Exercise for Analysis of Algorithms

Exercise 27

Use the symbolic method to calculate the number of words of length n that can be created by the following grammar:

$$P \to \quad \text{😀}\, P\, \text{😍} \quad | \quad \text{🤡}\, P\, \text{🙀} \quad | \quad \text{💩} \quad | \quad \text{💩}\, P$$

Solution:

The language of this grammar can be described by the following recursive definition.

$$P = (\{\text{😀}\} \times P \times \{\text{😍}\}) \cup (\{\text{🤡}\} \times P \times \{\text{🙀}\}) \cup \{\text{💩}\} \cup (\{\text{💩}\} \times P)$$

We need to define the weight of the atomic elements.

$$|\text{😀}| = |\text{😍}| = |\text{🤡}| = |\text{🙀}| = |\text{💩}| = 1$$

Let $T(z)$ be the generating function for the number of words of length n generated by the grammar. The symbolic method yields

$$T(z) = 2z^2 T(z) + z + zT(z).$$

We transform this into

$$T(z) = \frac{z}{1 - z - 2z^2}.$$

Notice that $1 - z - 2z^2 = (z+1)(1-2z)$. We want to find a partial fraction decomposition of the form

$$\frac{z}{1 - z - 2z^2} = \frac{A}{z+1} + \frac{B}{1-2z}.$$

By setting $z = 0$ we get $0 = A + B$ and by setting $z = 1$ we get $-1/2 = A/2 - B$. Together they yield $A = -1/3$ and $B = 1/3$. This means

$$T(z) = -\frac{1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{1}{1-2z}$$

and therefore $[z^n]T(z) = \frac{1}{3}(2^n - (-1)^n)$. There are $\frac{1}{3}(2^n - (-1)^n)$ words of length n.

Exercise 28

We are interested in *mountain ranges*. A mountain range is a sequence of rising, falling or plain fragments (rising and falling have same height difference) starting and ending on height zero and never dropping below height zero. As we are an collector of mountain pictures we are interested in the number r_n of mountain ranges with n fragments.

a) First, we take a look at the following recurrence relations. Let $A(z)$ be the generating function for the sequence a_n, i.e., $A(z) = \sum_{n=0}^{\infty} a_n z^n$. What are the generating functions $B(z)$, $C(z)$, and $D(z)$ for the following three sequences expressed as functions of $A(z)$ and z? Be aware of the different indexes and ranges of the summation.

$$b_n = \sum_{k=0}^{n} a_k a_{n-k} \qquad c_n = \sum_{k=1}^{n} a_{k-1} a_{n-k} \qquad d_n = \sum_{k=1}^{n-1} a_{k-1} a_{n-k}$$

b) Draw all mountain ranges of length 0, 1, 2 and 3 and list the number of mountain ranges of length 0, 1, 2 and 3 in a small table.

c) Find a recurrence relation for r_n, the number of mountain ranges of length n. *Hint:* For a case distinction, consider the first time the mountain range reaches height zero.

d) Derive a generating function $R(z)$ for the recurrence relation in c).

e) What is the exponential growth of r_n? Use your result from d). Explain your steps.

f) How could you proceed to find a closed formula $f(n)$ such that $r_n \sim f(n)$? An explanation is sufficient, you do not have to carry it out.

Solution:

a) It is easy to see that b_n is the convolution of a_n with itself. Hence $B(z) = A(z)^2$. For c_n first note that it does not matter if the summation starts at 0 or 1. So c_n is the convolution of a_{n-1} and a_n and $C(z) = zA(z)^2$. Finally, $d_n = c_n - a_0 a_{n-1}$ and $D(z) = C(z) - a_0 z A(z) = zA(z)^2 - a_0 z A(z)$.

c) Let k be the first time the mountain range hits the ground. Then k can be in the range $1, \ldots, n$. If $k = 1$ the first fragment must be straight. Otherwise, we start going up, then there is a mountain range of length $k-2$, and the go down, followed by a mountain range of length $n-k$. Alltogether this gives us the recurrence

$$r_n = r_{n-1} + \sum_{k=2}^{n} r_{k-2} r_{n-k} + (n = 0).$$

At this point it is a good idea to check the recurrence agains our small table. According to the recurrence $r_0 = 1$, $r_1 = r_0 + 0 = 1$, $r_2 = r_1 + r_0 r_0 = 2$, and $r_3 = r_2 + r_0 r_1 + r_1 r_0 = 2 + 1 + 1 = 4$.

d) Using the same argumentation as in a) we can read off an equation for generating function:

$$R(z) = zR(z) + z^2 R(z)^2 + 1$$

Solving for $R(z)$ yields

$$R(z) = \frac{1 - z \pm \sqrt{-3z^2 - 2z + 1}}{2z^2}$$

e) We can do a singularity analysis. We find the singularities where the square root becomes zero, i.e., where $-3z^2 - 2z + 1 = 0$. The dominant singularity is $1/3$ and the exponential growth 3^n.

f) We can use Darboux's theorem. For $z \to 1/3$ we get

$$R(z) \sim \frac{1-\frac{1}{3}}{2(1/3)^2} \pm \frac{\sqrt{-3(1+\frac{1}{3})(z-\frac{1}{3})}}{2(1/3)^2} \sim C\sqrt{1-3z},$$

for some C that we would have to compute. Then $r_n = [z^n]R(z) \sim C\binom{1/2}{n}(-3)^n$.

(If you want to finish the task: It is easy to see that $C = -3^{3/2}$ and we can use $\binom{1/2}{n} = \binom{2n}{n}\frac{(-1)^{n+1}}{2^{2n}(2n-1)}$ to write the result in a more readable way.

$$r_n \sim \frac{3^{3/2}}{2n-1}\binom{2n}{n}\left(\frac{3}{4}\right)^n$$

We can get an even nicer expression by replacing the binomial coefficient with an approximation based on Stirling's formula.)

Exercise 29

Find a generating function for the number of trees with exactly n internal and m external vertices $T_{n,m}$. For what values of n, m do we have $T_{n,m} = T_{m,n}$?
Hint: Do not do all the computations by hand. Seek the help of a computer algebra system. `maxima` can solve quadratic equations and can find the coefficients of a generating function via Taylor expansion.

Solution:

We consider a recursive definition of trees. A tree is either an external node or an internal node with at least one subtree. This yields

$$T = \square \cup \bigcirc \times T \times T^*$$

und and the generating function

$$T(u,z) = u + z\frac{T(u,z)}{1-T(u,z)},$$

where u is the number of external and z is the number of internal nodes. We need to solve this equation for $T(u,z)$. We delegate this task to `maxima`: The call `solve(T=u+z*T/(1-T),T)` yields

$$-\frac{\sqrt{z^2+(-2*u-2)*z+u^2-2*u+1}+z-u-1}{2}$$

and

$$+\frac{\sqrt{z^2+(-2*u-2)*z+u^2-2*u+1}-z+u+1}{2}.$$

We know that for $u = z = 0$ the solution needs to be 0. Thus only the first solution is correct.

We get the coefficients by doing a Taylor expansion of $T(u,z)$. We enter

```
T:-((sqrt(z^2+(-2*u-2)*z+u^2-2*u+1)+z-u-1)/2);
taylor(T,[z,u],0,5);
```

into `maxima`. We read the coefficients:

$n+m$	Term
1	u
2	uz
3	$uz^2 + u^2z$
4	$uz^3 + 3u^2z^2 + u^3z$
5	$uz^4 + 6u^2z^3 + 6u^3z^2 + u^4z$

We also want to find out for which values holds $T(u,z) = T(z,u)$. We do so by calculating $T(u,z) - T(z,u)$. We type into `maxima`:

```
TT(u,z):=-((sqrt(z^2+(-2*u-2)*z+u^2-2*u+1)+z-u-1)/2);
ev(TT(u,z)-TT(z,u),expand);
```

The answer is $u - z$. Therefore, the generating function of the difference is $u - z$. The all coefficients of this function are zero except for the case $u = 1, z = 0$ or $u = 0, z = 1$. This means $T_{u,z} = T_{z,u}$ for all other values. This makes sense as there is exactly one tree with a single external node and zero trees with a single internal node.

Exercise 30

```
f(int n){
int s=0;
if (n==0) return 1;
for (int i=0;i<n;i++)
s+=f(i);
return s;
}
```

Compute how often the 5th line of this program is executed using generating functions.

Solution:
Let A_n denote the how often the 5th line is called on input n. We immediately obtain $A_0 = 0$ and $A_n = n + \sum_{i=0}^{n-1} A_i$.
The corresponding generating function is thus

$$A(z) = \frac{z}{(1-z)^2} + \frac{zA(z)}{1-z}.$$

This implies

$$A(z)(1-2z) = \frac{z}{1-z}$$

and therefore

$$\begin{aligned} A(z) &= \frac{1}{(1-z)(1-2z)} z \\ &= \left(\frac{A}{1-z} + \frac{B}{1-2z} \right) z \end{aligned}$$

for some A, B. Setting $z = 0$ implies $A + B = 1$ and $z = -1$ implies $A/2 + B/3 = 1/6$. We easily obtain $A = -1$ and $B = 2$ and therefore

$$A(z) = \left(\frac{2}{1-2z} - \frac{1}{1-z} \right) z.$$

Since the multiplication with z is just a shift, the solution is therefore

$$a_{n+1} = 2 \cdot 2^n - 1 = 2^{n+1} - 1$$