RWTH Aachen	**WS 2025**
Theoretical Computer Science	Exercise 12
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Exercise for Analysis of Algorithms

Exercise 48

After convoluting a series $s_0, s_1, \ldots$ with itself, the result is a series with $b_n = 2^n$. How is s_n defined?

Solution:

$S(z)^2 = \frac{1}{1-2z}$ gives immediately $\frac{1}{\sqrt{1-2z}}$. This can then be solved using the table.

Exercise 49

Let

$$U(z) := \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

Prove that $[z^n]U(z) = 3^n n^{O(1)}$ without doing any computations. Then find out what the constant in the monomial is, i.e., for what c is $[z^n]U(z) = \Theta(n^c 3^n)$.

Solution:

The dominant singularity 1/3 is an algebraic singularity of order $c = 1/2$. Therefore $[z^n]U(z) = \Theta(n^{-c-1}3^n) = \Theta(3^n/n^{3/2})$.

Exercise 50

We continue with the last exercise where

$$U(z) = \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

and we found the constant c with $[z^n]U(z) = \Theta(n^c 3^n)$.

Now also find the multiplicative constant in the Θ-notation, i.e., find a simple function $f(n)$ such that $[z^n]U(z) \sim f(n)$.

Solution:

We estimate $U(z)$ in the vicinity of the dominant singularity:

$$U(z) \sim \frac{1 - \frac{1}{3} - \sqrt{(1-3z)(1+\frac{1}{3})}}{2 \cdot \frac{1}{3}} = 1 - \sqrt{3}\,\sqrt{1-3z} \text{ for } z \to \frac{1}{3}$$

The theorem about algebraic singularities says that

$$[z^n]U(z) \sim -\frac{\sqrt{3}\, n^{-3/2}\, 3^n}{\Gamma(-1/2)} = \frac{\sqrt{3}\, n^{-3/2}\, 3^n}{2\sqrt{\pi}}.$$

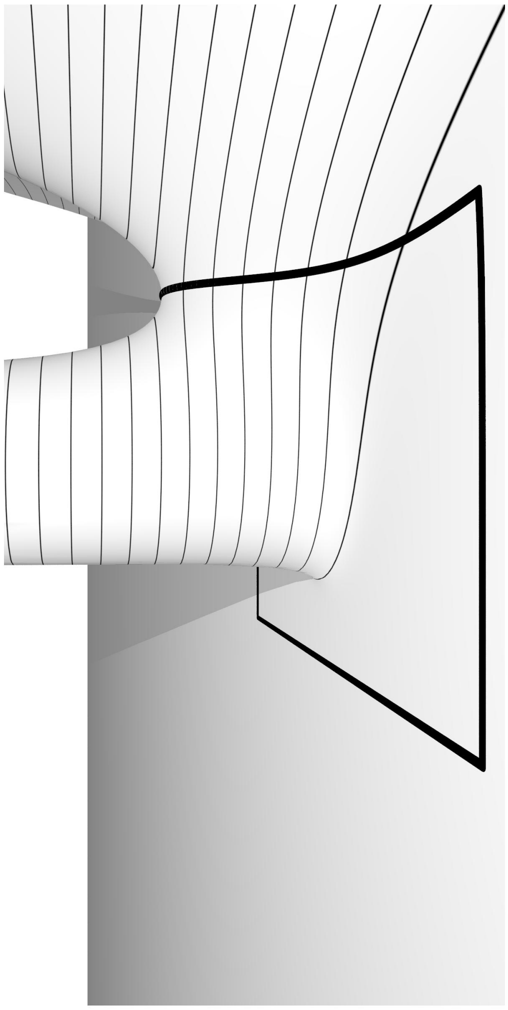
Exercise 51

In the lecture we used the saddle point method to approximate $[z^n]e^z$. In order to do it, we chose a circle as our integrating path.

Approximate now $[z^n]e^z$ using the same method but choosing a rectangular integrating path.

In order to simplify the calculation, you can use a degenerated rectangle.

Solution:



We choose a rectangle that goes through the points $\pm iD$, $-S$ and R. At the beginning have no need to fix the value of D and S, but we do have to fix $R = n + 1$, in order for the integrating path to go through the saddle point. The integral we need to calculate goes through the four edges of the rectangle, and we separate it in those four parts. Let us name them $I_1, \ldots, I_4$.

The first thing we observe is that if we take the limit value $D \to \infty$ then both integrals

$$I_2 = \int_{R+iD}^{-S+iD} \frac{e^z}{z^{n+1}} dz \text{ and } I_4 = \int_{-S-iD}^{R-iD} \frac{e^z}{z^{n+1}} dz \ ,$$

go to zero.
It is also clear that

$$I_3 = \int_{-S-iD}^{-S+iD} \frac{e^z}{z^{n+1}} dz = O(e^{-S}) \ ,$$

so if we set $S = n$, then we can ignore the term I_3 too.
So the only interesting integral is the remaining one

$$I_1 = \int_{R-iD}^{R+iD} \frac{e^z}{z^{n+1}} dz,$$

which we divide again into two parts

$$A = \int_{-\delta}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt$$

and

$$B = \int_{-\infty}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt + \int_{\delta}^{\infty} \frac{e^{R+it}}{(R+it)^{n+1}} i\, dt.$$

We first aproximate A:

$$\begin{aligned} iA &= \int_{-\delta}^{\delta} \frac{e^{R+it}}{(R+it)^{n+1}} dt = \frac{e^R}{R^{n+1}} \int_{-\delta}^{\delta} e^{it-(n+1)\ln(1+it/R)} dt = \\ &= \frac{e^R}{R^{n+1}} \int_{-\delta}^{\delta} e^{t^2/2(n+1)} \big(1 + O(t^3/n^2)\big)\, dt = \frac{e^R}{R^{n+1}} \big(1 + O(\delta^3/n^2)\big) \int_{-\delta}^{\delta} e^{t^2/2(n+1)}\, dt \end{aligned}$$

We choose δ such that $\delta^3/n^2 = o(1)$, and obtain

$$iA = \frac{e^R}{R^{n+1}} \sqrt{2(n+1)\pi} \ .$$

Through and simple subsitution and appending the tails (the value of B), we can take this to be the integral from $-\infty$ to ∞.
We obtain then finally

$$1/n! = [z^n]e^z = \frac{1}{2\pi i} \oint \frac{e^z}{z^{n+1}} dz \sim \frac{1}{2\pi} A = \frac{1}{2\pi} \frac{e^R}{R^{n+1}} \sqrt{2(n+1)\pi}$$