RWTH Aachen | **WS 2025**
Theoretical Computer Science | Exercise 8
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Exercise for Analysis of Algorithms

Exercise 27

Use the symbolic method to calculate the number of words of length n that can be created by the following grammar:

$P \to$ 😀 P 😍 | 🤡 P 🙀 | 💩 | 💩 P

Exercise 28

We are interested in *mountain ranges*. A mountain range is a sequence of rising, falling or plain fragments (rising and falling have same height difference) starting and ending on height zero and never dropping below height zero. As we are an collector of mountain pictures we are interested in the number r_n of mountain ranges with n fragments.

a) First, we take a look at the following recurrence relations. Let $A(z)$ be the generating function for the sequence a_n, i.e., $A(z) = \sum_{n=0}^{\infty} a_n z^n$. What are the generating functions $B(z)$, $C(z)$, and $D(z)$ for the following three sequences expressed as functions of $A(z)$ and z? Be aware of the different indexes and ranges of the summation.

$$b_n = \sum_{k=0}^{n} a_k a_{n-k} \qquad c_n = \sum_{k=1}^{n} a_{k-1} a_{n-k} \qquad d_n = \sum_{k=1}^{n-1} a_{k-1} a_{n-k}$$

b) Draw all mountain ranges of length 0, 1, 2 and 3 and list the number of mountain ranges of length 0, 1, 2 and 3 in a small table.

c) Find a recurrence relation for r_n, the number of mountain ranges of length n. *Hint:* For a case distinction, consider the first time the mountain range reaches height zero.

d) Derive a generating function $R(z)$ for the recurrence relation in c).

e) What is the exponential growth of r_n? Use your result from d). Explain your steps.

f) How could you proceed to find a closed formula $f(n)$ such that $r_n \sim f(n)$? An explanation is sufficient, you do not have to carry it out.

Exercise 29

Find a generating function for the number of trees with exactly n internal and m external vertices $T_{n,m}$. For what values of n, m do we have $T_{n,m} = T_{m,n}$?

Hint: Do not do all the computations by hand. Seek the help of a computer algebra system. `maxima` can solve quadratic equations and can find the coefficients of a generating function via Taylor expansion.

Exercise 30

```
f(int n){
int s=0;
if (n==0) return 1;
for (int i=0;i<n;i++)
s+=f(i);
return s;
}
```

Compute how often the 5th line of this program is executed using generating functions.