Exact Algorithms

Attempt any *four* questions. You may try the questions in any order.

Exercise 1 (10 Points)
Let U be a universe of size $|U| = n$ and $V \subseteq U \times U \times U$ be a set of three-tuples. We say a tuple $v = (x_1, x_2, x_3) \in V$ is covered by a set of elements $C \subseteq U$ if $x_1 \in C$ or $\{x_2, x_3\} \subseteq C$. Note that $(x_1, x_2, x_3) \neq (x_2, x_1, x_3)$. Design an algorithm that, given (U, V), finds a minimum sized $C \subseteq U$ such that all $v \in V$ are covered by C in time $O^*(\tau(1,2)^n)$, where $\tau(1,2)$ is the branching number for the branching vector $(1,2)$. Prove that your algorithm is correct and that its running time is indeed bounded by $O^*(\tau(1,2)^n)$.

Exercise 2 (10 Points)
Let $G = (V, E)$ be a graph. Design an algorithm that given G constructs a new weighted graph $G' = (V', E')$ with weight function $w\colon E \to \mathbf{N}$, such that $|V'| \in O(2^{|V|/3})$ and $|E'| \in O(2^{2|V|/3})$ and G' contains a triangle with edges e_1, e_2, e_3 and weight $w(e_1) + w(e_2) + w(e_3) = k$ if and only if G contains an independent set of size k. Prove that your algorithm is correct. Describe how the Independent Set problem for input G can be solved in time $O^*(2^{\omega|V|/3})$ using G', where $O(n^\omega)$ is the time required to compute the product of two $n \times n$ matrices.

Exercise 3 (10 Points)
The Domatic Number problem is defined as follows: given an undirected graph $G = (V, E)$, compute the largest integer k such that there is a partition of V into pairwise disjoint sets $V_1, \ldots, V_k$ such that $V_1 \cup \cdots \cup V_k = V$ and each V_i is a dominating set for G. The largest such integer is called the *domatic number* of G. Use dynamic programming to compute the domatic number of an n-vertex graph in time $O^*(3^n)$.

Exercise 4 (10 Points)
The Weighted Hamiltonian Cycle problem is defined as follows: We are given a graph $G = (V = \{s, u_1, \ldots, u_n\}, E)$ whose edges are weighted by a positive integer-valued function $w\colon E \to \{1, \ldots, k\}$. The question is to decide whether G has a simple cycle starting and ending at s, containing all the vertices of G and of total weight at most k. Formulate this problem as an Inclusion-Exclusion formula and show that it can be solved in time $O^*(2^n \cdot \log k)$ and space $O^*(k)$.

Exercise 5 (10 points)
Let $f, g, h\colon 2^U \to \mathbf{Z}$ where $U = \{1, \ldots, n\}$. Recall that the convolution product $f * g\colon 2^U \to \mathbf{Z}$ is defined as $(f * g)(S) = \sum_{X \uplus Y = S} f(X)g(Y)$ for all $S \subseteq U$.

1. Prove that the convolution product is associative, that is, $(f * g) * h = f * (g * h)$.
2. The zeta and Möbius transforms of f are defined as: for all $S \subseteq U$,
$$\zeta(f)(S) = \sum_{X \subseteq S} f(X); \qquad \mu(f)(S) = \sum_{X \subseteq S} (-1)^{|S \setminus X|} f(X).$$
Show that $\mu(\zeta(f)) = f$ for any f.
3. Given a graph $G = (V, E)$, define a function $f\colon 2^V \to \{0, 1\}$ such that its k-fold convolution $(\underbrace{f * \cdots * f}_{k \text{ times}})(V)$ is strictly positive iff there exists a k-coloring of G. Show that your answer is correct by inducting on k.