RWTH Aachen **WS 2025**
Theoretical Computer Science Exercise 11
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Exercise for Analysis of Algorithms

Exercise 43

In this exercise we consider the following (regular) CFG G:

$$S \to abA \mid bS \mid a$$
$$A \to bA \mid aS$$

1. Find a generating function for number of words s_n in $L(G)$ that have length n.
2. What is the dominant singularity and what kind of singularity is it?
3. What is the exponential growth of s_n?
4. How precisely can you estimate s_n with just the knowledge of the dominating singularity and its nature?
5. Find a closed formula for s_n with an additive error of at most $O(0.8^n)$.

Solution:

1. Since the grammar is unique, the symbolic method gives us
$$S(z) = z^2A(z) + zS(z) + z,$$
$$A(z) = zA(z) + zS(z).$$
We solve the latter for $A(z)$ and obtain $A(z) = zS(z)/(1-z)$. Now we can insert it into the former. This yields
$$S(z) = z^3S(z)/(1-z) + zS(z) + z.$$
We then solve for $S(z)$ and get the generating function
$$S(z) = \frac{z}{1-z-z^3/(1-z)} = \frac{z(1-z)}{(1-z)^2-z^3}.$$

2. The singularities are the roots of of the denominator. We ask Maxima `solve((1-z)^2-z^3,z)` and get
$$z = -\frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}\left(\sqrt{3}\,i+1\right)+5\,2^{\frac{2}{3}}\,3^{\frac{5}{6}}\,i-2^{\frac{4}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}-5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{4}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}},$$
$$z = \frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}\left(\sqrt{3}\,i-1\right)+5\,2^{\frac{2}{3}}\,3^{\frac{5}{6}}\,i+2^{\frac{4}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}+5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{4}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}},$$
$$z = \frac{\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{2}{3}}+2^{\frac{1}{3}}\,3^{\frac{1}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}-5\,2^{\frac{2}{3}}\,3^{\frac{1}{3}}}{2^{\frac{1}{3}}\,3^{\frac{7}{6}}\left(9\sqrt{23}+11\sqrt{3}\right)^{\frac{1}{3}}}$$

Wolfram Alpha even gives us a nice diagram

`http://www.wolframalpha.com/input/?i=(1-z)%5E2-z%5E3+%3D+0`

from which we see that we have a small real and two larger complex conjugated singularities. We evaluate them numerically and see that their magitude are $\beta \approx 1.32471$ and $\alpha \approx 0.5698402909980533$. The dominant singularity is α. We decomposed the denominator of $S(z)$ into three roots of degree one. The function $S(z)(z-\alpha)$ is therefore analytical at α. This means that α is a pole of first order.

For a meromorph generating function $B(z)$ with poles $\alpha_1, \ldots, \alpha_m$ we know that there exist polynomials $P_1(n), \ldots, P_m(n)$ such that

$$[z^n]B(z) = \sum_{j=1}^{m} P_j(n)\alpha_j^n$$

and the degree of the polynomial $P_j(n)$ is one smaller than the order of the pole α_j. Since we have three poles of first order the polynomials are constants. We have

$$s_n = c_1(\alpha^{-n}) + c_2(\beta^{-n}).$$

for some constants c_1 and c_2.

3. The exponetial growth is $\alpha^{-n} \approx 1.754877666246692655^n$.

4. Notice that $c_2(\beta^{-n}) = O(0.8^n)$. If we can find the hidden factor c_1 we get a good approximation with vanishing additive error for s_n. We can decompose

$$S(z) = \sum_{n=1}^{\infty} c_1\alpha^{-n}z^n + c_2\beta^{-n}z^n = \frac{c_1}{1-z/\alpha} + \frac{c_2}{1-z/\beta}$$

Let

$$B(z) = \frac{1}{1-z/\alpha} \quad \text{and} \quad E(z) = \frac{1}{1-z/\beta}$$

Notice that $\lim_{z\to\alpha} B(z) = \infty$ while $\lim_{z\to\alpha} E(z)$ is constant. Then

$$\lim_{z\to\alpha} \frac{S(z)}{B(z)} = \lim_{z\to\alpha} \frac{c_1B(z) + c_2E(z)}{B(z)} = c_1.$$

We use maxima to approximate

$$\lim_{z\to\alpha} \frac{S(z)}{B(z)} = \lim_{z\to\alpha} \frac{z(1-z)(1-z/\alpha)}{(1-z)^2 - z^3} \approx 0.23448675.$$

We use the following programm to verify the correctness

```
s = range(0,1000)
a = range(0,1000)
s[0] = 0
s[1] = 1
a[0] = 0
a[1] = 0
for n in range(2, 100):
s[n] = a[n-2]+s[n-1]
a[n] = a[n-1]+s[n-1]
print n, s[n], 1.0*s[n]/s[n-1],
0.23448675*1.754877666246692655**n/s[n]
```

The last line states

```
99 355268071453933228439241 1.75487766625 0.999999931817.
```

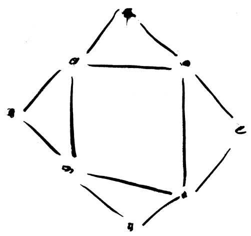
Indeed, after 100 iterations we only make a multiplicative error of 0.999999931817.

Exercise 44

An algorithm I computes an optimal independent set for an undirected graph $G = (V, E)$ of size n as follows: It picks a vertex v with maximal degree. If this degree is at most two, then the graph is a collection of cycles and paths and the solution is computed in linear time.

Otherwise, the optimal independent set either contains v (and then cannot contain any vertex in $N(v)$) or it does not. Hence, the algorithm recursively computes the two independent sets $I(G[V - N(v)])$ and $I(G[V - \{v\}])$ and then chooses the bigger one, or the first if they have the same size.

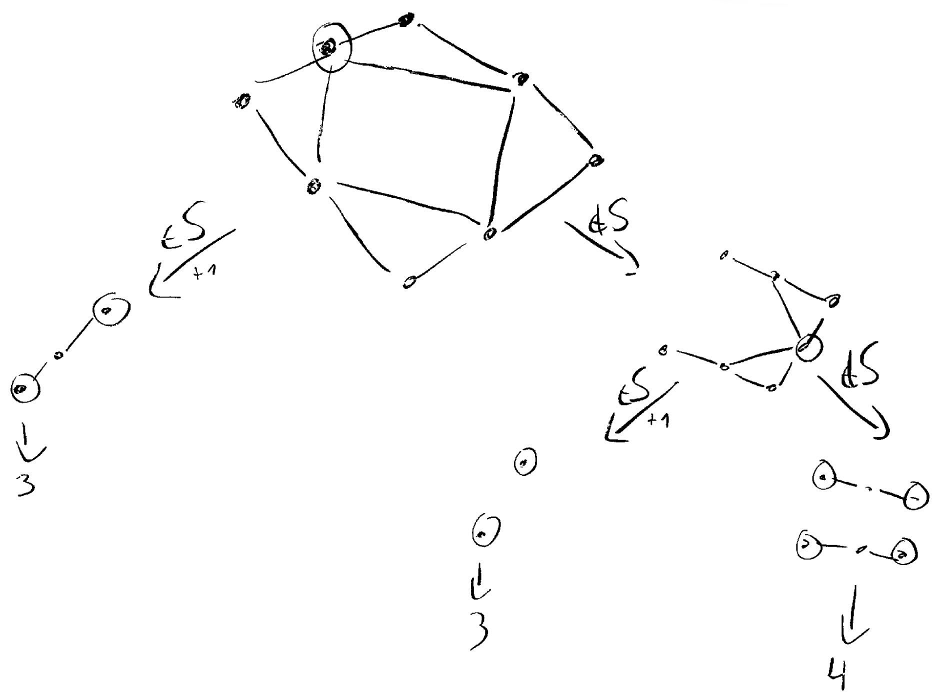
1. Simulate the algorithm on this graph:



2. Estimate its asymptotic running time up to a constant factor.

Solution:

1.



The best solution is therefore obtained by picking the 4 corner points.

2.

The smallest reduction happens if the graph has max degree 3, in this case we get the following recurrence relation:

$$t(n) = t(n-4) + t(n-1) + (n = 0)$$

We find a generating function:

$$\begin{aligned} S(z) &= z^4 S(z) + zS(z) + 1 \\ (1 - z^4 - z)S(z) &= 1 \\ S(z) &= \frac{1}{1 - z^4 - z} \end{aligned}$$

Finding the dominant singularity α of $S(z)$ by setting $1 - z^4 - z = 0$, yields the numerical result $\alpha \approx 0.72449$. And a runtime of the algorithm of $O(\alpha^{-n}) = O(1.38028^n)$.

Exercise 45

Prove that

$$[z^n](1-z)^w \sim \frac{n^{-w-1}}{\Gamma(-w)}$$

for $w \in \mathbf{C}$ without using the theorem of the lecture. (The idea of this assignment is to get a deeper insight into the theorem.)

Hint: Use Newton's formula. Now replace the binomial coefficient by factorials or the gamma function. In the first case, you need to be careful with a definition of factorials for real numbers. In general, however, $\Gamma(n+1) = n!$.

Solution:

$$\begin{aligned} [z^n](1-z)^w &= \binom{w}{n}(-1)^n \\ &= \binom{n-w-1}{n} \\ &= \frac{(n-w-1)!}{n!(-w-1)!} \\ &= \frac{1}{\underline{n^{w+1}}(-w-1)!} \\ &= \frac{1}{\underline{n^{w+1}}\Gamma(-w)} \\ &= \frac{n^{-w-1}}{\Gamma(-w)}\left(1 + O\left(\frac{1}{n}\right)\right) \end{aligned}$$

Exercise 46

Approximate $[z^n]\frac{1}{2-e^z}$ up to an error of $O(12^{-n})$.

Solution:
In the lecture, we found the first term (of the dominant singularity), namely $\frac{1}{2}(\frac{1}{\ln 2})^{n+1}$. So we take a look at the singularity with the second highest absolute value, which is $\ln 2 \pm 2\pi i$. Both are poles of order 1. Let us see how $S(z)$ behaves asymptotically for $z \to \ln 2 \pm 2\pi i$. We have that $e^z \sim 2(1 - \ln 2 \mp 2\pi i + z)$ for $z \to \ln 2 \pm 2\pi i$ and therefore

$$\begin{aligned} \frac{1}{2-e^z} &\sim \frac{1}{2-(2-2\ln 2 \mp 4\pi i + 2z)} \\ &= \frac{1}{2}\frac{1}{\ln 2 \mp 2\pi i}\frac{1}{1-\frac{z}{\ln 2 \mp 2\pi i}} \\ &= \frac{1}{2}\frac{1}{\ln 2 \mp 2\pi i}\sum_{n=0}^{\infty}(\frac{1}{\ln 2 \mp 2\pi i})^n z^n \end{aligned}$$

With Theorem 9 we get:

$$\begin{aligned} [z^n]S(z) &= \frac{1}{2}\left(\left(\frac{1}{\ln 2}\right)^{n+1} + (\frac{1}{\ln 2 + 2\pi i})^{n+1} + (\frac{1}{\ln 2 - 2\pi i})^{n+1}\right) + O(r)^{-n} \\ &= \frac{1}{2}\left(\left(\frac{1}{\ln 2}\right)^{n+1} + r^{n+1}(e^{i\phi(n+1)} + e^{-i\phi(n+1)})\right) + O(r)^{-n} \\ &= \frac{1}{2}\left(\left(\frac{1}{\ln 2}\right)^{n+1} + 2r^{n+1}\cos(\phi(n+1))\right) + O(r)^{-n} \end{aligned}$$

with $r = 1/\sqrt{\ln^2 2 + 4\pi^2} \approx 12.58547409739904$, $\phi = \arctan(\frac{2\pi}{\ln 2})$.

Exercise 47
Determine g_n up to an additive error of $O(4^n)$ for

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = \frac{15z^2 + 8z + 1}{15z^2 - 8z + 1}.$$

Solution:
We have

$$G(z) = \frac{15z^2 + 8z + 1}{(3z-1)(5z-1)},$$

the singularities are at $\frac{1}{3}$ and $\frac{1}{5}$, and both are poles of first order. Because of

$$G(z) \sim 8\frac{1}{(z-\frac{1}{5})} \text{ for } z \to \frac{1}{5}$$

the difference between g_n and $[z^n]\frac{8}{1-5z}$ is at most $O(3^n)$. Therefore $g_n = \frac{8}{.}5^n + O(3^n)$.