$.

- $\delta(x) = 3$ and there is at least one clause $\{x, l_1\}$ of length 2. In the branch $x = 0$, we gain α_2 for x and at least α_2 for l_1, since only l_1 is left to satisfy the first clause.

  In the branch $x = 1$, we gain α_2 for x.

  This yields a branching vector of at least $(2\alpha_2, \alpha_2)$.

- $\delta(x) = 3$ and x is contained only in clauses of length 3, e.g., $\{x, l_1, l_2\}$ and $\{\bar{x}, l_2, l_3\}$.

  In the branch $x = 0$, we gain α_2 for x.

  In the branch $x = 1$, we gain α_2 for x.

  This yields a branching vector of at least (α_2, α_2).

By the last case, it is now not hard to see that $\alpha_2 = 1$ is the best choice, and the algorithms requires at most $O^*(2^{\Phi(F)}) = O^*(2^n)$ steps. So unfortunately, using this measure, we could not show an improvement over the trivial analysis.

Homework Assignment H7 (10 Points)

Consider a formula in F in 4-CNF and the same algorithm as in Exercise T6. Let again $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbf{R}$ and

$$\Phi(F) := \sum_{x \in vars(F)} \alpha_{\delta(x)}.$$

Compute, in terms of $\alpha_1, \alpha_2, \alpha_3$ and α_4, the branching vectors for all possible cases when measured in $\Phi(F)$.

Homework Assignment H8 (10 Points)

Find optimal values for $\alpha_1, \alpha_2, \alpha_3$ and α_4.