Old Exam (2014) with solutions 0

This is an old exam from 2014.

Task T1
Consider the following algorithm for searching an array $a[1,\ldots,n]$ for an element x. We assume that the array is sorted in increasing order and that the element x is at some random location in the array. Let B_n be the expected number of comparisons on an n-element array. Write down a recurrence for B_n. What is B_3?

Algorithm: Binary Search with randomly chosen pivot element

1. Choose randomly and with uniform probability an $i \in \{1,\ldots,n\}$.
2. If $a[i] = x$, output i and halt.
3. Continue recursively on the left subarray, if $x < a[i]$, or the right subarray, if $x > a[i]$.

Solution
There are two cases to consider here: The first is that the element x is found at the randomly chosen location i. This happens with a probability of $1/n$. With a probability of $1-1/n$, the search continues and the element is found at the recursive step. Now if the element x is found at the recursive step then the expected number of comparisons made is:

$$1+\frac{1}{n}\left(\sum_{k=1}^{n}\frac{k-1}{n-1}B_{k-1}+\sum_{k=1}^{n}\frac{n-k}{n-1}B_{n-k}\right).$$

This may be explained as follows: In this case, one comparison is made and the search is carried on in either the left or right subarray. Now the probability that the index chosen is k is $1/n$. The probability that the element being searched for is in the left subarray is $(k-1)/(n-1)$, since there are $n-1$ possibilities and there are $k-1$ of them to the left. The last term above is the expected number of comparisons made if the element is in the right subarray. Now the expected number of comparions is:

$$B_n=\frac{1}{n}+\frac{n-1}{n}\left(1+\sum_{k=1}^{n}\left(\frac{k-1}{n-1}B_{k-1}+\frac{n-k}{n-1}B_{n-k}\right)\right).$$

This may be written as follows:

$$B_n = 1+\frac{2}{n^2}\sum_{k=0}^{n-1}kB_k.$$

Now, $B_1 = 1$, $B_2 = \frac{3}{2}$, and $B_3 = 17/9$.

Task T2
An alphabet Σ consists of two numeric characters $1, 2$ and four alphabetic characters a, b, c, d. Find and solve a recurrence relation for the number of words of length n in Σ^*, where there are no consecutive (identical or distinct) numeric characters.

Solution
Let the number of n-length strings be A_n. Then $A_0 = 1$ and $A_1 = 6$. If the first letter is alphabetic, then there are $4A_{n-1}$ strings. If the first letter is numeric, then the second letter must be alphabetic and there are $8A_{n-2}$ strings. Thus the recurrence we are seeking is:

$$A_n = 4A_{n-1} + 8A_{n-2}, \text{ with } A_0 = 1 \text{ and } A_1 = 6.$$

Task T3
Find an expression for

$$[z^n]\frac{1}{(1-z)^2}\ln\frac{1}{1-z}.$$

Your solution can include a sum!

Solution
Define $\bar{H}_n = 0$ if $n = 0$ and $\bar{H}_n = H_n$ for $n \geq 1$. We may write down the given function as:

$$\begin{aligned}\frac{1}{(1-z)^2}\ln\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n \sum_{n=0}^{\infty} \bar{H}_n z^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \bar{H}_k z^n.\end{aligned}$$

Thus the coefficient of z^n is $\sum_{k=0}^{n} \bar{H}_k$.

Task T4
Sort the series with the following generating functions by their asymptotic growth. Justify your steps!

1. $A(z) = \frac{1}{\sqrt{2-\frac{1}{z}}}$.
2. $B(z) = \frac{z}{2-3z+z^2}$.
3. $C(z) = \frac{e^{-z-z^2/2}}{1-z}$.

Solution
We first determine the exponential growth of each series. Maybe we can derive an order from it.
The dominant singularity of $A(z)$ is $1/2$, hence $A_n \asymp 2^n$.
The series $B(z)$ has the singularities 1 and 2. Hence $B_n \asymp 1$.
The dominant singularity of $C(z)$ is 1, so $[z^n]C(z)$ has the same exponential growth.
So far we have determined that $[z^n]A(z)$ grows faster asymptotically than the other two. Now we have to compare both in more detail using singularity analysis.
As 1 is a singularity of first order, we compute $\lim_{z\to 1}(1-z)B(z)$ which is 1. Hence we get $B_n = 1 + o(1)$. Doing the same for $C(z)$ we get that $\lim_{z\to 1}(1-z)C(z)$ is $e^{-3/2}$ and $C_n = e^{-3/2} + o(1)$ which is asymptotically smaller than B_n.
Hence we get that following order of asymptotic growth: $C_n \preceq B_n \preceq A_n$.